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Ricci solitons in manifolds with quasi-constant curvature

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This paper is dedicated to the memory of Professor Stere Ianus (1939–2010)

Abstract. The Eisenhart problem of finding parallel tensors treated already in the framework of quasi-constant curvature manifolds in [Jia] is reconsidered for the symmetric case and the result is interpreted in terms of Ricci solitons. If the generator of the manifold provides a Ricci soliton then this is i) expanding on para-Sasakian spaces with constant scalar curvature and vanishing D-concircular tensor field and ii) shrinking on a class of orientable quasi-umbilical hypersurfaces of a real projective space=elliptic space form.

1. Introduction

In 1923, Eisenhart [Eisenhart] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1926, Levy [Levy] proved that a parallel second order symmetric non-degenerated tensor α in a space form is proportional to the metric tensor. Note that this question can be considered as the dual to the the problem of finding linear connections making parallel a given tensor field; a problem which was considered by Wong in [Wong]. Also, the former question implies topological restrictions namely if the (pseudo) Riemannian manifold M admits a parallel symmetric $(0, 2)$ tensor field then M is locally the direct product of a number of (pseudo) Riemannian

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manifolds, [Wu] (cited by [Zhao]). Another situation where the parallelism of α is involved appears in the theory of totally geodesic maps, namely, as is point out in [Oniciuc, p. 114], $\nabla\alpha = 0$ is equivalent with the fact that $1 : (M, g) \to (M, \alpha)$ is a totally geodesic map.

While both Eisenhart and Levy work locally, Ramesh Sharma gives in [Sharma1] a global approach based on Ricci identities. In addition to space-forms, Sharma considered this *Eisenhart problem* in contact geometry [Sharma2]–[Sharma4], for example for K-contact manifolds in [Sharma3]. Since then, several other studies appeared in various contact manifolds, see for example, the bibliography of [CalinCrasm].

Another framework was that of quasi-constant curvature in [Jia]; recall that the notion of *manifold with quasi-constant curvature* was introduced by Bang-yen Chen and Kentaro Yano in 1972, [ChenYano], and since then, was the subject of several and very interesting works, [Bernardini], [DeGhosh], [Wang], in both local and global approaches. Unfortunately, the paper of Jia contains some typos and we consider that a careful study deserves a new paper. There are two remarks regarding Jia result: i) it is in agreement with what happens in all previously recalled contact geometries for the symmetric case, ii) it is obtained in the same manner as in Sharma's paper [Sharma1]. Our work improves the cited paper with a natural condition imposed to the generator of the given manifold, namely to be of torse-forming type with a regularity property.

Our main result is connected with the recent theory of Ricci solitons [Cao], a subject included in the Hamilton–Perelman approach (and proof) of Poincaré Conjecture. A connection between Ricci flow and quasi-constant curvature manifolds appears in [CaiZhao]; thus our treatment for Ricci solitons in quasi-constant curvature manifolds seems to be new.

Our work is structured as follows. The first section is a very brief review of manifolds with quasi-constant curvature and Ricci solitons. The next section is devoted to the (symmetric case of) Eisenhart problem in our framework and the relationship with the Ricci solitons is pointed out. A technical conditions appears, which we call *regularity*, and is concerning with the non-vanishing of the Ricci curvature with respect to the generator of the given manifold. Let us remark that in the Jia's paper this condition is involved, but we present a characterization of these manifolds as well as some remarkable cases which are out of this condition namely: quasi-constant curvature locally symmetric and Ricci semi-symmetric metrics. A characterization of Ricci soliton is derived for dimension greater that 3.

Four concrete examples involved in possible Ricci solitons on quasi-constant manifolds are listed at the end. For the second example, we pointed out some consequences which are yielded by the hypothesis of compacity, used in paper [DragomirGrimaldi], in connection with (classic by now) papers of T. Ivey and Perelman.

2. Quasi-constant curvature manifolds. Ricci solitons

Fix a triple (M, g, ξ) with M_n a smooth $n(>2)$-dimensional manifold, g a Riemannian metric on M and ξ an unitary vector field on M. Let η the 1-form dual to ξ with respect to g.

