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Two generalisations of the symmetric inverse semigroups

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Abstract. We study two generalisations of the full inverse symmetric semigroup $\mathcal{I}_X$ and its dual semigroup $\mathcal{I}_X^*$ – inverse semigroups $\mathcal{PI}_X^*$ and $\overline{\mathcal{PI}^*}_X$. Both of them have the same carrier and contain $\mathcal{I}_X$. Binary operations on $\mathcal{PI}_X^*$ and $\overline{\mathcal{PI}^*}_X$ are reminiscent of the multiplication in $\mathcal{I}_X$. We use a convenient geometric way to realise elements from these two semigroups. This enables us to study efficiently their inner properties and to compare them with the corresponding properties of $\mathcal{I}_X$ and $\mathcal{I}_X^*$.

1. Introduction

One of the most natural examples of proper inverse semigroups (i.e., except groups) is the symmetric inverse semigroup $\mathcal{I}_X$. Beside pure combinatorial interest in this semigroup, it plays an important role for the class of all inverse semigroups similar to that played by the symmetric group $\mathcal{S}_X$ for the class of all groups. For some facts about semigroup and combinatorial properties of $\mathcal{I}_X$ we refer the reader to [6].

Seeking for further natural examples of inverse semigroups, FitzGerald and Leech [5], using categorical methods, introduced the *dual symmetric inverse semigroup* $\mathcal{I}_X^*$ (see also [4]). Using more general categorical approach, $\mathcal{I}_X^*$ also appeared in [11]. This semigroup also has a useful geometric realisation, which was exploited in [3], [13] to study some inner properties of $\mathcal{I}_X^*$.

In a recent work [10] there was found a new, representation theoretic, link between $\mathcal{I}_X$ and $\mathcal{I}_X^*$.

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In addition, both $\mathcal{I}_X^*$ and $\mathcal{I}_X$ belong to the class of the so-called *partition semigroups* [15], [20] and are contained in the "full partition semigroup". The latter semigroup was studied mainly in the context of representation theory and cellular algebras [7], [9], [14], [20]. Some of its pure semigroup aspects were studied in [7], [15].

In the present paper we aim at constructing two inverse semigroups $\mathcal{PI}_X^*$ and $\overline{\mathcal{PI}^*}_X$, which are strongly related to $\mathcal{I}_X$ and $\mathcal{I}_X^*$, though have more complicated structure. We give transparent geometric definitions for these two semigroups and then study their inner properties, focusing on combinatorial aspects and their resemblance to $\mathcal{I}_X$ and $\mathcal{I}_X^*$.

The semigroups $\mathcal{PI}_X^*$ and $\overline{\mathcal{PI}^*}_X$ are natural also from the representation theoretic point of view: $\overline{\mathcal{PI}^*}_X$ is contained in a bigger semigroup, the "deformation" of the full partition semigroup, whose semigroup algebra naturally arises in the representation theory, see, e.g., [7]. Some other representation theoretic aspects, where $\mathcal{PI}_X^*$ and $\overline{\mathcal{PI}^*}_X$ appeared naturally, can be found in [10].

Both semigroups $\mathcal{PI}_X^*$ and $\overline{\mathcal{PI}^*}_X$ admit realisations as semigroups of *difunctional binary relations* (see Section 2). These special relations have been studied in a series of works [1], [2], [4], [17], [18]. Using this realisation $\overline{\mathcal{PI}^*}_X$ has already appeared in [19].

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2. Definitions

Let X, X' be two disjoint sets of the same cardinality, and $'$ be a bijection from X to X'. We denote the inverse bijection from X' to X by the same symbol $'$, so that $(x')' = x$ for every $x \in X$.

2.1. $\mathcal{I}_X^*$**.** The carrier of $\mathcal{I}_X^*$ is the set of all partitions of $X \cup X'$ into subsets which have non-empty intersections with both X and X'. We realise these partitions as diagrams with two strands of vertices, top vertices indexed by X and bottom vertices indexed by X' so that for each $x \in X$ the vertices x and x' are in the same vertical column of the diagram. For $\alpha \in \mathcal{I}_X^*$ two vertices of the corresponding diagram belong to the same block or "connected component" if and only if they belong to the same set of the partition α. Notice that there may be many different ways of presenting an element $\alpha \in \mathcal{I}_X^*$ as a diagram, we treat two diagrams corresponding to the same α as equal. In the case when X is n-set we assume that $X = \mathcal{N} = \{1, 2, \ldots, n\}$ and in the notation for our semigroups replace lower

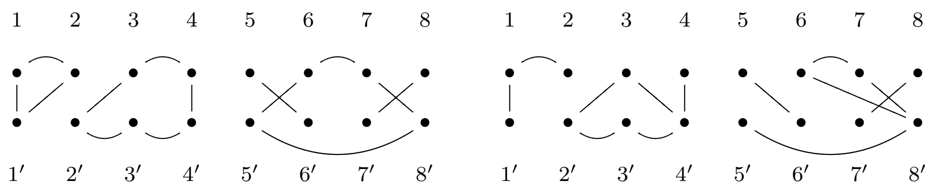

Figure 1. Two diagrams for $\{1,2,1'\}\cup\{3,4,2',3',4'\}\cup\{5,6'\}\cup\{6,7,5',8'\}\cup\{8,7'\}\in\mathcal{I}_8^*$.

index X by n. An example of two diagrams corresponding to the same element of $\mathcal{I}_8^*$ is given in Figure 1.

The elements of $\mathcal{I}_X^*$ can be interpreted as equivalence relations on $X\cup X'$ such that each equivalence class has a non empty intersection with both X and X'. Namely, the equivalence relation $\sim_\alpha$ corresponding to $\alpha\in\mathcal{I}_X^*$ is defined as follows. For $x,y\in X\cup X'$ we have $x\sim_\alpha y$ if and only if x and y belong to the same block of α.

Less immediate, the elements of $\mathcal{I}_X^*$ are in the bijective correspondence with the *full difunctional binary relations* on X. A binary relation $\equiv$ on a set Y is called *difunctional* if it can be presented as a union $\cup_{j\in J}(A_j\times B_j)$, where A_j, B_j are subsets of Y, the sets A_j are pairwise disjoint, and the sets B_j are pairwise disjoint. The difunctional relation $\cup_{j\in J}(A_j\times B_j)$ is called *full* if $Y=\cup_{j\in J}A_j=\cup_{j\in J}B_j$. To $\alpha\in\mathcal{I}_X^*$ we assign the full difunctional relation $\equiv_\alpha$ on X as follows. For $x,y\in X$ we set $x\equiv_\alpha y$ if and only if x and y' belong to the same block of the partition α which is equivalent to $x\sim_\alpha y'$. Using the diagram language, we have that $x\equiv_\alpha y$ if and only if x and y' are in the same connected component of the diagram representing α.

The multiplication on $\mathcal{I}_X^*$ can be formally defined as follows. We set $\equiv_{\alpha\beta}$ to be the smallest full difunctional binary relation which contains the product of the binary relations $\equiv_\alpha$ and $\equiv_\beta$ ([4], [19]).

Using the interpretation with equivalence relations, the multiplication on $\mathcal{I}_X^*$ can be defined as follows. Let α, $\beta\in\mathcal{I}_X^*$, and $\sim_\alpha$ and $\sim_\beta$ be the correspondent equivalence relations on $X\cup X'$. Then the relation $\sim_{\alpha\beta}$ is defined by:

- For $i,j\in X$ we have $i\sim_{\alpha\beta} j$ if and only if $i\sim_\alpha j$ or there exists a sequence $s_1,\dots,s_m$, m even, such that $i\sim_\alpha s_1'$, $s_1\sim_\beta s_2$, $s_2'\sim_\alpha s_3'$, and so on, $s_{m-1}\sim_\beta s_m$, $s_m'\sim_\alpha j$.
- For $i,j\in X$ we have $i'\sim_{\alpha\beta} j'$ if and only if $i'\sim_\beta j'$ or there exists a

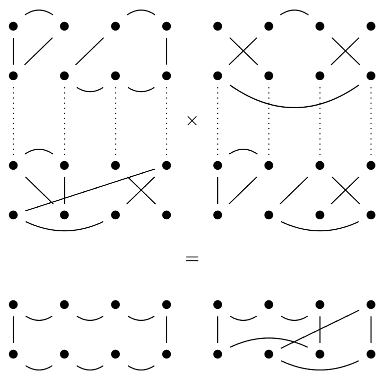

Figure 2. Elements of $\mathcal{I}_8^*$ and their multiplication.

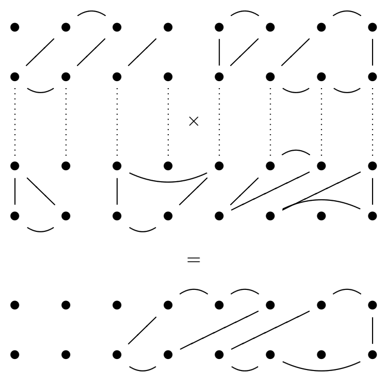

Figure 3. Elements of $\mathcal{PI}_8^*$ and their multiplication.

sequence $s_1, \dots, s_m$, m even, such that $i' \sim_\beta s_1$, $s_1' \sim_\alpha s_2'$, $s_2 \sim_\beta s_3$, and so on, $s_{m-1}' \sim_\alpha s_m'$, $s_m \sim_\beta j'$.

- For $i, j \in X$ we have $i \sim_{\alpha\beta} j'$ if and only if there exists a sequence $s_1, \dots, s_m$, m odd, such that $i \sim_\alpha s_1'$, $s_1 \sim_\beta s_2$, $s_2' \sim_\alpha s_3'$, and so on, $s_{m-1}' \sim_\alpha s_m'$, $s_m \sim_\beta j'$.

Using the interpretation with diagrams, to get the diagram of the product $\alpha\beta$ we identify the bottom vertices of α with the corresponding top vertices of β, which uniquely defines the connection of the remaining vertices (which are the top vertices of α and the bottom vertices of β). On Figure 2.1 we give an example of multiplication of the elements of $\mathcal{I}_8^*$.

2.2. $\mathcal{PI}_X^*$. Let $\mathcal{PI}_X^*$ be the set of all partitions of the set $X \cup X'$ into subsets which are either one-element or have a non-empty intersection with both X and X'. As a set, $\mathcal{I}_X^*$ is a subset of $\mathcal{PI}_X^*$. Similarly as we did above for the elements of $\mathcal{I}_X^*$, we represent the elements of $\mathcal{PI}_X^*$ as diagrams. The connected components of such diagrams now can be also one-point. There is a bijective correspondence between the elements of $\mathcal{PI}_X^*$ and difunctional relations on X. We define the difunctional relation $\equiv_\alpha$ similarly as we did in the case of $\mathcal{I}_X^*$ (but now it need not be full). For $x, y \in X$ we set $x \equiv_\alpha y$ if and only if x and y' belong to the same block of α.

We can define an associative multiplication on $\mathcal{PI}_X^*$ as follows. Let $x \notin X$. For every $\alpha \in \mathcal{PI}_X^*$ set $\overline{\alpha} \in \mathcal{I}_{X\cup\{x\}}^*$ to be the element such that its blocks are the blocks of α whose cardinality is at least two, plus one more block consisting of x,

x' and all elements of one-element blocks of α. In terms of the corresponding difunctional binary relations, we construct the full difunctional relation $\overline{\equiv_\alpha}$ on $X \cup \{x\}$ given a (not necessarily full) difunctional relation $\equiv_\alpha$ as follows. For $y, z \in X$ we set $y\overline{\equiv_\alpha}z$ if and only if y and z' belong to the same block of the partition α or both y and z' constitute one-element blocks of α. We also set $x\overline{\equiv_\alpha}x$, $x\overline{\equiv_\alpha}y$ for all y such that $\{y'\}$ is a one-element block of α, and $y\overline{\equiv_\alpha}x$ for all y such that $\{y\}$ is a one-element block of α. Denote by φ the injection, which maps $\alpha \in \mathcal{PI}_X^*$ to $\overline{\alpha} \in \mathcal{I}_{X\cup\{x\}}^*$. Observe that $\gamma \in \mathcal{I}_{X\cup\{x\}}^*$ belongs to the image of φ if and only if $x \equiv_\gamma x$. This enables us to define an associative multiplication on $\mathcal{PI}_X^*$ as follows:

$$\alpha * \beta = \varphi^{-1}(\overline{\alpha}\overline{\beta}).$$

The construction above is based on the idea of the well-known embedding of the partial transformation semigroup on the set X into the full transformation semigroup on the set $X \cup \{x\}$.

In terms of the diagrams we have the following interpretation of the operation $*$. Connect the bottom vertices of α with the top vertices of β. Then two elements a, b from the union of the top vertices of α and the bottom ones of β belong to the same block of $\alpha*\beta$ if and only if $a = b$, or a and b are connected and neither of them is connected to a vertix which constitutes a one-element block. On Figure 2.1 we give an example of multiplication of the elements from $\mathcal{PI}_8^*$.

2.3. $\overline{\mathcal{PI}^*}_X$. There is another way to define a multiplication on the set $\mathcal{PI}_X^*$. For a binary relation α on X and $x \in X$ we set $x\alpha$ to be the set of all $y \in X$ such that $(x, y) \in \alpha$, and αx to be the set of all $y \in X$ such that $(y, x) \in \alpha$. Given α, β from the set $\mathcal{PI}_X^*$, there is a unique element $\gamma = \alpha \circ \beta \in \mathcal{PI}_X^*$ such that for $y, z \in X$, we have $y \equiv_\gamma z$ if and only if $y(\equiv_\alpha) = (\equiv_\beta)z$ and $y(\equiv_\alpha) \neq \varnothing$. It is easy to see that $\circ$ gives rise to a semigroup $\overline{\mathcal{PI}^*}_X$ on the set $\mathcal{PI}_X^*$ [19]. An example of the multiplication in $\overline{\mathcal{PI}^*}_X$ is given in Figure 2.3.