If there exist two smooth functions a, $b \in C^\infty(M)$ such that:

$$R(X,Y)Z = a[g(Y,Z)X - g(X,Z)Y] + b[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ + b\eta(Z)[\eta(Y)X - \eta(X)Y] \tag{2.1}$$

then (M, g, ξ) is called *manifold of quasi-constant curvature* and ξ is *the generator*, [ChenYano]. Using the notation of [DragomirTomassini, p. 325] let us denote $M^n_{a,b}(\xi)$ this manifold.

It follows:

$$R(X,Y)\xi = (a+b)[\eta(Y)X - \eta(X)Y] \tag{2.2}$$

$$R(X,\xi)Z = (a+b)[\eta(Z)X - g(X,Z)\xi] \tag{2.3}$$

while the Ricci curvature $S(X,Y) = Tr(Z \to R(Z,X)Y)$ is:

$$S(X,Y) = [a(n-1)+b]g(X,Y) + b(n-2)\eta(X)\eta(Y) \tag{2.4}$$

which means that (M, g, ξ) is an *eta-Einstein manifold*; in particular, if a, b are scalars, then (M, g, ξ) is an *quasi-Einstein manifold*, [GhoshDeBinh]. The scalar curvature is:

$$r = (n-1)(na+2b), \tag{2.5}$$

and we derive:

$$a = \frac{r - 2S(\xi,\xi)}{(n-1)(n-2)}, \quad b = \frac{nS(\xi,\xi) - r}{(n-1)(n-2)}. \tag{2.6}$$

Then $a + b = \frac{S(\xi,\xi)}{n-1}$. Let us consider also the Ricci $(1,1)$ tensor field Q given by: $S(X,Y) = g(QX,Y)$. From (2.4) we get:

$$Q(X) = [a(n-1)+b]X + b(n-2)\eta(X)\xi \tag{2.7}$$

which yields:

$$Q(\xi) = (a+b)(n-1)\xi \tag{2.8}$$

and then ξ is an eigenvalue of Q.

In the last part of this section we recall the notion of Ricci solitons according to [Sharma5, p. 139]. On the manifold M, a *Ricci soliton* is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that:

$$\mathcal{L}_V g + 2S + 2\lambda g = 0. \tag{2.9}$$

The Ricci soliton is said to be *shrinking, steady* or *expanding* according as λ is negative, zero or positive.

Also, we adopt the notion of η-*Ricci soliton* introduced in the paper [ChoKimura] as a data (g, V, λ, μ):

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0. \tag{2.10}$$

3. Parallel symmetric second order tensors and Ricci solitons

Fix α a symmetric tensor field of $(0, 2)$-type which we suppose to be parallel with respect to the Levi–Civita connection ∇ i.e. $\nabla\alpha = 0$. Applying the Ricci identity $\nabla^2\alpha(X, Y; Z, W) - \nabla^2\alpha(X, Y; W, Z) = 0$ we obtain the relation (1.1) of [Sharma1, p. 787]:

$$\alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W) = 0 \tag{3.1}$$

which is fundamental in all papers treating this subject. Replacing $Z = W = \xi$ and using (2.2) it results, by the symmetry of α:

$$(a + b)[\eta(Y)\alpha(X, \xi) - \eta(X)\alpha(Y, \xi)] = 0. \tag{3.2}$$

Definition 3.1. $M^n_{a,b}(\xi)$ is called *regular* if $a + b \neq 0$.

In order to obtain a characterization of such manifolds we consider:

Definition 3.2 ([RachunekMikes]). ξ is called *semi-torse forming vector field* for (M, g) if, for all vector fields X:

$$R(X, \xi)\xi = 0. \tag{3.3}$$

From (2.2) we get: $R(X, \xi)\xi = (a + b)(X - \eta(X)\xi)$ and therefore, if $X \in \ker\eta = \xi^\perp$, then $R(X, \xi)\xi = (a + b)X$ and we obtain:

Proposition 3.3. *For $M^n_{a,b}(\xi)$ the following are equivalent:*

i) *is regular,*

ii) ξ *is not semi-torse forming,*

iii) $S(\xi,\xi) \neq 0$ *i.e.* ξ *is non-degenerate with respect to* S,

iv) $Q(\xi) \neq 0$ *i.e.* ξ *does not belong to the kernel of* Q.

In particular, if ξ *is parallel* ($\nabla\xi = 0$) *then* M *is not regular.*

Remarks 3.4. i) From Theorems 2 and 3 of [Wang, p. 175] a regular $M^n_{a,b}(\xi)$ is neither recurrent nor locally symmetric.

ii) From Theorem 3 of [DragomirGrimaldi, p. 228] a regular $M^n_{a,b}(\xi)$ with a and b constants is not Ricci semi-symmetric.

In the following we restrict to the regular case. Returning to (3.2), with $X = \xi$ in:

$$\eta(Y)\alpha(X,\xi) = \eta(X)\alpha(Y,\xi) \tag{3.4}$$

we derive:

$$\alpha(Y,\xi) = \eta(Y)\alpha(\xi,\xi) = \alpha(\xi,\xi)g(Y,\xi). \tag{3.5}$$

The parallelism of α implies also that $\alpha(\xi,\xi)$ is a constant:

$$X(\alpha(\xi,\xi)) = 2\alpha(\nabla_X\xi,\xi) = 2\alpha(\xi,\xi)g(\nabla_X\xi,\xi) = 2\alpha(\xi,\xi)\cdot 0 = 0. \tag{3.6}$$

Making $Y = \xi$ in (3.1) and using (2.3) we get:

$$\eta(Z)\alpha(X,W) - g(X,Z)\alpha(\xi,W) + \eta(W)\alpha(X,Z) - g(X,W)\alpha(\xi,Z) = 0$$

which yield, via (3.5) and $W = \xi$:

$$\alpha(X,Z) = \alpha(\xi,\xi)g(X,Z). \tag{3.7}$$

In conclusion:

Theorem 3.5. *A parallel second order symmetric covariant tensor in a regular* $M^n_{a,b}(\xi)$ *is a constant multiple of the metric tensor.*

At the end of this section we include some applications of the above Theorem to Ricci solitons:

Naturally, two remarkable situations appear regarding the vector field V: $V \in \operatorname{span}\xi$ or $V \perp \xi$ but the second class seems far too complex to analyse in practice. For this reason it is appropriate to investigate only the case $V = \xi$. So, we can apply the previous result for $\alpha := \mathcal{L}_\xi g + 2S$ which yields $\lambda = -S(\xi,\xi)$.

Theorem 3.6. *Fix a regular* $M^n_{a,b}(\xi)$.

i) *A Ricci soliton* $(g,\xi,-S(\xi,\xi) \neq 0)$ *can not be steady but is shrinking if the constant* $S(\xi,\xi)$ *is positive or expanding if* $S(\xi,\xi) < 0$.

ii) *An η-Ricci soliton (g, ξ, λ, μ) provided by the parallelism of $\alpha + 2\mu\eta \otimes \eta$ is given by:*

$$\lambda + \mu = -S(\xi, \xi) \neq 0. \tag{3.8}$$

iii) *If $n \geq 4$ and $b \neq 0$ then $(g, \xi, -S(\xi, \xi))$ is a Ricci soliton if and only if ξ is geodesic i.e. $\nabla_\xi \xi = 0$ and:*

$$\frac{\xi(a+b)}{4b} + a(n-1) + b = \frac{a+b}{n-1}. \tag{3.9}$$

Proof. iii) We have three cases:

I) $\alpha + 2\lambda g = 0$ on $\operatorname{span} \xi$ yields the above expression of λ.

II) $\alpha + 2\lambda g = 0$ on $\ker \eta = \xi^\perp$ gives:

$$\frac{\xi(a+b)}{4b} + \lambda + a(n-1) + b = 0 \tag{3.10}$$

where we use the formula (3.5) of [GanchevMihova, p. 123].

III) $\alpha + 2\lambda g = 0$ on $(U, \xi) \in \ker \eta \oplus \operatorname{span} \xi$ gives:

$$g(\nabla_U \xi, \xi) + g(U, \nabla_\xi \xi) = 0.$$

But the first term is zero since ξ is unitary while the second implies that $\nabla_\xi \xi \in \operatorname{span} \xi$. But again, ξ being unitary we have that $\nabla_\xi \xi$ is orthogonal to ξ. □

Example 3.7. A para-Sasakian manifold with constant scalar curvature and vanishing D-concircular tensor is an $M^n_{a,b}(\xi)$ with [DragomirGrimaldi, p. 186]:

$$a = \frac{r + 2(n-1)}{(n-1)(n-2)}, \quad b = \frac{-r - n(n-1)}{(n-1)(n-2)}$$

and then, a Ricci soliton (g, ξ) on it is expanding. This result can be considered as a version in para-contact geometry of the Corollary of [Sharma5, p. 140] which states that a Ricci soliton g of a compact K-contact manifold is Einstein, Sasakian and shrinking.