Observe, that while being closed under $*$, $\mathcal{I}_X^*$ is not closed under $\circ$, which is illustrated on Figure 2.3. In addition, the $\circ$-product of the two elements of $\mathcal{PI}_8^*$ from Figure 2.1 is the element, all the blocks of which are one-element. This element is a zero with respect to both $*$ and $\circ$. In the sequel we will denote this element just by 0.

In what follows we will use the following notation. Let $\alpha \in \mathcal{PI}_X^*$. Call the blocks of α whose cardinality is at least two *generalized lines*, and the blocks of the cardinality two *points*. Let $\alpha \in \mathcal{PI}_X^*$ be the element whose generalised lines are $\{(A_i \cup B_i')\}_{i\in I}$, $A_i, B_i \subseteq X$. Since α is uniquely defined by its generalised lines, we will write $\alpha = \{(A_i \cup B_i')\}_{i\in I}$. When we write $\{(A_i \cup B_i')\}_{i\in I}$ we always

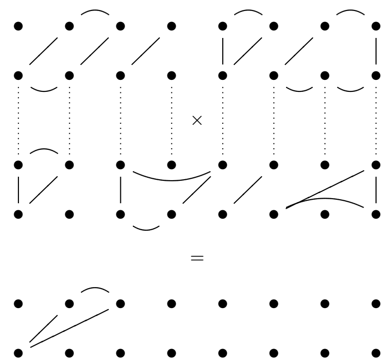

Figure 4. Elements of $\overline{\mathcal{PI}^*}_8$ and their multiplication.

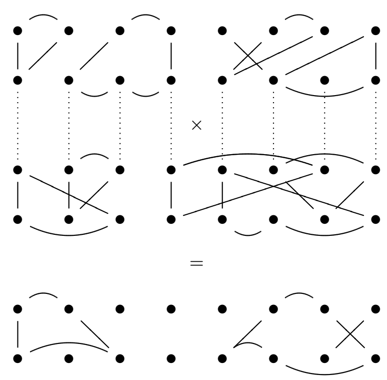

Figure 5. $\mathcal{I}_8^*$ is not closed under the operation $\circ$.

presume that this is the notation for some element of our semigroups, that is that the sets A_i are non empty and pairwise disjoint, and the same for the sets B_i. We also set $\mathrm{rank}(\alpha) = |I|$, $\mathrm{codom}(\alpha) = X \setminus \bigcup_{i\in I} A_i$, $\mathrm{coran}(\alpha) = X \setminus \bigcup_{i\in I} B_i$, $\mathrm{dom}(\alpha)$ to be the partition $\bigcup_{i\in I} A_i$ of the set $X \setminus \mathrm{codom}(\alpha)$, $\mathrm{ran}(\alpha)$ – the partition $\bigcup_{i\in I} B_i'$ of the set $X' \setminus \mathrm{coran}(\alpha)$.

3. $\mathcal{PI}_X^*$ and $\overline{\mathcal{PI}^*}_X$ are inverse semigroups

It was proved in [5], [13] that $\mathcal{I}_X^*$ is an inverse semigroup, and in [19] that $\overline{\mathcal{PI}^*}_X$ is an inverse semigroup. In this section we are going to prove that $\mathcal{PI}_X^*$ is an inverse semigroup as well. We also provide a proof that $\overline{\mathcal{PI}^*}_X$ is an inverse semigroup, for the sake of completeness. Our proofs are based on our geometric approach.

Lemma 1. *Idempotents of $\mathcal{PI}_X^*$ or of $\overline{\mathcal{PI}^*}_X$ are elements of the form* $\{(A_i \cup A_i')\}_{i\in I}$.

Proof. Traditionally, for a semigroup S by $E(S)$ we denote the set of idempotents of S. It follows immediately from the definitions that the sets $\{(A_i \cup A_i')\}_{i\in I}$ are idempotents for both $\mathcal{PI}_X^*$ and $\overline{\mathcal{PI}^*}_X$.

Let $\{(A_i \cup B_i')\}_{i\in I}$ be an idempotent in $\mathcal{PI}_X^*$. Suppose that for some i, $A_i \neq B_i$. Assume that $x \in A_i \setminus B_i$ is such that $\{x'\}$ is a one-element block of α. Then in the element α^2 we have one-element blocks $\{y'\}$ for every $y \in B_i$, and therefore, $\alpha^2 \neq \alpha$. In the case when there is $x \in B_i \setminus A_i$ such that $\{x\}$ is a one-element block of α we reason similarly.

Now we can suppose there is no $x \in A_i \setminus B_i$ such that $\{x'\}$ is a one-element block of α and no $x \in B_i \setminus A_i$ such that $\{x\}$ is a one-element block of α. If $B_i \setminus A_i \neq \varnothing$, we fix $y \in B_i \setminus A_i$. Then y belongs to some block $A_j \cup B'_j$ of α with $B_j \neq B_i$. It follows that α^2 has some block $A \cup B'$ where $A \supseteq A_i$ and $B \supseteq B_j$. Therefore, $\alpha^2 \neq \alpha$. The case when $A_i \setminus B_i \neq \varnothing$ is treated similarly.

Suppose $e = \{(A_i \cup B'_i)\}_{i \in I}$ is an idempotent in $\overline{\mathcal{PI}^*}_X$. Suppose that for some i, $A_i \neq B_i$. Let $x \in A_i$. Then in the element α^2, $\{x\}$ is not a one element block if and only if $B_i = A_j$ for some j. Then α^2 contains the block $A_i \cup B'_j$, and hence $\alpha^2 \neq \alpha$. □

Proposition 2. *$\mathcal{PI}^*_X$ and $\overline{\mathcal{PI}^*}_X$ are inverse semigroups.*

Proof. It is sufficient to prove that the semigroups are regular and idempotents commute (see [16, Theorem II.1.2, p. 78]). By Lemma 1 the idempotents in $\mathcal{PI}^*_X$ and $\overline{\mathcal{PI}^*}_X$ are of the form $\big\{(A_i \cup A'_i)\big\}_{i \in I}$. It follows from this and the definitions of the multiplication that both $E(\mathcal{PI}^*_X)$ and $E(\overline{\mathcal{PI}^*}_X)$ are semilattices.

It remains to show that $\mathcal{PI}^*_X$ and $\overline{\mathcal{PI}^*}_X$ are regular. Let $\alpha = \big\{(A_i \cup B'_i)\big\}_{i \in I} \in \mathcal{PI}^*_X$. Set $\alpha^{-1} = \big\{(B_i \cup A'_i)\big\}_{i \in I}$. Then we have $\alpha * \alpha^{-1} * \alpha = \alpha$, $\alpha^{-1} * \alpha * \alpha^{-1} = \alpha^{-1}$ and $\alpha \circ \alpha^{-1} \circ \alpha = \alpha$, $\alpha^{-1} \circ \alpha \circ \alpha^{-1} = \alpha^{-1}$. □

Recall that we call the cardinality of the set of all generalised lines in $s \in \mathcal{PI}^*_X$ the *rank* of s and denote it by $\operatorname{rank}(s)$. The following proposition describing the structure of the Green's relations on our semigroups is a routine to check.

Proposition 3. *Let a, b be from $\mathcal{PI}^*_X$ or from $\overline{\mathcal{PI}^*}_X$.*

(1) *$a\mathcal{R}b$ if and only if $\operatorname{dom}(a) = \operatorname{dom}(b)$.*

(2) *$a\mathcal{L}b$ if and only if $\operatorname{ran}(a) = \operatorname{ran}(b)$.*

(3) *$a\mathcal{D}b$ if and only if $a\mathcal{J}b$ if and only if $\operatorname{rank}(a) = \operatorname{rank}(b)$.*

(4) *All the ideals of $\mathcal{PI}^*_X$ (respectively $\overline{\mathcal{PI}^*}_X$) have the form*

$$J_\xi = \big\{\alpha \in \mathcal{PI}^*_X : \operatorname{rank}(\alpha) < \xi\big\}$$

for certain cardinal $\xi \leq |X|'$, where $|X|'$ is the successor cardinal of $|X|$.

4. Fundamentality

Recall that an inverse semigroup S is said to be *fundamental* if the *maximal idempotent-separating congruence*

$$\mu = \big\{(a,b) \in S \times S : a^{-1}ea = b^{-1}eb \quad \text{for all } e \in E(S)\big\}$$

is trivial. It is well-known that μ is the largest congruence contained in $\mathcal{H}$. For $x \in X$ set $\alpha_x = \big\{(\{t\} \cup \{t'\})\big\}_{t \in X \setminus \{x\}}$.

Proposition 4. *Let X be non-singleton. Then $\mathcal{PI}^*_X$ and $\overline{\mathcal{PI}^*}_X$ are fundamental.*

Proof. We will prove the statement for $\mathcal{PI}^*_X$; for $\overline{\mathcal{PI}^*}_X$ the proof is similar. Suppose $(a,b) \in \mu$ for some $a, b \in \mathcal{PI}^*_X$. Since $a\mathcal{H}b$, there are two collections of pairwise disjoint sets A_i, $i \in I$, B_i, $i \in I$, such that

$$a = \big\{(A_i \cup B'_i)\big\}_{i \in I}, \quad b = \big\{(A_i \cup B'_{\pi(i)})\big\}_{i \in I}$$

for some bijection $\pi : I \to I$. Let $i \in I$ and $u_i \in A_i$. Then $\big(\alpha_{u_i} * a, \alpha_{u_i} * b\big) \in \mu$ and so $(\alpha_{u_i} * a)\mathcal{H}(\alpha_{u_i} * b)$. On the other hand $\operatorname{coran}(\alpha_{u_i} * a) = \operatorname{coran}(a) \cup B'_i$ and $\operatorname{coran}(\alpha_{u_i} * b) = \operatorname{coran}(a) \cup B'_{\pi(i)}$. Therefore $B_i = B_{\pi(i)}$. Thus π is the identity mapping. It follows that $a = b$. □

Remark 5. Let X be non-singleton. $\mathcal{I}^*_X$ is not fundamental.

Proof. For $Y \subseteq X$ define the idempotent $\eta_Y = \{Y \cup Y', (X \setminus Y) \cup (X \setminus Y)'\}$. Let $x \in X$, $a = \eta_x$ and

$$b = \big\{\{x\} \cup (X \setminus \{x\})', (X \setminus \{x\}) \cup \{x'\}\big\}\mathcal{H}a.$$

Observe that either $a^{-1}ea = \eta_X$ or $a^{-1}ea = a$, for every $e \in E(\mathcal{I}^*_X)$. In particular, $a^{-1}ea = a$ if and only if e contains the block $\{x, x'\}$. Analogously, we have that either $b^{-1}eb = \eta_X$ or $b^{-1}eb = a$, for $e \in E(\mathcal{I}^*_X)$. In particular, $b^{-1}eb = a$ if and only if e contains the block $\{x, x'\}$. Therefore $(a,b) \in \mu$ which implies that μ is not trivial. □

Note that $\mathcal{I}_X$ is fundamental, [8, p. 215, ex. 22].

5. A generating set for $\mathcal{PI}_n^*$

In the following sections we will need to use some generating sets for $\mathcal{PI}_n^*$. Let $x, y, z \in X$ be pairwise distinct. Set

$$\gamma_{x,y} = \big\{\{x, y\} \cup \{x'\}, \{\{t\} \cup \{t'\}\}_{t \in X\setminus\{x,y\}}\big\};$$

$$\xi_{x,y,z} = \big\{\{x, y\} \cup \{x'\}, \{z\} \cup \{y', z'\}, \{\{t\} \cup \{t'\}\}_{t \in X\setminus\{x,y,z\}}\big\};$$

$$\tau_{x,y} = \big\{\{x, y\} \cup \{x, y\}', \{\{t\} \cup \{t\}\}'_{t \in X\setminus\{x,y\}}\big\}.$$

Notice that $\xi_{x,y,z} \in \mathcal{I}_X^*$. The elements $\gamma_{x,y}$ and $\xi_{x,y,z}$ satisfy the following equalities:

$$\gamma_{x,y}\gamma_{z,y}^{-1} = \xi_{x,y,z}, \quad \gamma_{x,y}\gamma_{x,y}^{-1} = \tau_{x,y}, \quad \gamma_{x,y}^{-1}\gamma_{x,y} = \alpha_y; \tag{1}$$

$$g^{-1}\gamma_{x,y}g = \gamma_{g(x),g(y)}, \quad \text{for any } g \in \mathcal{S}_X. \tag{2}$$

Lemma 6. *Let u be an element of $\mathcal{PI}_n^*$ of rank $n-1$. There are $\pi, \tau \in \mathcal{S}_n$ such that $\pi u \tau \in \{\tau_{1,2}, \alpha_1, \xi_{1,2,3}, \gamma_{1,2}, \gamma_{1,2}^{-1}\}$.*

Proof. It is enough to observe that every element of rank $n-1$ coincides with some element of the form $\tau_{x,y}\pi$, $\alpha_x\pi$, $\xi_{x,y,z}\pi$, $\gamma_{x,y}\pi$, or $\gamma_{x,y}^{-1}\pi$, where $x, y, z \in X$ and $\pi \in \mathcal{S}_X$. $\square$

It is known from [13, Proposition 12] that for $n \geq 3$, $\mathcal{I}_n^* = \langle \mathcal{S}_n, \xi_{1,2,3}\rangle$.

Lemma 7. *Let $n \geq 3$. Then $\mathcal{PI}_n^* = \langle \mathcal{S}_n, \gamma_{1,2}, \gamma_{1,2}^{-1}\rangle$.*

Proof. Let $a \in \mathcal{PI}_n^*$. Consider four possible cases.

Case 1. Suppose $a \in \mathcal{I}_n^*$. Then from [13, Proposition 12], (1) and (2) it follows that $a \in \langle \mathcal{S}_n, \gamma_{1,2}, \gamma_{1,2}^{-1}\rangle$.

Case 2. Suppose a has a block $\{x\}$, $x \in \mathcal{N}$, and a block $\{y'\}$, $y \in \mathcal{N}$. Let $A = \text{codom}(a)$ and $B' = \text{coran}(a)$. Construct an element q as follows: it contains all the generalised lines of a and, in addition, the generalised line $A \cup B'$. Then $q \in \mathcal{I}_n^* \subseteq \langle \mathcal{S}_n, \gamma_{1,2}, \gamma_{1,2}^{-1}\rangle$. This, $a = \alpha_x q \alpha_y$ and (1) imply $a \in \langle \mathcal{S}_n, \gamma_{1,2}, \gamma_{1,2}^{-1}\rangle$.

Case 3. Suppose a has a block $\{x'\}$, $x \in \mathcal{N}$, and has no blocks $\{y\}$, $y \in \mathcal{N}$. Then there exists a generalised line $A \cup B'$ in a such that $|A| \geq 2$. Fix $i, j \in A$. Set $M' = \text{coran}(a) \neq \varnothing$. Construct the element p as follows: it contains the blocks $\{j\} \cup M'$, $(A \setminus \{j\}) \cup B'$ and all the other blocks of p are all the generalised lines of a except $A \cup B'$. By the construction, $p \in \mathcal{I}_n^*$. Moreover, $\gamma_{i,j}p = a$. From what we have proved in the first case now follows $p \in \langle \mathcal{S}_n, \gamma_{1,2}, \gamma_{1,2}^{-1}\rangle$, which implies $a \in \langle \mathcal{S}_n, \gamma_{1,2}, \gamma_{1,2}^{-1}\rangle$.

Case 4. a has a block $\{x\}$, $x \in \mathcal{N}$, and has no blocks $\{y'\}$, $y \in \mathcal{N}$. This case is dual to Case 3. $\square$

Lemma 8. $\gamma_{1,2}^{-1} \notin \langle \mathcal{S}_n, \gamma_{1,2}, \tau_{1,2}, \xi_{1,2,3}, \alpha_1 \rangle$.

Proof. Assume that there are elements $a_1, \dots, a_k$ in $\mathcal{S}_n\{\gamma_{1,2}, \tau_{1,2}, \xi_{1,2,3}, \alpha_1\}\mathcal{S}_n$ such that $\gamma_{1,2}^{-1} = a_1 \cdot \dots \cdot a_k$. Since $\mathrm{coran}(\gamma_{1,2}^{-1}) = \varnothing$, it follows that $\mathrm{coran}(a_k) = \varnothing$. Thus $a_k \in \mathcal{S}_n\{\tau_{1,2}, \xi_{1,2,3}\}\mathcal{S}_n \subseteq \mathcal{I}_n^*$. This, in turn, gives $\mathrm{coran}(a_{k-1}) = \varnothing$, whereas $a_{k-1} \in \mathcal{I}_n^*$. Then $a_i \in \mathcal{I}_n^*$ for all $i \leq k$ by induction. Therefore $\gamma_{1,2}^{-1} = a_1 \cdot \dots \cdot a_k \in \mathcal{I}_n^*$. This is a contradiction, which completes the proof. $\square$

Theorem 9. *Let $n \geq 3$.*

1) *$\mathcal{PI}_n^*$ as an inverse semigroup is generated by $\mathcal{S}_n$ and $\gamma_{1,2}$.*
2) *$\mathcal{PI}_n^*$ is generated (as an inverse semigroup) by $\mathcal{S}_n$ and some $u \in \mathcal{PI}_n^* \setminus \mathcal{S}_n$ if and only if $u \in \mathcal{S}_n\{\gamma_{1,2}, \gamma_{1,2}^{-1}\}\mathcal{S}_n$.*

Proof. The first claim follows from Lemma 7. To prove the second one it suffices to show that $\mathcal{PI}_n^* = \langle \mathcal{S}_n, u, u^{-1} \rangle$ implies $u \in \mathcal{S}_n\{\gamma_{1,2}, \gamma_{1,2}^{-1}\}\mathcal{S}_n$. Let $\mathcal{PI}_n^* = \langle \mathcal{S}_n, u, u^{-1} \rangle$. Then u is of rank $n-1$. From Lemma 6 we have $u \in \mathcal{S}_n\{\tau_{1,2}, \alpha_1, \xi_{1,2,3}, \gamma_{1,2}, \gamma_{1,2}^{-1}\}\mathcal{S}_n$. Observe that $u \notin \mathcal{S}_n\{\alpha_1\}\mathcal{S}_n$ since otherwise we would have $\langle \mathcal{S}_n, u, u^{-1} \rangle \subseteq \mathcal{I}_n$, and $u \notin \mathcal{S}_n\{\tau_{1,2}, \xi_{1,2,3}\}\mathcal{S}_n$ since otherwise we would have $\langle \mathcal{S}_n, u, u^{-1} \rangle \subseteq \mathcal{I}_n^*$. The statement follows. $\square$

The situation with the generating sets for $\overline{\mathcal{PI}^*}_n$ is much more complicated: one can show that $\overline{\mathcal{PI}^*}_n$ can not be generated by adding to $\mathcal{S}_n$ some natural and 'compact' set of elements.

6. Maximal and maximal inverse subsemigroups

Theorem 10. *Maximal subsemigroups of $\mathcal{PI}_n^*$ are exhausted by the following list:*

1) $\mathcal{S}_n \cup J_{n-1} \cup \mathcal{S}_n\{\tau_{1,2}, \alpha_1, \gamma_{1,2}, \xi_{1,2,3}\}\mathcal{S}_n$;
2) $\mathcal{S}_n \cup J_{n-1} \cup \mathcal{S}_n\{\tau_{1,2}, \alpha_1, \gamma_{1,2}^{-1}, \xi_{1,2,3}\}\mathcal{S}_n$;
3) *$G \cup J_n$, where G runs through the set of all maximal subgroups of $\mathcal{S}_n$.*

Maximal inverse subsemigroups of $\mathcal{PI}_n^*$ are exhausted by the following list:

1) $\mathcal{S}_n \cup J_{n-1} \cup \mathcal{S}_n\{\tau_{1,2}, \alpha_1, \xi_{1,2,3}\}\mathcal{S}_n = \langle \mathcal{I}_n^*, \mathcal{I}_n \rangle$,
2) *$G \cup J_n$, where G runs through the set of all maximal subgroups of $\mathcal{S}_n$.*

Proof. That the semigroups listed in items 1) and 2) are maximal follows from Lemma 6, Lemma 7 and Lemma 8. That the semigroups given in item 3) are maximal is obvious.

Let T be a maximal subsemigroup of $\mathcal{PI}_n^*$. Then $J_{n-1} \subseteq T$ and $G \subseteq T$, where G is either $\mathcal{S}_n$ or a maximal subgroup of $\mathcal{S}_n$. If $G \neq \mathcal{S}_n$ then $T \subseteq S$, where S is one of the semigroups listed in item 3). Since both T and S are maximal, it follows that $T = S$. Let $G = \mathcal{S}_n$. Observe that we can not have $\gamma_{1,2} \in T$ and $\gamma_{1,2}^{-1} \in T$, since otherwise we would have $T = \mathcal{PI}_n^*$ by Lemma 7. Suppose $\gamma_{1,2} \in T$ and $\gamma_{1,2}^{-1} \notin T$. Then $T \subseteq S$, where $S = J_{n-1} \cup \mathcal{S}_n\{\tau_{1,2}, \alpha_1, \gamma_{1,2}, \xi_{1,2,3}\}\mathcal{S}_n$. Since both T and S are maximal, it follows that $T = S$. The case $\gamma_{1,2}^{-1} \in T$ and $\gamma_{1,2} \notin T$ is treated similarly.

The proof of the claim about maximal inverse subsemigroups is analogous and is left to the reader. $\square$

7. Congruences on $\mathcal{I}_n^*$

Let S be an inverse semigroup and $E = E(S)$. We recall the definitions from [16, p. 118]. A subsemigroup K of S is said to be a *normal subsemigroup* of S if $E \subseteq K$ and $s^{-1}Ks \subseteq K$ for all $s \in S$. A congruence Λ on E is said to be *normal* provided that for all $e, f \in E$ and $s \in S$, $e\Lambda f$ implies $s^{-1}es\Lambda s^{-1}fs$. The pair (K, Λ) is said to be a *congruence pair* of S if K is a normal subsemigroup of S, Λ is a normal congruence on E and

- $ae \in K$, $e\Lambda a^{-1}a$ imply $a \in K$ for all $a \in S$ and $e \in E$;
- $k \in K$ implies $kk^{-1}\Lambda k^{-1}k$.

For congruence pair (K, Λ) of S define the relation $\rho_{(K,\Lambda)}$:

$$\big(a\rho_{(K,\Lambda)}b\big) \Leftrightarrow \big(a^{-1}a\Lambda b^{-1}b \quad \text{and} \quad ab^{-1} \in K\big).$$

It is known (see [16, Theorem III.1.5, p. 119]) that $\rho_{(K,\Lambda)}$ is a congruence on S, and every congruence on S is of the form $\rho_{(K,\Lambda)}$, where (K, Λ) is a congruence pair of S.

In this section we describe all normal congruences, all normal subsemigroups and all congruence pairs on $\mathcal{I}_n^*$. Set $E_n = E(\mathcal{I}_n^*)$.

Lemma 11. *Let $e, f \in E_n$ be such that* $\operatorname{rank}(f) \leq \operatorname{rank}(e)$. *Then there exists $s \in \mathcal{I}_n^*$ such that $s^{-1}es = f$.*

Proof. Suppose $e = \{E_1 \cup E_1', \dots, E_k \cup E_k'\}$, $f = \{F_1 \cup F_1', dots, F_l \cup F_l'\}$ where $k \geq l$. Then $f = s^{-1}es$ for $s = \{E_1 \cup F_1', \dots, E_{l-1} \cup F_{l-1}', (\bigcup_{i=l}^{k} E_i) \cup F_l'\}$. $\square$

Let $Y \subseteq \mathcal{N}$. Let $\tau_Y = \{Y \cup Y', \{t\} \cup \{t\}'_{t \in \mathcal{N} \setminus Y}\}$. Observe that $\tau_{\mathcal{N}} = \mathcal{N} \cup \mathcal{N}'$ is the zero of $\mathcal{I}_n^*$, $\operatorname{rank}(\tau_{\mathcal{N}}) = 1$ and $\tau_{\mathcal{N}}$ is the only element in $\mathcal{I}_n^*$ of rank 1. For a set M let ι_M denote the identity relation on M. Set also

$$I_k = \{a \in \mathcal{I}_n^* : \operatorname{rank}(a) \leq k\} \quad \text{and}$$

$$E_n^{(k)} = \{e \in E_n : \operatorname{rank}(e) \leq k\} = E_n \cap I_k.$$

Lemma 12. *Let Λ be a normal congruence on E_n, $e \in E_n$ and $\operatorname{rank}(e) = m$. If $e \Lambda \tau_{\mathcal{N}}$ then $(E_n^{(m)} \times E_n^{(m)}) \subseteq \Lambda$.*

Proof. Let $f \in E_n^{(m)}$. By Lemma 11 there exists $t \in \mathcal{I}_n^*$ such that $f = t^{-1}et$. This and the definition of a normal congruence imply $f = t^{-1}et \Lambda t^{-1}\tau_{\mathcal{N}}t = \tau_{\mathcal{N}}$. □

The following lemma characterises normal congruences on $E(\mathcal{I}_n^*)$:

Lemma 13. *Let Λ be a normal congruence on E_n. Then there is k such that $\Lambda = \iota_{E_n} \cup (E_n^{(k)} \times E_n^{(k)})$.*

Proof. Suppose $\Lambda \neq \iota_{E_n}$ (otherwise we can put $k = 1$). Let $e, f \in E_n$ be such that $e \neq f$ and $e \Lambda f$. Assume $\operatorname{rank}(e) \geq \operatorname{rank}(f)$. Then $e \Lambda ef$ and $\operatorname{rank}(e) \geq \operatorname{rank}(ef)$. Moreover $\operatorname{rank}(e) > \operatorname{rank}(ef)$. Indeed, otherwise we would have $\operatorname{rank}(f) \geq \operatorname{rank}(ef) = \operatorname{rank}(e) \geq \operatorname{rank}(f)$ which would imply $\operatorname{rank}(ef) = \operatorname{rank}(e) = \operatorname{rank}(f)$ and then $e = ef = f$, a contradiction. Let $\operatorname{rank}(e) = m \geq 2$. We will show that $(E_n^{(m)} \times E_n^{(m)}) \subseteq \Lambda$. Set $B = \{1, \ldots, n - m + 1\}$. We have $\operatorname{rank}(\tau_B) = m$. Lemma 11 implies that there is $t \in \mathcal{I}_n^*$ such that $t^{-1}et = \tau_B$. Observe that

$$\operatorname{rank}(t^{-1}eft) < m \quad \text{and} \quad \tau_B \Lambda t^{-1}eft. \tag{3}$$

Let $u = \tau_B t^{-1}eft\tau_B = (U_i \cup U_i')_{i \in I}$. Then there exists $i_0 \in I$ such that $B \subseteq U_{i_0}$. We also have $\tau_B \Lambda u$. Consider two possible cases.