From (3.9) we get $r = -n$ and returning to formulae above it results:

$$a = \frac{1}{n-1}, \quad b = \frac{-n}{n-1}.$$

Example 3.8. Let $N_{n+1}(c)$ be a space form with the metric g and M a *quasi-umbilical hypersurface* in N, [ChenYano], [Wang, p. 175], i.e. there exist two smooth functions α, β on M and a 1-form η of norm 1 such that the second fundamental form is:

$$h_{ij} = \alpha g_{ij} + \beta \eta_i \eta_j.$$

According to the cited papers M is an $M^n_{a,b}(\xi)$ with:

$$a = c + \alpha^2, \quad b = \alpha\beta$$

and ξ the g-dual of η. This $M^n_{a,b}(\xi)$ is regular if and only if $c + \alpha^2 + \alpha\beta \neq 0$. Therefore, a Ricci soliton (g, ξ) on this $M^n_{a,b}(\xi)$ is shrinking if $c + \alpha^2 + \alpha\beta > 0$ and expanding if $c + \alpha^2 + \alpha\beta < 0$.

Inspired by Theorem 3 of [DragomirGrimaldi, p. 185] let $N = \mathbb{R}P^{n+1}(c)$, $c > 0$ and M an orientable quasi-umbilical hypersurface with $b = \alpha\beta > 0$. Then:

i) a Ricci soliton (g, ξ) on it is shrinking and M is a real homology sphere (all Betti numbers vanish) if it is also compact,

ii) using the result of [Ivey], for $n = 3$ the manifold is of constant curvature being compact; so the case $n = 4$ is the first important in any conditions or the case $n = 3$ without compactness when we (possible) give up at the topology of real homology sphere,

iii) using again a classic result, now due to Perelman [Perelman], the compactness implies that the Ricci soliton is gradient i.e. η is exact.

Example 3.9. Let (M^{2n}_0, ω_0, B) be a generalized Hopf manifold [DragomirOrnea], and M^n an n-dimensional anti-invariant and totally geodesic submanifold. We set $\|\omega_0\| = 2c$ and suppose that B is unitary. Then, formula (12.40) of [DragomirOrnea, p. 162] gives that if $R^\perp = 0$ then M^n is of quasi-constant curvature with $a = c^2$ and $b = -\frac{1}{4}$. Therefore, M^n is regular for $\|\omega_0\| \neq 1$ and a Ricci soliton is shrinking if $\|\omega_0\| > 1$ and expanding if $\|\omega_0\| < 1$.

Example 3.10. Suppose that ξ is a *torse-forming vector field* i.e. there exist a smooth function f and a 1-form ω such that:

$$\nabla_X \xi = fX + \omega(X)\xi. \tag{3.11}$$

From the fact that ξ has unitary length it results $f + \omega(\xi) = 0$ which means that ξ is exactly a geodesic vector field.

Particular cases:

i) ([RachunekMikes]) If ω is exact then ξ is called *concircular*; let $\omega = -du$ with u a smooth function on M. Then $f = -\omega(\xi) = \xi(u)$.

ii) If $\omega = -f\eta$ then we call ξ of *Kenmotsu type* since (3.11) becomes similar to a expression well-known in Kenmotsu manifolds, [CalinCrasm].

Let us restrict to ii). From (3.11) a straightforward computation gives:

$$R(X,Y)\xi = X(f)[Y-\eta(Y)\xi]-Y(f)[X-\eta(X)\xi]+f^2[\eta(X)Y-\eta(Y)X] \tag{3.12}$$

and a comparison with (2.2) yields $a + b = -f^2$ and f must be a constant, different from zero from regularity of the manifold. So, a possible Ricci soliton in a Kenmotsu type case must be expanding and with $S(\xi, \xi)$ and the scalar curvature constants, a result similar to Propositions 3 and 4 of [CalinCrasm].

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