Case 1. $B = U_{i_0}$. Since $\operatorname{rank}(u) < \operatorname{rank}(e)$, it follows that there is $j \in I \setminus \{i_0\}$ such that $U_j \subseteq \mathcal{N} \setminus B = \overline{B}$ and $|U_j| \geq 2$. Fix $x, y \in U_j$, $x \neq y$. It follows from $u\tau_{x,y} = u$ that $\tau_B \Lambda u = u\tau_{x,y} \Lambda \tau_B \tau_{x,y}$. Let now $p, q \in \overline{B}$, $p \neq q$. There is $g \in \mathcal{S}_n$ such that $g(i) = i$ for all $i \in B$ and $g(x) = p$, $g(y) = q$. Then $\tau_B = g^{-1}\tau_B g \Lambda g^{-1}\tau_B\tau_{x,y}g = \tau_B\tau_{p,q}$. Therefore we obtain

$$\tau_B \Lambda \prod_{p,q \in \overline{B}, p \neq q} \tau_B \tau_{p,q} = \tau_B \tau_{\overline{B}}.$$

This implies that

$$\tau_{B \cup \{x\}} = \tau_B \tau_{1,x} \Lambda \tau_B \tau_{\overline{B}} \tau_{1,x} = \tau_{\mathcal{N}}.$$

Observe that $\mathrm{rank}(\tau_{B\cup\{x\}}) = m-1$. We have $\big(E_n^{(m-1)} \times E_n^{(m-1)}\big) \subseteq \Lambda$ by Lemma 12. The latter, (3) and Lemma 12 imply $\big(E_n^{(m)} \times E_n^{(m)}\big) \subseteq \Lambda$, as required.

Case 2. B is a proper subset of U_{i_0}. Take $w \in U_{i_0} \setminus B$. We have

$$\tau_B \Lambda u = u\tau_{B\cup\{w\}}\Lambda\tau_B\tau_{B\cup\{w\}} = \tau_{B\cup\{w\}}.$$

Let $j \in \overline{B}$. There is $g \in \mathcal{S}_n$ such that $g(i) = i$ for all $i \in B$ and $g(w) = j$. Then $\tau_B = g^{-1}\tau_B g \Lambda g^{-1}\tau_{B\cup\{w\}} g = \tau_{B\cup\{j\}}$ and

$$\tau_B \Lambda \prod_{j\in\overline{B}} \tau_{B\cup\{j\}} = \tau_{\mathcal{N}}.$$

Applying Lemma 12 we obtain $\big(E_n^{(m)} \times E_n^{(m)}\big) \subseteq \Lambda$, as required.

We have shown that $\big(E_n^{(m)} \times E_n^{(m)}\big) \subseteq \Lambda$ whenever $e\Lambda f$ for all idempotents e, f such that $e \neq f$ and $\mathrm{rank}(e) = m$. Let $k \in \mathbb{N}$, $k \leq n$, be such that there is $e \in E_n$ of rank k satisfying the condition

$$e\Lambda f \quad \text{for some } f \in E_n,\ f \neq e, \tag{4}$$

while there is no $e \in E_n$ with $\mathrm{rank}(e) > k$ satisfying (4). It follows that $\Lambda = \iota_{E_n} \cup \big(E_n^{(k)} \times E_n^{(k)}\big)$. □

Let $e \in E_n$, $k = \mathrm{rank}(e)$ and $A \lhd \mathcal{S}_k$. It is easy to see that $H_e \cong \mathcal{S}_k$. It follows that there is unique $A_e \lhd H_e$, such that $A_e \simeq A$. Set $N_k(A)$ to be the union of all subgroups A_e, where e runs through all idempotents of rank k. We also set $N_{n+1}(A) = \varnothing$ whenever $A \lhd \mathcal{S}_{n+1}$.

Proposition 14. *Let K be a normal subsemigroup of $\mathcal{I}_n^*$ and Λ a normal congruence on E_n. Then (K,Λ) is a congruence pair of $\mathcal{I}_n^*$ if and only if there is $k \in \mathcal{N}$ such that $\Lambda = \iota_{E_n} \cup \big(E_n^{(k)} \times E_n^{(k)}\big)$ and $K = E_n \cup N_{k+1}(A) \cup I_k$ for some $A \lhd \mathcal{S}_{k+1}$.*

Proof. The sufficiency follows from Lemma 13 and the observation that $E_n \cup N_{k+1}(A) \cup I_k$, $k \in \mathcal{N}$, is a normal subsemigroup of $\mathcal{I}_n^*$.

Suppose (K,Λ) is a congruence pair of $\mathcal{I}_n^*$. Lemma 13 implies that there is $k \in \mathcal{N}$ such that $\Lambda = \iota_{E_n} \cup \big(E_n^{(k)} \times E_n^{(k)}\big)$.

Assume that $k = n$. Then $\Lambda = E_n \times E_n$. In this case we have $K = \mathcal{I}_n^*$. Indeed, let $a \in \mathcal{I}_n^*$. Since $E_n \subset K$, it follows that, in particular, $a\tau_{\mathcal{N}} = \tau_{\mathcal{N}} \in K$. We also have $\tau_{\mathcal{N}}\Lambda aa^{-1}$. The first condition of the definition of a congruence pair yields $a \in K$. Hence $\mathcal{I}_n^* \subseteq K$.

Assume now that $k \leq n-1$. Since $a\tau_{\mathcal{N}} = \tau_{\mathcal{N}} \in E_n \subseteq K$ and $\tau_{\mathcal{N}}\Lambda a^{-1}a$ for all $a \in I_k$, it follows that $I_k \subseteq K$. Since $d \in K$ implies $dd^{-1}\Lambda d^{-1}d$ for all $d \in K$, it follows that all elements $d \in K$ such that $\operatorname{rank}(d) \geq k+1$, belong to certain subgroups of $\mathcal{I}_n^*$. Let $b \in K$ and $\operatorname{rank}(b) = m \geq k+2$. Observe that $m \geq 3$. Show that b must be an idempotent. Since b is a group element, there exists a partition $\mathcal{N} = \bigcup_{i \in I} B_i$ such that $b = \big\{(B_i \cup B'_{\pi(i)})_{i\in I}\big\}$, $|\, I \,| \geq 3$, for some bijection $\pi : I \to I$. Show that π is the identity transformation of I. Consider π as a permutation from $\mathcal{S}_I$. Suppose π is not the identity map. Consider a cycle $(i_1, i_2, \dots, i_l)$ of π, where $i_1, \dots, i_l \in I$ and $l \geq 2$. If $l \geq 3$ then $b\tau_{B_{i_1} \cup B_{i_2}}$ is of rank $m-1$, is not a group element and belongs to K, which is impossible. If $l = 2$ consider $j \in I \setminus \{i_1, i_2\}$ and $b\tau_{B_{i_1} \cup B_j}$. This element is again of rank $m-1$, is not a group element and belongs to K, which is also impossible. Thus π is the identity transformation of I. Therefore b is an idempotent. It follows that $K \setminus I_{k+1} = E_n \setminus I_{k+1}$.

Fix $e, f \in E_n$ such that $\operatorname{rank}(e) = \operatorname{rank}(f) = k+1$. Set $A'_e = K \cap H_e$, $A'_f = K \cap H_f$. Since K is self-conjugate it follows that $A'_e \lhd H_e$ and $A'_f \lhd H_f$. Take any $s \in \mathcal{I}_n^*$ such that $s^{-1}es = f$ and $sfs^{-1} = e$ (it is easily seen that such an element exists). Further, from $s^{-1}Ks \subseteq K$ and $sKs^{-1} \subseteq K$, it follows that the maps $x \mapsto s^{-1}xs$ from A'_e onto A'_f and $y \mapsto sys^{-1}$ from A'_f onto A'_e are mutually inverse bijections, whence $|\, A'_e \,|=|\, A'_f \,|$. It follows that an element of K has rank $k+1$ if and only if it lies in $N_{k+1}(A)$ for some $A \lhd \mathcal{S}_{k+1}$. Thus $K = E_n \cup N_{k+1}(A) \cup I_k$, and the proof is complete. □

For $1 \leq k \leq n$ denote by D_k the set of elements of $\mathcal{I}_n^*$ of rank k. Let $A \lhd \mathcal{S}_{k+1}$, $1 \leq k \leq n$. Let $F_k(A)$ be the relation on D_{k+1} that is defined by $(x, y) \in F_k(A)$ if and only if $x\mathcal{H}y$ and $xy^{-1} \in N_{k+1}(A)$. Set $\rho_{k,A} = \iota_{\mathcal{I}_n^*} \cup F_k(A) \cup \big(I_k \times I_k\big)$. The construction implies that $\rho_{k,A}$ coincides with $\rho_{(K,\Lambda)}$, corresponding to the congruence pair (K, Λ), where $K = E_n \cup N_{k+1}(A) \cup I_k$ and $\Lambda = \iota_{E_n} \cup \big(E_n^{(k)} \times E_n^{(k)}\big)$.

Theorem 15. *Let ρ be a relation on $\mathcal{I}_n^*$. Then ρ is a congruence on $\mathcal{I}_n^*$ if and only if $\rho = \rho_{k,A}$ for some $k, 1 \leq k \leq n$ and normal subgroup $A \lhd \mathcal{S}_{k+1}$.*

Proof. The claim follows from Proposition 14 and from [16, Theorem III.1.5]. □

We note that the formulation of Theorem 15 resembles the one of the corresponding classic Liber's result [12] for $\mathcal{I}_n$.

8. Congruences on $\mathcal{PI}^*_n$ and $\overline{\mathcal{PI}^*}_n$

8.1. Congruences on $\mathcal{PI}^*_n$. Let $Y \subseteq X$. Set $\alpha_Y = \big\{\{t,t'\}_{t\in X\setminus Y}\big\}$. Notice that α_Y is an idempotent for any $Y \subseteq X$ and that the element $0 = \alpha_X$ is the zero element of both $\mathcal{PI}^*_X$ and $\overline{\mathcal{PI}^*}_X$.

Let $\widetilde{E}_n = E(\mathcal{PI}^*_n)$ and $\widetilde{E}^{(k)}_n = \big\{e \in \widetilde{E}_n : \operatorname{rank}(e) \le k\big\} = \widetilde{E}_n \cap J_{k+1}$.

Lemma 16. *Let $e, f \in \widetilde{E}_n$ and $\operatorname{rank}(f) \le \operatorname{rank}(e)$. Then there exists $s \in \mathcal{PI}^*_n$ such that $s^{-1} * e * s = f$.*

Proof. The proof is analogous to that of Lemma 11. □

As an immediate consequence we obtain the following lemma.

Lemma 17. *Let Λ be a normal congruence on $(\widetilde{E}_n, *)$. Then $a\Lambda 0$ implies $b\Lambda 0$ for all idempotents $b \in J_{\operatorname{rank}(a)+1}$.*

Lemma 18. *Let a and b of $\mathcal{I}_n$ be two idempotents with $\operatorname{rank}(a) > \operatorname{rank}(b)$ and Λ – a normal congruence on $E(\mathcal{I}_n)$. Then a is Λ-related to 0.*

Proof. The proof is similar to that of Lemma 13. □

Lemma 19. *Let Λ be a normal congruence on $(\widetilde{E}_n, *)$. Then there is $k \in \mathcal{N}$ such that $\Lambda = \iota_{\widetilde{E}_n} \cup \big(\widetilde{E}^{(k)}_n \times \widetilde{E}^{(k)}_n\big)$.*

Proof. Suppose $\Lambda \ne \iota_{\widetilde{E}_n}$. Take distinct $e, f \in \widetilde{E}_n$ such that $e \ne f$ and $m = \operatorname{rank}(e) \ge \operatorname{rank}(f)$. Show that $\big(\widetilde{E}^{(m)}_n \times \widetilde{E}^{(m)}_n\big) \subseteq \Lambda$. Similarly to as it was done in the proof of Lemma 13 we show that $\tau_B \Lambda u$, where $B = \{1, \dots, n-m+1\}$ and $u \in \widetilde{E}_n$ are such that $\operatorname{rank}(u) < m$ and $u = \tau_B u = u\tau_B$. Show that there exists an element of rank m which is Λ-related to 0. Set

$$d = \big\{B \cup \{1'\}, \{k, k'\}_{k\in\mathcal{N}\setminus B}\big\}.$$

Consider three possible cases.

Case 1. Suppose u contains a block $C \cup C'$, where C strictly contains B. Then $\tau_B \Lambda u = u\tau_C \Lambda \tau_B\tau_C = \tau_C$. This and Lemma 13 imply $\tau_B \Lambda \tau_{\mathcal{N}}$. It follows that $\alpha_B = \tau_B\alpha_1 \Lambda \tau_{\mathcal{N}}\alpha_1 = 0$. Since $\operatorname{rank}(u) \le \operatorname{rank}(\alpha_B)$ it follows from Lemma 17 that $u\Lambda 0$, whence $\tau_B \Lambda 0$.

Case 2. Suppose u contains a block $\{t\}$ for some $t \in B$. Then

$$\tau_B \Lambda u = \alpha_t u \Lambda \alpha_t \tau_B = \alpha_B = \alpha_1 \dots \alpha_{n-m+1}.$$

Therefore

$$\alpha_{B\setminus\{1\}} = \alpha_2 \dots \alpha_{n-m+1} = d^{-1}\tau_B d \Lambda d^{-1}\alpha_1 \dots \alpha_{n-m+1} d = \alpha_1 \dots \alpha_{n-m+1} = \alpha_B.$$

Both $\alpha_{B\setminus\{1\}}$ and α_B belong to $\mathcal{I}_n$. In addition, $\mathrm{rank}(\alpha_{B\setminus\{1\}}) = m$ and $\mathrm{rank}(\alpha_B) = m-1$. Applying Lemma 18 we obtain $\alpha_2 \dots \alpha_{n-m+1}\Lambda 0$.

Case 3. Suppose u contains a block $B \cup B'$. If $u \in \mathcal{I}_n^*$ then Lemma 13 ensures that $\tau_B \Lambda \tau_{\mathcal{N}}$. Applying the same arguments as in the first case, we conclude that $\tau_B \Lambda 0$. Otherwise there is $j \in \mathcal{N} \setminus B$ such that $\tau_B \Lambda \tau_B \alpha_j$. Then

$$\alpha_{B\setminus\{1\}} = \alpha_2 \dots \alpha_{n-m+1} = d^{-1}\tau_B d \Lambda d^{-1}\tau_B \alpha_j d = \alpha_B \alpha_j.$$

Observe that $\mathrm{rank}(\alpha_B\alpha_j) < \mathrm{rank}(\alpha_{B\setminus\{1\}}) = m$. This and Lemma 18 imply $\alpha_{B\setminus\{1\}}\Lambda 0$.

Lemma 17 implies that $\widetilde{E}_n^{(m)} = \widetilde{E}_n \cap J_{m+1}$ lies in some Λ-class. Applying the same arguments as at the end of the proof of Lemma 13, we obtain that there is $k \in \mathcal{N}$ such that $\Lambda = \iota_{\widetilde{E}_n} \cup \big(\widetilde{E}_n^{(k)} \times \widetilde{E}_n^{(k)}\big)$. $\square$

For $A \lhd \mathcal{S}_k$ we construct the set $\widetilde{N}_k(A)$ and the relation $\widetilde{F}_k(A)$ similarly to as we constructed $N_k(A)$ and $F_k(A)$ in Section 7. Set $\widetilde{\rho}_{k,A} = \iota_{\mathcal{PI}_n^*} \cup \widetilde{F}_k(A) \cup \big(J_{k+1} \times J_{k+1}\big)$. The proof of the following statement is analogous to that of Proposition 14.

Proposition 20. *Let K be a normal subsemigroup of $\mathcal{PI}_n^*$ and Λ be a normal congruence on $\widetilde{E}_n$. Then (K,Λ) is a congruence pair of $\mathcal{PI}_n^*$ if and only if there is $k \in \mathcal{N}$ such that $\Lambda = \iota_{\widetilde{E}_n} \cup \big(\widetilde{E}_n^{(k)} \times \widetilde{E}_n^{(k)}\big)$ and $K = \widetilde{E}_n \cup \widetilde{N}_{k+1}(A) \cup J_{k+1}$ for some $A \lhd \mathcal{S}_{k+1}$.*

The description of congruences on $\mathcal{PI}_n^*$ can be formulated now in the same way as Theorem 15.

For the semigroup $\overline{\mathcal{PI}^*}_n$ the arguments are similar. In particular, we observe that an analogue of Lemma 19 holds. After this, it is easy to conclude that sets of congruences on $\mathcal{PI}_n^*$ and $\overline{\mathcal{PI}^*}_n$ coincide.

9. Completely isolated subsemigroups of $\mathcal{I}_n^*$, $\mathcal{PI}_n^*$ and $\overline{\mathcal{PI}^*}_n$

From now on suppose that $n \geq 2$. Recall that a subsemigroup T of a semigroup S is called *completely isolated* provided that $ab \in T$ implies either $a \in T$ or $b \in T$ for all $a, b \in S$. A subsemigroup T of a semigroup S is called *isolated* provided that $a^k \in T$, $k \geq 1$, implies $a \in T$ for all $a \in T$. A completely isolated subsemigroup is isolated, but the converse is not true in general.

We begin this section with several general observations, which will be needed for the sequel and are also interesting on their own.

Lemma 21. *Let S be a semigroup with an identity element 1 and the group of units G. Suppose $S \setminus G$ is a subsemigroup of S. Then G is completely isolated and the map $T \mapsto T \cup G$ is a bijection from the set of all completely isolated subsemigroups, which are disjoint with G, to the set of all completely isolated subsemigroups, which contain G as a proper subsemigroup.*

Proof. Obviously, G is a completely isolated subsemigroup. Suppose that T is a completely isolated subsemigroup such that $T \cap G = \varnothing$. Observe that $T \cup G$ is a subsemigroup of S. Indeed, let $g \in G$ and $t \in T$. Since T is completely isolated and disjoint with G, the inclusion $g^{-1} \cdot gt = t \in T$ implies $gt \in T \subset T \cup G$. Similarly, $tg \cdot g^{-1} = t \in T$ implies $tg \in T \subset T \cup G$. Let now $ab \in T \cup G$. Consider two possible cases.

Case 1. Suppose $ab \in G$. Since $S \setminus G$ is a subsemigroup of S, it follows that either $a \in G$ or $b \in G$.

Case 2. Suppose $ab \in T$. Since T is completely isolated, it follows that either $a \in T$ or $b \in T$.

Therefore, either $a \in G \cup T$ or $b \in G \cup T$. Hence $T \cup G$ is completely isolated.

Now suppose that T is a completely isolated subsemigroup with $T \supset G$ and prove that $T \setminus G$ is completely isolated as well. Let $a, b \in T \setminus G = T \cap (S \setminus G)$. Then $ab \in T \setminus G$ as both T and $S \setminus G$ are subsemigroups of S, proving that $T \setminus G$ is a semigroup. Suppose $ab \in T \setminus G$ and show that at least one of the elements a, b lies in $T \setminus G$. Since $T \setminus G \subset T$ and T is completely isolated, it follows that at least one of the elements a, b belongs to T. Suppose $a \in T$ (the case when $b \in T$ is treated similarly). If $a \in T \setminus G$, we are done. If $a \in G$ we have $b = a^{-1} \cdot ab \in T$. Moreover, $b \in T \setminus G$ as the inclusion $b \in G$ would imply $ab \in G$. Hence $T \setminus G$ is completely isolated. $\square$

Lemma 22. *Let S be a semigroup, $e \in E(S)$ and $G = G(e)$ – the maximal subgroup of S with the identity element e. Suppose G is periodic and T is an isolated subsemigroup of S such that $T \cap G \neq \varnothing$. Then $T \supseteq G$.*

Proof. Let $a \in T \cap G$. There is $m \in \mathbb{N}$ such that $a^m = e$, which implies $e \in T$. Let $b \in G$. Since G is periodic, $b^k = e$ for certain $k \in \mathbb{N}$. The statement follows. $\square$

Corollary 23. *Let S be a semigroup with the group of units G. Suppose that $S \setminus G$ is a subsemigroup of S and that G is periodic.*

(1) *If $T_i, i \in I$, is the full list of completely isolated subsemigroups of S, which are disjoint with G, then T_i, $i \in I, T_i \cup G$, $i \in I, G$ is the full list of completely isolated subsemigroups of S.*

(2) *If T_i, $i \in I$, is the full list of completely isolated subsemigroups of S, which contain G as a proper subsemigroup, then T_i, $i \in I$, $T_i \setminus G$, $i \in I, G$ is the full list of completely isolated subsemigroups of S.*

Proof. The proof follows from Lemma 21 and Lemma 22. □

9.1. Completely isolated subsemigroups of $\mathcal{I}_n^*$.

Theorem 24. *Let $n \geq 2$. The semigroups $\mathcal{I}_n^*$, $\mathcal{S}_n$ and $\mathcal{I}_n^* \setminus \mathcal{S}_n$ and only them are completely isolated subsemigroups of the semigroup $\mathcal{I}_n^*$.*

Proof. For $n = 2$ the proof is easy. Suppose $n \geq 3$. That all the subsemigroups given in the formulation are completely isolated follows from the definition.

Let T be a completely isolated subsemigroup of $\mathcal{I}_n^*$ containing $\mathcal{S}_n$ as a proper subsemigroup. Applying Corollary 23, it is enough to prove that $T = \mathcal{I}_n^*$. Show that T contains some element from $\mathcal{S}_n \xi_{1,2,3} \mathcal{S}_n$. Indeed, consider $g \in T \setminus \mathcal{S}_n$. Due to $\mathcal{I}_n^* = \langle \mathcal{S}_n, \xi_{1,2,3} \rangle$ ([13, Proposition 12]) we can write

$$g = g_1 \xi_{1,2,3} g_2 \xi_{1,2,3} \cdots \xi_{1,2,3} g_{k+1},$$

where $k \geq 1$ and $g_1 \dots, g_{k+1} \in \mathcal{S}_n$. If $k > 1$ we have that either $g_1 \xi_{1,2,3} g_2 \xi_{1,2,3} \cdots \xi_{1,2,3} g_k \in T$ or $\xi_{1,2,3} g_{k+1} \in T$, since T is completely isolated. The claim follows by induction.

Now we can assert that $\xi_{1,2,3} \in T$ as $T \supset \mathcal{S}_n$ by the assumption. This together with $\mathcal{I}_n^* = \langle \mathcal{S}_n, \xi_{1,2,3} \rangle$ implies $T = \mathcal{I}_n^*$. □

9.2. Completely isolated subsemigroups of $\mathcal{PI}_n^*$.

Theorem 25. *Let $n \geq 2$. The semigroups $\mathcal{PI}_n^*$, $\mathcal{S}_n$ and $\mathcal{PI}_n^* \setminus \mathcal{S}_n$ and only them are the completely isolated subsemigroups of the semigroup $\mathcal{PI}_n^*$.*

For the proof of Theorem 25 we will need two auxiliary lemmas:

Lemma 26. *Let $a \in \mathcal{PI}_n^* \setminus \mathcal{I}_n^*$. Then there are $k \geq 1$ and $g_1, \dots, g_k$ of $\mathcal{S}_n$ such that $a g_1 a g_2 a \dots a g_k a = 0$.*

Proof. The statement follows from the observation that a has at least one point. □

Lemma 27. *Let T be a completely isolated subsemigroup of $\mathcal{PI}_n^*$ such that $(\{0\} \cup \mathcal{S}_n) \subset T$. Then $\mathcal{PI}_n^* \setminus \mathcal{I}_n^* \subset T$.*

Proof. Let $a \in \mathcal{PI}_n^* \setminus \mathcal{I}_n^*$. By Lemma 26 we have $a g_1 a \dots g_k a = 0$ for some $g_1, \dots, g_k \in \mathcal{S}_n$. Since T is completely isolated, it follows that either $a g_1 a g_2 \dots a g_k \in T$ or $a \in T$. If $a \in T$ then we are done. Otherwise, we have $a g_1 a g_2 \dots a g_{k-1} a \in T$. The statement follows by induction. □

Proof of Theorem 25. That all the listed semigroups are completely isolated is checked directly. Let T be a completely isolated subsemigroup of $\mathcal{PI}_n^*$ strictly containing $\mathcal{S}_n$. In view of Corollary 23 it is enough to show that $T = \mathcal{PI}_n^*$.

First assume that $T \setminus \mathcal{I}_n^* \neq \varnothing$. Take any $a \in T \setminus \mathcal{I}_n^*$. Since $a \in \mathcal{PI}_n^* \setminus \mathcal{I}_n^*$, it follows from Lemma 26 that $0 \in T$. Applying Lemma 27 we obtain the inclusion $\mathcal{PI}_n^* \setminus \mathcal{I}_n^* \subseteq T$.

Consider the element

$$w = \gamma_{1,2} = \big\{\{1, 2, 1'\}, \{t, t'\}_{t\in\mathcal{N}\setminus\{1,2\}}\big\} \in \mathcal{PI}_n^*.$$

Since $w^2 = (w^{-1})^2 = \alpha_1\alpha_2 \in \mathcal{I}_n \subseteq T$, we conclude that $w \in T$ and $w^{-1} \in T$, which implies $ww^{-1} \in T$. From the other hand, $ww^{-1} = \tau_{1,2} \in \mathcal{I}_n^* \setminus \mathcal{S}_n$. It follows that $ww^{-1} \in T \cap \big(\mathcal{I}_n^* \setminus \mathcal{S}_n\big)$. Observe that $\tau_{\mathcal{N}} \in \langle \mathcal{S}_n, \tau_{1,2}\rangle \subseteq T$. It is easy to see that $T \cap \mathcal{I}_n^*$ is a completely isolated subsemigroup of $\mathcal{I}_n^*$. In addition, $T \cap \mathcal{I}_n^*$ contains $\mathcal{S}_n$ as a proper subsemigroup. Applying Theorem 24 we obtain $\mathcal{I}_n^* \subseteq T$. It follows that $T = \mathcal{PI}_n^*$.

Assume now that $T \setminus \mathcal{I}_n^* = \varnothing$, that is, $T \subseteq \mathcal{I}_n^*$. Let $a = \big\{(A_i \cup B_i')_{i\in I}\big\} \in T \setminus \mathcal{S}_n$. Since $a \notin \mathcal{S}_n$, there exists $j \in I$ such that $|B_j| \geq 2$. Fix some $x \in B_j$ and consider the elements

$$b = \big\{(A_i \cup B_i')_{i\in I\setminus\{j\}}, A_j \cup \{x'\}\big\} \quad \text{and} \quad c = \big\{(B_i \cup B_i')_{i\in I\setminus\{j\}}, \{x\} \cup B_j'\big\}$$

of $\mathcal{PI}_n^* \setminus \mathcal{I}_n^*$. We have $bc = a \in T$ by the construction. Therefore, $b \in T \subset \mathcal{I}_n^*$ or $c \in T \subset \mathcal{I}_n^*$. We obtained a contradiction, which shows that the inclusion $T \subseteq \mathcal{I}_n^*$ is impossible. The proof is complete. □

9.3. Completely isolated subsemigroups of $\overline{\mathcal{PI}^*}_n$.

Theorem 28. *Let $n \geq 2$. All completely isolated subsemigroups of $\overline{\mathcal{PI}^*}_n$ are exhausted by the following list: $\overline{\mathcal{PI}^*}_n$, $\mathcal{S}_n$ and $\overline{\mathcal{PI}^*}_n \setminus \mathcal{S}_n$.*

Lemma 29. *Let $e \in E(\overline{\mathcal{PI}^*}_n) \setminus \mathcal{S}_n$. Then there exists $a \in \overline{\mathcal{PI}^*}_n$ such that $aa^{-1} = e$ and $a^2 = (a^{-1})^2 \in \mathcal{I}_n \setminus \mathcal{S}_n$.*

Proof. If $e \in \mathcal{I}_n$ we can set $a = e$. Otherwise, let $e = \big\{(A_i \cup A_i')_{i\in I}, \{t, t'\}_{t\in J}\big\}$, where $\mathcal{N} \setminus \big((\bigcup_{i\in I} A_i) \cup J\big)$ is non-empty and $|A_i| \geq 2$, $i \in I$. Since $e \notin \mathcal{I}_n$, it follows that $I \neq \varnothing$. Take $x_i \in A_i$, $i \in I$. Set $a = \big\{(A_i \cup x_i')_{i\in I}, \{t, t'\}_{t\in J}\big\}$. We have that $aa^{-1} = e$ and $a^2 = (a^{-1})^2 = \big\{\{t, t'\}_{t\in J}\big\} \in \mathcal{I}_n \setminus \mathcal{S}_n$. □

The following statement follows from Lemma 29.

Corollary 30. *Let T be an isolated subsemigroup of $\overline{\mathcal{PI}^*}_n$. If $\mathcal{I}_n \setminus \mathcal{S}_n \subseteq T$, then $\overline{\mathcal{PI}^*}_n \setminus \mathcal{S}_n \subseteq T$.*

We will need the following fact, see [6, Chapter 5].

Lemma 31. *All completely isolated subsemigroups of $\mathcal{I}_n$ are exhausted by the following list: $\mathcal{I}_n$, $\mathcal{S}_n$ and $\mathcal{I}_n \setminus \mathcal{S}_n$.*

Proof of Theorem 28. It is straightforward to verify that $\overline{\mathcal{PI}^*}_n$, $\mathcal{S}_n$ and $\overline{\mathcal{PI}^*}_n \setminus \mathcal{S}_n$ are completely isolated. Let now T be a completely isolated subsemigroup of $\overline{\mathcal{PI}^*}_n$. If $T \cap \mathcal{S}_n \neq \varnothing$ then $T \supset \mathcal{S}_n$ by Lemma 22. Assume that $T \setminus \mathcal{S}_n \neq \varnothing$. It is enough to prove that $\overline{\mathcal{PI}^*}_n \setminus \mathcal{S}_n \subseteq T$.

Let $b \in T \setminus \mathcal{S}_n$. There is k such that $b^k = e$ is an idempotent. Let $a \in \overline{\mathcal{PI}^*}_n$ be such that $aa^{-1} = e$ and $f = a^2 = (a^{-1})^2 \in \mathcal{I}_n \setminus \mathcal{S}_n$ (such an element exists by Lemma 29). Then $f \in T \cap \mathcal{I}_n$. Applying Lemma 31 we have $\mathcal{I}_n \setminus \mathcal{S}_n \subseteq T$. Finally, $\overline{\mathcal{PI}^*}_n \setminus \mathcal{S}_n \subseteq T$ by Corollary 30. $\square$

10. Isolated subsemigroups of $\mathcal{I}_n^*$, $\mathcal{PI}_n^*$ and $\overline{\mathcal{PI}^*}_n$

10.1. Isolated subsemigroups of $\mathcal{I}_n^*$.

Proposition 32. *Let $e \in \mathcal{I}_n^*$ be an idempotent of rank $n-1$, that is, $e = \tau_A$ for some $A \subset \mathcal{N}$ with $|A| = 2$. Then $G(e)$ is an isolated subsemigroup of $\mathcal{I}_n^*$.*

Proof. Assume that $a \in \mathcal{I}_n^*$ is such that $a^k \in G(e)$ for some $k \geq 1$. Since $G(e)$ is finite, we can assume that $a^k = e$. We are to show that $a \in G(e)$. Since $\mathrm{rank}(a^k) = n-1$, it follows that $\mathrm{rank}(a) \geq n-1$. Hence $\mathrm{rank}(a) = n-1$. But $\mathrm{rank}(a) = \mathrm{rank}(a^k)$ implies that $a\mathcal{D}a^2$, which implies that $a\mathcal{H}a^2$ (since $\mathcal{I}_n^*$ is finite), which means that $a \in G(e)$. $\square$

Theorem 33. *The semigroups $\mathcal{I}_n^*$, $\mathcal{S}_n$, $\mathcal{I}_n^* \setminus \mathcal{S}_n$ and $G(e)$, where e is an idempotent of rank $n-1$ and only them are isolated subsemigroups of $\mathcal{I}_n^*$.*

Proof. That all the listed subsemigroups are isolated follows from Proposition 32 and Theorem 24.

Assume that $T \neq \mathcal{S}_n$ is an isolated subsemigroup of $\mathcal{I}_n^*$. Then $T \setminus \mathcal{S}_n \neq \varnothing$. Let $a \in T \setminus \mathcal{S}_n$. Going, if necessary, to some power of a, we may assume that a is an idempotent. Let us show that T contains some idempotent of rank $n-1$.

Suppose first that a has some block $A \cup A'$ with $A \subseteq \mathcal{N}$, $|A| \geq 3$. Let $A = \{t_1, \dots t_k\}$. Consider $b \in \mathcal{I}_n^*$ such that it contains all the blocks of a, except $A \cup A'$, and instead of $A \cup A'$ it has two blocks: $\{t_1, \dots, t_{k-1}, t_1'\}$ and $\{t_k, t_2', \dots, t_k'\}$. The construction implies $b^2 = (b^{-1})^2 = a$, whence $b, b^{-1} \in T$. It follows that $bb^{-1} \in T$. This element is an idempotent, contains all the blocks of a, except $A \cup A'$, and instead of $A \cup A'$ it contains two blocks: $(A \setminus \{t_k\}) \cup (A \setminus \{t_k\})'$ and $\{t_k, t_k'\}$.

Applying the described procedure as many times as needed we obtain that there T contains an idempotent e such that $|A| \leq 2$ for each block $A \cup A'$, $A \subseteq \mathcal{N}$, of e.

Suppose now that $e \in E(T)$ contains two blocks $\{t_1, t_2\} \cup \{t_1, t_2\}'$ and $\{t_3, t_4\} \cup \{t_3, t_4\}'$, $t_1, t_2, t_3, t_4 \in \mathcal{N}$. Let $a \in \mathcal{I}_n^*$ be the element whose blocks are all the blocks of e, except $\{t_1, t_2\} \cup \{t_1, t_2\}'$ and $\{t_3, t_4\} \cup \{t_3, t_4\}'$, and instead of these two blocks it contains the following three blocks: $\{t_1, t_3'\}$, $\{t_2, t_4'\}$, $\{t_3, t_4, t_1', t_2'\}$. The construction of a implies that $a^2 = (a^{-1})^2 = e$, which implies $a, a^{-1} \in T$. It follows that $aa^{-1} \in T$. Observe that $aa^{-1} \in E(T)$. This element contains all the blocks of e, except $\{t_1, t_2\} \cup \{t_1, t_2\}'$. In addition, it has two blocks $\{t_1, t_1'\}$ and $\{t_2, t_2'\}$. Therefore, aa^{-1} has fewer blocks of the form $A \cup A'$ with $A \subset \mathcal{N}$, $|A| = 2$ than e. Applying this procedure as many times as required we obtain that T contains some idempotent $e = \tau_A$ with $|A| = 2$. Therefore, T contains some idempotent e of rank $n-1$.

If e is the only idempotent of T we have $T = G(e)$. Suppose now that, except e, T has some other idempotent, say, f. We will show that $\tau_{\mathcal{N}} \in T$. If $n = 2$ this is obvious. Suppose $n \geq 3$. In view of Lemma 22 $G(e), G(f) \subset T$. Let $e = \tau_A$, where $A = \{t_1, t_2\}$. Consider two possible cases.

Case 1. Suppose $\mathrm{rank}(f) \leq n-1$. Since $f \neq e$ it follows that f has a block $B \cup B'$ with $B \subseteq \mathcal{N}$, $|B| \geq 2$ and $B \setminus A \neq \varnothing$. Fix some $t_3 \in B \setminus A$ and $s \in B$, $s \neq t_3$. For each $i \in \mathcal{N} \setminus \{t_1, t_2\}$ consider the transposition π_i of $G(e)$ which swaps i and t_3. Then the idempotent $e_i = (\pi_i f)(\pi_i f)^{-1}$ has a block $C \cup C'$, $C \subseteq \mathcal{N}$ with $i, s \in C$. Now consider the transposition $\pi_1 \in G(e)$ which switches the blocks $\{t_1, t_2\}$ and $\{t_3\}$. Then the idempotent $e_1 = (\pi_1 f)(\pi_1 f)^{-1}$ has a block $C \cup C'$, $C \subseteq \mathcal{N}$, with $t_1, t_2, s \in C$. The product of all the constructed idempotents e_i, $i \in \mathcal{N} \setminus \{t_2\}$, equals $\tau_{\mathcal{N}}$.

Case 2. Suppose $\mathrm{rank}(f) = n$, that is, $f = 1$. Then $\mathcal{S}_n \subseteq T$. Conjugating e by each transposition of $\mathcal{S}_n$, that moves t_1, and taking the product all the obtained elements outputs $\tau_{\mathcal{N}}$.

Show that $E_n^{(n-1)} \subseteq T$. Take $e \in E_n^{(n-1)}$. Suppose

$$e = \big\{A_1 \cup A_1', \dots, A_k \cup A_k'\big\},$$

where $k = \mathrm{rank}(e) \leq n-1$ and $|A_1| \geq 2$. Let $A_i = \{t_1^i, \dots t_{m_i}^i\}$, $1 \leq i \leq k$. Construct the blocks $B_1, \dots, B_k$ as follows: $B_1 = \{t_1^1\}$, B_2 consists of $|A_2|$ elements of

$$t_1^1, \dots, t_{m_1}^1, \dots, t_1^k, \dots, t_{m_k}^k \tag{5}$$

which follow t_1^1, B_3 consists of $|A_3|$ elements of (5) which follow the last element of B_2, and so on, finally B_k consists of the remaining $|A_k| + |A_1| - 1$ elements of (5). Set

$$a = \{A_1 \cup B_1', \dots, A_k \cup B_k'\}.$$

The construction implies that some powers of a and of a^{-1} equal $\tau_{\mathcal{N}}$. Hence, $a, a^{-1} \in T$, and thus $e = aa^{-1} \in T$.

Finally, since some power of every element of $\mathcal{I}_n^* \setminus \mathcal{S}_n$ is an idempotent of $E_n^{(n-1)} \subset T$ and T is isolated, we have $\mathcal{I}_n^* \setminus \mathcal{S}_n \subseteq T$. The statement follows. $\square$

10.2. Isolated subsemigroups of $\overline{\mathcal{PI}^*}_n$.

Theorem 34. *The semigroups $\overline{\mathcal{PI}^*}_n$, $\mathcal{S}_n$, $\overline{\mathcal{PI}^*}_n \setminus \mathcal{S}_n$ and $G(e)$, e is an idempotent with* $\operatorname{corank}(e) \leq 1$, *and only them, are isolated subsemigroups of $\overline{\mathcal{PI}^*}_n$.*

For the proof of this theorem we need some preparation. The observation below follows from the definition of $\circ$.

Lemma 35. *Let $a \in \overline{\mathcal{PI}^*}_n$. Then every block of* $\operatorname{dom}(a^k)$ *coincides with some block of* $\operatorname{dom}(a)$ *and every block of* $\operatorname{ran}(a^k)$ *coincides with some block of* $\operatorname{ran}(a)$ *for each $k \geq 1$.*

Let $e \in E(\mathcal{PI}_n^*)$. Set $\operatorname{corank}(e) = |\operatorname{codom}(e)| = |\operatorname{coran}(e)|$.

Lemma 36. *Let $e \in E(\overline{\mathcal{PI}^*}_n)$ be such that* $\operatorname{corank}(e) \leq 1$. *Then $G(e)$ is an isolated subsemigroup of $\overline{\mathcal{PI}^*}_n$.*

Proof. Similarly to as in the proof of Proposition 32 it is enough to prove that $a \in G(e)$ under the assumption that $a^k = e$ for some $k \geq 1$. Consider two possible cases.

Case 1. $\operatorname{corank}(e) = 0$. Since $\operatorname{coran}(e) \supseteq \operatorname{coran}(a)$ and $\operatorname{codom}(e) \supseteq \operatorname{codom}(a)$ it follows that $|\operatorname{coran}(a)| = |\operatorname{codom}(a)| = 0$. Thus $\operatorname{dom}(a)$, $\operatorname{dom}(e)$, $\operatorname{ran}(a)$, $\operatorname{ran}(e)$ are some partitions of $\mathcal{N}$. This and Lemma 35 imply $\operatorname{dom}(a) = \operatorname{dom}(e)$ and $\operatorname{ran}(a) = \operatorname{ran}(e)$. Therefore, $a\mathcal{H}e$, implying $a \in G(e)$.

Case 2. $\operatorname{corank}(e) = 1$. Assume that $\operatorname{codom}(e)=\{t\}$. By Lemma 35 there are two possibilities: either $\operatorname{dom}(a) = \operatorname{dom}(e)$ and $\operatorname{ran}(a) = \operatorname{ran}(e)$, or $\operatorname{dom}(a) = \operatorname{dom}(e) \cup \{t\}$ and $\operatorname{ran}(a) = \operatorname{ran}(e) \cup \{t'\}$. In the first case we have $a\mathcal{H}e$, which yields $a \in G(e)$, as required. In the second case we would have $a\mathcal{H}f$ and then $e \in G(f)$, where f is an idempotent such that each generalised line of e is a generalised line of f and, besides, f has the block $\{t, t'\}$, which is impossible. $\square$

To proceed, we need to recall the description of isolated subsemigroups of $\mathcal{I}_n$ which is taken from [6, Chapter 5]:

Lemma 37. *The semigroups $\mathcal{I}_n$, $\mathcal{S}_n$, $\mathcal{I}_n \setminus \mathcal{S}_n$, and $G(e)$, where e is an idempotent of rank $n-1$, and only them are isolated subsemigroups of $\mathcal{I}_n$.*

Proof of Theorem 34. Applying Lemma 36 and Theorem 28, it is enough to prove the sufficiency. Let T be an isolated subsemigroup of $\overline{\mathcal{PI}^*}_n$, such that $T \neq \mathcal{S}_n$ and $T \neq G(e)$ for any idempotent e of corank 0 or 1. We are to show that $T \supseteq \overline{\mathcal{PI}^*}_n \setminus \mathcal{S}_n$.

First show that T has an idempotent of corank at least 2. Assume the converse. Then T contains at least two distinct idempotents e, f such that $\mathrm{corank}(e) \leq 1$, $\mathrm{corank}(f) \leq 1$. Since $ef \in T$ and $\mathrm{corank}(ef) \leq 1$, one of e, f must be equal to ef. Hence we can assume that $e \geq f$. We have $G(e), G(f) \subseteq T$ by Lemma 22. Observe that among all the products of elements of $G(e)$ and $G(f)$ there are elements some powers of which are idempotents of corank at least 2.

Let $f \in T$ be an idempotent of corank at least 2. Fix $t_1, t_2 \in \mathcal{N}$, $t_1 \neq t_2$, such that $t_1, t_2 \in \mathrm{codom}(f)$. Define $a \in \overline{\mathcal{PI}^*}_n$ as follows. Each generalised line of f is a generalised line of a. Besides, a has one more generalised line: $\{t_1, t_2'\}$. Then $a^2 = (a^{-1})^2 = f$, the element $\widetilde{f} = aa^{-1}$ is an idempotent, and each generalised line of f is a generalised line of $\widetilde{f}$. In addition, $\widetilde{f}$ has exactly one more line: $\{t_1, t_1'\}$. Since T is isolated, $G(\widetilde{f}) \subseteq T$. Multiplying all the products of elements from $G(\widetilde{f})$ by f we obtain 0. This shows that $0 \in T$.

Since $T \cap \mathcal{I}_n \neq \varnothing$, it follows that $T \cap \mathcal{I}_n$ is an isolated subsemigroup of $\mathcal{I}_n$, which by Lemma 37 and $0 \in T$ implies $\mathcal{I}_n \setminus \mathcal{S}_n \subseteq T$. Thus $\overline{\mathcal{PI}^*}_n \setminus \mathcal{S}_n \subseteq T$ by Corollary 30. □

10.3. Isolated subsemigroups of $\mathcal{PI}_n^*$. Let $Y \subset \mathcal{N}$ and $a \in \mathcal{PI}_n^*$. We will call the set Y *invariant* with respect to a if either $A \subset Y \cup Y'$ or $A \cap (Y \cup Y') = \varnothing$ for each block A of a. If Y is invariant with respect to a denote by $a|_Y$ the element of $\mathcal{PI}_Y^*$ whose blocks are all blocks of a which are contained in $Y \cup Y'$. The element $a|_Y$ will be called the *restriction* of a to Y. The semigroup $\mathcal{I}_Y^*$ embeds into $\mathcal{I}_n^*$ via the map sending $a \in \mathcal{I}_Y^*$ to the element of $\mathcal{I}_n^*$ whose generalised lines are precisely the generalised lines of a, and all the other blocks are points. We will identify $\mathcal{I}_Y^*$ with its image under this embedding.

Lemma 38. *Let $n \geq 3$. The semigroups*

1) $\mathcal{I}_n^*$, $\mathcal{I}_n^* \setminus \mathcal{S}_n$, $\mathcal{S}_n$, $G(e)$, *where e is an idempotent of rank $n-1$ of $\mathcal{I}_n^*$;*
2) $\mathcal{I}_Y^*$, $\mathcal{I}_Y^* \setminus \mathcal{S}_Y$, $\mathcal{S}_Y$, $G(e)$, *where e is an idempotent of rank $n-2$ of $\mathcal{I}_Y^*$, where $Y = \mathcal{N} \setminus \{t\}$, $t \in \mathcal{N}$;*
3) $\mathcal{PI}_n^*$, $\mathcal{PI}_n^* \setminus \mathcal{S}_n$

are isolated subsemigroups of $\mathcal{PI}_n^*$.

Proof. The proof is a straightforward verification. It resembles the proofs of Proposition 32 and Lemma 36. □

Theorem 39. *Let $n \geq 3$. The semigroups listed in Lemma 38 and only them are isolated subsemigroups of $\mathcal{PI}_n^*$.*

Proof. Let T be an isolated subsemigroup of $\mathcal{PI}_n^*$. If $T \subset \mathcal{I}_n^*$ then T must be an isolated subsemigroup of $\mathcal{I}_n^*$. Therefore, applying Theorem 33, we see that T is one of the semigroups listed in the first item of Lemma 38.

Suppose $T \setminus \mathcal{I}_n^* \neq \varnothing$. Then T contains an idempotent of corank 1 (this can be shown using arguments similar to those from the third paragraph of the proof of Theorem 34, where an idempotent $\widetilde{f}$ is being constructed by f). It follows that there is $Y \subset \mathcal{N}$, $Y = \mathcal{N} \setminus \{t\}$, $t \in \mathcal{N}$, such that $T \cap \mathcal{I}_Y^* \neq \varnothing$. It follows that $T \cap \mathcal{I}_Y^*$ is an isolated subsemigroup of $\mathcal{I}_Y^*$. If $T \subseteq \mathcal{I}_Y^*$ then T is one of the semigroups of the second item of Lemma 38.

Suppose that $T \setminus \mathcal{I}_Y^* \neq \varnothing$. Then T has at least two idempotents e and f such that there is no proper subset Z of $\mathcal{N}$ for which $e, f \in \mathcal{I}_Z^*$. Since $e, f, ef \in T$ it follows that we may assume $e > f$. Now, $G(e), G(f) \subset T$ imply $0 \in T$. Hence $T \cap \mathcal{I}_n$ is an isolated subsemigroup of $\mathcal{I}_n$ containing the zero. This and Lemma 37 show that $\mathcal{I}_n \setminus \mathcal{S}_n \subseteq T$.

To complete the proof show that $\mathcal{PI}_n^* \setminus \mathcal{S}_n \subseteq T$. It is enough to show that $\widetilde{E}_n^{(n-1)} \subseteq T$. Let $e \in \mathcal{PI}_n^* \setminus \mathcal{I}_n$ be an idempotent. Let $Z = \mathcal{N} \setminus \operatorname{codom}(e)$. If $Z = \mathcal{N}$ then $e \in T$ by arguments at the end of the proof of Theorem 33. Let $\mathcal{N} \setminus Z \neq \varnothing$. We have that $e|_Z \in E(\mathcal{I}_Z^* \setminus \mathcal{S}_Z)$. We claim that it is enough to show that the element $\widetilde{\tau}_Z$, having the only generalised line $Z \cup Z'$ and all the other blocks points, belongs to T. Indeed, if $\widetilde{\tau}_Z \in T$ then applying the arguments similar to those at the end of the proof of Theorem 33, we obtain that $e|_Z \in T|_Z$, implying that $f \in T$ for some $f \in \mathcal{PI}_n^*$ with $e|_Z = f|_Z$. Since we also know that $1|_Z \in \mathcal{I}_n \setminus \mathcal{S}_n \subseteq T$, we have that $e = 1|_Z f \in T$ as well. Take $t \in Z$. Set a to be the element of $\mathcal{PI}_n^*$ with the only one generalised line $Z \cup \{t'\}$, and all the other blocks points. Then $a^2 = (a^{-1})^2 = 0$, while $aa^{-1} = \widetilde{\tau}_Z$. The statement follows. $\square$

11. Automorphisms of $\mathcal{PI}_X^*$ and $\overline{\mathcal{PI}^*}_X$

11.1. Automorphisms of $\mathcal{PI}_X^*$. Let $Y \subset X$. We will need to consider the following subsemigroups of $\mathcal{PI}_X^*$:

$$\widetilde{\mathcal{S}}_Y = \big\{a \in \mathcal{S}_X : a \text{ contains the blocks } \{t, t'\},\ t \in X \setminus Y\big\},$$

$$\widetilde{\mathcal{I}}_Y = \big\{a \in \mathcal{I}_X : a \text{ contains the blocks } \{t, t'\},\ t \in X \setminus Y\big\} \quad \text{and}$$

$$\widetilde{\mathcal{I}^*}_Y = \big\{a \in \mathcal{I}^*_X : a \text{ contains the blocks } \{t,t'\},\ t \in X \setminus Y\big\}.$$

Let $\mathrm{Aut}(S)$ denote the group of automorphisms of a semigroup S.

Theorem 40. $\mathrm{Aut}(\mathcal{PI}^*_X) \cong \mathcal{S}_X$. *Moreover, for every* $\varphi \in \mathrm{Aut}(\mathcal{PI}^*_X)$ *there is* $\pi \in \mathcal{S}_X$ *such that* $a^{\varphi} = \pi^{-1}a\pi$, $a \in \mathcal{PI}^*_X$.

Proof. Let $\varphi \in \mathrm{Aut}(\mathcal{PI}^*_X)$. Take $x \in X$. Since $\mathcal{S}_X$ is the group of units of $\mathcal{PI}^*_X$, in should be preserved by φ: $\varphi(\mathcal{S}_X) = \mathcal{S}_X$. For $u \in \mathcal{PI}^*_X$ and a subsemigroup $T \subseteq \mathcal{PI}^*_X$ let

$$\mathrm{St}^r_T(u) = \{s \in T \mid us = u\}, \quad \mathrm{St}^l_T(u) = \{s \in T \mid su = u\}.$$

Recall that for $x \in X$ by α_x we denote the idempotent $\big\{\{t,t'\}_{t \in X\setminus\{x\}}\big\} \in \mathcal{PI}^*_X$.

Observe that for an idempotent $u \in \mathcal{PI}^*_X$ $|\mathrm{St}^r_{\mathcal{S}_X}(u)| = 1$ if and only if $u = \alpha_z$ for some $z \in X$. It follows that for each $x \in X$ there is $g(x) \in X$ such that $\varphi(\alpha_x) = \alpha_{g(x)}$. This defines a permutation $g \in \mathcal{S}_X$.

Show that $\varphi(\mathcal{I}_X) = \mathcal{I}_X$. Let $a = \big\{\{t,\pi(t)'\}_{t\in I}\big\} \in \mathcal{I}_X$, where $\pi : I \to \pi(I)$ is a bijection. For all $z \in X \setminus I$ and $r \in X \setminus \pi(I)$ we have $\alpha_z a = a = a\alpha_r$. Passing in this equality to φ-images, we see that $\varphi(a)$ should contain the blocks $\{q\}$, $q \in g(X \setminus I)$, and $\{r'\}$, $r \in g(X \setminus \pi(I))$. Let $t_0 \in I$. Notice that the equality

$$\alpha_{t_0}a = \alpha_z \cdot \alpha_{t_0}a \cdot \alpha_r \tag{6}$$

holds if and only if $z \in (X \setminus I) \cup \{t_0\}$ and $r \in \big(X \setminus \pi(I)\big) \cup \{\pi(t_0)\}$. Going in (6) to φ-images, we obtain

$$\alpha_{g(t_0)}\varphi(a) = \alpha_{g(z)} \cdot \alpha_{g(t_0)}\varphi(a) \cdot \alpha_{g(r)}.$$

Similarly as above we have that the equality

$$\alpha_{g(t_0)}\varphi(a) = \alpha_z \cdot \alpha_{g(t_0)}\varphi(a) \cdot \alpha_r$$

holds if and only if $z \in g\big((X \setminus I) \cup \{t_0\}\big)$ and $r \in g\big(\big(X \setminus \pi(I)\big) \cup \{\pi(t_0)\}\big)$. The latter implies that $\varphi(a)$ contains a block $\{g(t_0), g(\pi(t_0))\}$. Now we can assert that $\varphi(a) = \big\{\{g(t), g(\pi(t))'\}\big\}_{t\in I}$. It follows that $\varphi(\mathcal{I}_X) = \mathcal{I}_X$. Moreover, for every $Y \subset X$ we have

$$\varphi(\widetilde{\mathcal{I}}_Y) = \widetilde{\mathcal{I}}_{g(Y)}. \tag{7}$$

Show that $\varphi(\mathcal{I}^*_X) = \mathcal{I}^*_X$. Observe that the elements of $\mathcal{I}^*_X$ may be characterized as follows: $b \in \mathcal{I}^*_X$ if and only if $\alpha_x b \neq b$ and $b\alpha_x \neq b$ for all $x \in X$. Let $b = \big\{(A_i \cup B'_i)_{i\in I}\big\} \in \mathcal{I}^*_X$. The equality $\alpha_u b = \alpha_v b$ holds if and only if u and v belong to A_i for some $i \in I$, the equality $b\alpha_u = b\alpha_v$ holds if and only if u and v belong to B_i for some $i \in I$, and the equality $\alpha_u b = b\alpha_v$ holds if and only if $u \in A_i$ and $v \in B_i$ for some $i \in I$. Going to φ-images and using the fact that

$\varphi(\alpha_x) = \alpha_{g(x)}$, $x \in X$, we can assert that $\varphi(b) = \{(g(A_i) \cup g(B_i)')_{i\in I}\}$. Thus $\varphi(\mathcal{I}_X^*) = \mathcal{I}_X^*$ and, moreover,

$$\varphi(\widetilde{\mathcal{I}^*}_Y) = \widetilde{\mathcal{I}^*}_{g(Y)} \tag{8}$$

for every $Y \subset X$. Since $\widetilde{\mathcal{S}}_Y = \widetilde{\mathcal{I}}_Y \cap \widetilde{\mathcal{I}^*}_Y$, applying (7) and (8) we obtain

$$\varphi(\widetilde{\mathcal{S}}_Y) = \varphi(\widetilde{\mathcal{I}}_Y) \cap \varphi(\widetilde{\mathcal{I}^*}_Y) = \widetilde{\mathcal{S}}_{g(Y)}. \tag{9}$$

Let $a = \{(U_i \cup V_i')_{i\in I}\} \in \mathcal{PI}_X^*$. Observe that

$$\mathrm{St}^l_{\mathcal{I}_X^*}(a) = \Big(\widetilde{\mathcal{I}^*}_{X\setminus \bigcup_{i\in I} U_i}\Big) \oplus \Big(\bigoplus_{i\in I} \widetilde{\mathcal{I}^*}_{U_i}\Big); \quad \mathrm{St}^r_{\mathcal{I}_X^*}(a) = \Big(\widetilde{\mathcal{I}^*}_{X\setminus \bigcup_{i\in I} V_i}\Big) \oplus \Big(\bigoplus_{i\in I} \widetilde{\mathcal{I}^*}_{V_i}\Big);$$

$$\mathrm{St}^l_{\mathcal{I}_X}(a) = \Big(\widetilde{\mathcal{I}}_{X\setminus \bigcup_{i\in I} U_i}\Big) \oplus \Big(\bigoplus_{i\in I} \widetilde{\mathcal{S}}_{U_i}\Big); \quad \mathrm{St}^r_{\mathcal{I}_X}(a) = \Big(\widetilde{\mathcal{I}}_{X\setminus \bigcup_{i\in I} V_i}\Big) \oplus \Big(\bigoplus_{i\in I} \widetilde{\mathcal{S}}_{V_i}\Big).$$

We observe that the equalities

$$\mathrm{St}^l_{\mathcal{I}_X^*}(a) = \mathrm{St}^l_{\mathcal{I}_X^*}(b), \quad \mathrm{St}^r_{\mathcal{I}_X^*}(a) = \mathrm{St}^r_{\mathcal{I}_X^*}(b),$$
$$\mathrm{St}^l_{\mathcal{I}_X}(a) = \mathrm{St}^l_{\mathcal{I}_X}(b), \quad \mathrm{St}^r_{\mathcal{I}_X}(a) = \mathrm{St}^r_{\mathcal{I}_X}(b)$$

hold for some $b \in \mathcal{PI}_X^*$ if and only if $\mathrm{dom}(a) = \mathrm{dom}(b)$ and $\mathrm{ran}(a) = \mathrm{ran}(b)$, which by Proposition 3, is equivalent to $a\mathcal{H}b$.

By (7), (8) and (9) we have

$$\varphi(a) = \{g(U_i) \cup g(V_{\pi(i)})'\}_{i\in I} \tag{10}$$

for some bijection $\pi : I \to I$. Let us show that π should be the identity map. Let $j \in I$. Fix $u_j \in U_j$. We compute

$$\alpha_{u_j} a = \{(U_i \cup V_i')\}_{i\in I\setminus\{j\}}.$$

By (7), (8) and (9) we have

$$\mathrm{coran}(\varphi(\alpha_{u_j} a)) = \Big\{\{t'\},\ t \notin \bigcup_{i\in I} V_i, \{t'\},\ t \in g(V_j)\Big\}. \tag{11}$$

From the other hand, $\varphi(\alpha_{u_j} a) = \alpha_{g(u_j)}\varphi(a)$, and thus

$$\mathrm{coran}(\varphi(\alpha_{u_j} a)) = \Big\{\{t'\},\ t \notin \bigcup_{i\in I} V_i,\ \{t'\},\ t \in g(V_{\pi(j)})\Big\}. \tag{12}$$

It follows from (11) and (12) that $\pi(j) = j$, and then π is the identity map. Hence $\varphi(a) = g^{-1}ag$, $a \in \mathcal{PI}_X^*$. The proof is completed. $\square$

11.2. Automorphisms of $\overline{\mathcal{PI}^*}_X$. Let $Y \subseteq X$. Set $\varepsilon_Y = \big\{Y \cup Y'\big\}$. The element ε_Y is an idempotent of rank 1. If ε is an idempotent of rank 1, denote by $Y(\varepsilon)$ such a subset $Y \subseteq X$ that $\varepsilon_{Y(\varepsilon)} = \varepsilon$.

Theorem 41. $\mathrm{Aut}(\overline{\mathcal{PI}^*}_X) \cong \mathcal{S}_X$.

Proof. Let $\varphi \in \mathrm{Aut}(\overline{\mathcal{PI}^*}_X)$. The maps $\varepsilon_Y \mapsto Y$ and $Y \mapsto \varepsilon(Y)$ are mutually inverse bijections between the idempotents of rank 1 of $\overline{\mathcal{PI}^*}_X$ and nonempty subsets of X. It follows that φ induces some permutation π on $2^X \setminus \{\varnothing\}$.

Show that $A \cap B = \varnothing$ implies $\pi(A) \cap \pi(B) = \varnothing$ for all $A, B \subseteq X$. Consider the idempotent $e = \big\{A \cup A', B \cup B'\big\}$. Let $f = \varphi(e) = \{C \cup C', D \cup D'\}$ ($\mathrm{rank}(f) = 2$ because $\mathrm{rank}(e) = 2$, and ranks of idempotents are preserved by automorphisms as they may be characterised in terms of the natural order). Since $\varepsilon_A e = \varepsilon_A$ and $\varepsilon_B e = \varepsilon_B$, going to φ-images, we obtain $\varepsilon_{\pi(A)} f = \varepsilon_{\pi(A)}$ and $\varepsilon_{\pi(B)} f = \varepsilon_{\pi(B)}$. It follows that f has the blocks $\pi(A) \cup \pi(A)'$ and $\pi(B) \cup \pi(B)'$. Taking into account that $\mathrm{rank}(f) = 2$, we see that $\{C, D\} = \{\pi(A), \pi(B)\}$. Since $C \cap D = \varnothing$, than also $\pi(A) \cap \pi(B) = \varnothing$.

Show now that π maps one-element subsets of X to one-element subsets. Assume the converse. Let $x \in X$ be such that $\pi(\{x\}) = M$, where $|M| \geq 2$. Take $y, z \in M$, $y \neq z$. Let M_y and M_z denote the sets satisfying $\pi(M_y) = \{y\}$ and $\pi(M_z) = \{z\}$, respectively. Since $\{y\} \cap \{z\} = \varnothing$, by the argument from the previous paragraph we obtain $M_y \cap M_z = \varnothing$. On the other hand, using $\{y\} \cap M \neq \varnothing$ and $\{z\} \cap M \neq \varnothing$, we obtain that it must be $M_y \cap \{x\} \neq \varnothing$ and $M_z \cap \{x\} \neq \varnothing$. But then $x \in M_y \cap M_z$, which is impossible. The restriction of π to one-element subsets of X defines a permutation $g \in \mathcal{S}_X$.

We proceed by showing that $\pi(M) = g(M) = \{g(m) \mid m \in M\}$ for each subset M of X. Indeed, since $M \cap \{t\} = \varnothing$, $t \in X \setminus M$, it follows that $\pi(M) \subseteq g(M)$. Similar arguments applied for the automorphism φ^{-1} ensure that $\pi^{-1}(g(M)) \subseteq M$, and thus $g(M) \subseteq \pi(M)$. The reverse inclusion is established similarly.

Let $a \in \overline{\mathcal{PI}^*}_X$. Suppose that a has a block $A \cup B'$. Show that $\varphi(a)$ has the block $g(A) \cup g(B)'$. Indeed, $\varepsilon_A a \varepsilon_B \neq 0$. Going to φ-images, we obtain $\varepsilon_{g(A)} \varphi(a) \varepsilon_{g(B)} \neq 0$. The latter implies that $\varphi(a)$ has the block $g(A) \cup g(B)'$, as required. It follows that $A \cup B'$ is a generalised line of a if and only if $g(A) \cup g(B)'$ is a generalised line of $\varphi(a)$, which completes the proof. $\square$

12. $\mathcal{PI}_n^*$ and $\overline{\mathcal{PI}^*}_n$ are embeddable into $\mathcal{I}_{2^n-1}$

Let S be an inverse semigroup with the natural partial order ϱ on it. The following definitions are taken from [8, p. 188]. An inverse subsemigroup H of S is called a *closed inverse subsemigroup* of S if $H\varrho = H$. Let

$$\mathcal{C} = \mathcal{C}_H = \big\{(Hs)\varrho : ss^{-1} \in H\big\} \tag{13}$$

be the set of all *right ϱ-cosets* of H.

Let, further,

$$\phi_H(s) = \big\{\big((Hx)\varrho, (Hxs)\varrho\big) : (Hx)\varrho, (Hxs)\varrho \in \mathcal{C}\big\} \tag{14}$$

be the *effective transitive representation* $\phi_H : S \to \mathcal{I}_{\mathcal{C}}$. If K and H are two closed inverse subsemigroups of S, the representations ϕ_K and ϕ_H are *equivalent* if and only if there exists $a \in S$ such that $a^{-1}Ha \subseteq K$ and $aKa^{-1} \subseteq H$ (see [16, Proposition IV.4.13]).

Theorem 42. *Let $n \geq 2$. Up to equivalence, there is only one faithful effective transitive representation of $\mathcal{PI}_n^*$ (respectively $\overline{\mathcal{PI}^*}_n$), namely to $\mathcal{I}_{2^n-1}$. In particular, $\mathcal{PI}_n^*$ and $\overline{\mathcal{PI}^*}_n$ embed into $\mathcal{I}_{2^n-1}$.*

Proof. We prove the statement for the case of $\mathcal{PI}_n^*$, the other case being treated analogously. Suppose H is a closed inverse subsemigroup of $\mathcal{PI}_n^*$. Denote by ω the natural partial order on $\mathcal{PI}_n^*$. First we observe that $H = G\omega$ for some subgroup G of $\mathcal{PI}_n^*$. Indeed, since $\mathcal{PI}_n^*$ is finite, $E(H)$ contains a zero element. It remains to apply [16, Proposition IV.5.5], which claims that if the set of idempotents of a closed inverse subsemigroup contains a zero element, then this subsemigroup is a closure of some subgroup of the original semigroup. Denote by f the identity element of the group G.

Now we prove that if $f = 0$ then ϕ_H is not faithful. We have $H = 0\omega = \mathcal{PI}_n^*$ and hence $(Hx)\omega \supseteq 0\omega = \mathcal{PI}_n^*$ for all $x \in \mathcal{PI}_n^*$. Thus $(Hx)\omega = \mathcal{PI}_n^*$ for all $x \in \mathcal{PI}_n^*$. Then $|\,\phi_H(\mathcal{PI}_n^*)\,| = 1$ and so ϕ_H is not faithful.

Let now $\mathrm{rank}(f) \geq 2$. We will show that in this case ϕ_H is not faithful either. Take $b \in D_1$ where D_1 denotes the set of elements of $\mathcal{PI}_n^*$ of rank 1. Since $bb^{-1} \in D_1$ we have that $bb^{-1} \notin H$ and therefore $(Hb)\omega \notin \mathcal{C}$. This implies that $\phi_H(b)$ is equal to the zero element of $\mathcal{I}_{\mathcal{C}}$. Then due to $|\,D_1\,| \geq 2$ we obtain that ϕ_H is not faithful.

Let finally $\mathrm{rank}(f) = 1$. We will show that in this case ϕ_H is faithful. Observe that $H = f\omega$. Let $f = \varepsilon_E = \big\{E \cup E'\big\}$ where $E \neq \varnothing$. Suppose that $\phi_H(s) = \phi_H(t)$

for some s and t from $\mathcal{PI}_n^*$. Without loss of generality assume that $s \neq 0$. Suppose that s contains a block $A \cup B'$. Consider the element $x = \{E \cup A'\}$. Then $(Hx)\omega$ and $(Hxs)\omega$ belong to $\mathcal{C}$. This implies that $(Hxs)\omega = (Hxt)\omega$. The latter means that t contains some generalised lines whose union is the block $A \cup B'$. Changing the roles of s and t we obtain that both s and t contain the block $A \cup B'$. Thus $s = t$, as required.

Observe that all the idempotents of $\mathcal{PI}_X^*$ of rank 1 are precisely the primitive idempotents. Let g be a primitive idempotent of $\mathcal{PI}_n^*$. We will show that $|\mathcal{C}_{g\omega}| = 2^n - 1$. Note that $\mathcal{C}_{g\omega} = \{(gs)\omega : ss^{-1} \geq g\}$. We have $(gs)\omega = (gt)\omega$ if and only if $gs = gt$, that is, the number of different sets $(gs)\omega$, $ss^{-1} \geq g$, is equal to the number of different nonempty subsets of $\mathcal{N}$, which equals $2^n - 1$.

To complete the proof we note that for two primitive idempotents $f_1, f_2 \in \mathcal{PI}_n^*$ we have that $\phi_{f_1\omega}$ and $\phi_{f_2\omega}$ are equivalent by the definition of equivalent representations. □

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