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Multilinear Calderón–Zygmund operators on Morrey space with non-doubling measures

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Abstract. Under the assumption that μ is a non-negative Radon measure on $\mathbb{R}^d$ which only satisfies some growth condition, the authors proved the multilinear Calderón–Zygmund operators are bounded from $\mathcal{M}_{q_1}^{p_1}(k,\mu)\times\cdots\times\mathcal{M}_{q_m}^{p_m}(k,\mu)$ into $\mathcal{M}_q^p(k,\mu)$ for some fixed $q_1,\dots,q_m\in(1,\infty)$ and $1/q=1/q_1+\cdots+1/q_m$. Furthermore, the authors established the same bounded estimates for the commutators generated by multilinear Calderón–Zygmund operators and RBMO(μ) functions. Some of the results are also new even when the measure μ is the d-dimensional Lebesgue measure.

1. Introduction

We will work on $\mathbb{R}^d$ with a non-negative Radon measure μ which only satisfies the following growth condition that there exists a positive constant C_0 and fixed $n\in(0,d]$ such that

$$\mu(B(x,l))\le C_0l^n \tag{1.1}$$

for all $x\in\mathbb{R}^d$ and $l>0$, where $B(x,l)=\{y\in\mathbb{R}^d:|y-x|<l\}$. Such a measure μ is not necessary to be doubling. We recall that μ is said to satisfy the doubling condition if there exists a positive constant C such that $\mu(B(x,2l))\le C\mu(B(x,l))$ for all $x\in\operatorname{supp}\mu$ and $l>0$. It is well known that the doubling condition is an essential assumption in the classical theory of harmonic analysis. But recently,

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many classical results have been proved still valid if the underlying measure μ is substituted by a non-doubling Radon measure as in (1.1); see [1]–[8] and their references. The analysis on spaces with non-doubling measures is proved to play a striking role in solving the famous Painlevé's problem by X. Tolsa in [3]; see also [4] for more background.

Let $K(x, y_1, \dots, y_m)$ be a locally function defined away from the diagonal on $x = y_1 = \dots = y_m$ in $(\mathbb{R}^d)^{m+1}$, which satisfies the size condition that

$$|K(x, y_1, \dots, y_m)| \le \frac{C}{\left(\sum_{i=1}^m |x - y_i|\right)^{nm}} \tag{1.2}$$

for all $(x, y_1, \dots, y_m) \in (\mathbb{R}^d)^{m+1}$ with $x \ne y_j$ for some j. Furthermore, assume that

$$\begin{aligned} &|K(x, y_1, \dots, y_j, \dots, y_m) - K(x, y_1, \dots, y_j', \dots, y_m)| \\ &\qquad \le C \frac{|y_j - y_j'|^\delta}{\left(\sum_{i=1}^m |x - y_i|\right)^{nm+\delta}} \end{aligned} \tag{1.3}$$

for $\max_{1 \le i \le m}\{|x - y_i|\} \ge 2|y_j - y_j'|$, where $1 \le j \le m$, $0 < \delta \le 1$ and $C > 0$. We define the multilinear Calderón–Zygmund operator T by

$$T(\vec{f})(x) := T(f_1, \dots, f_m)(x) = \int\limits_{\mathbb{R}^m} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m) d\mu(\vec{y}), \tag{1.4}$$

where $f_1, \dots, f_m$ are C^∞−functions with compact supports and $x \notin \cap_{j=1}^m \mathrm{supp} f_j$.

For $\epsilon > 0$, the truncated operator $T_\epsilon(f_1, \dots, f_m)$ is defined as

$$T_\epsilon(\vec{f})(x) = \int_{\sum_{i=1}^m |x-y_i|^2 > \epsilon^2} K(x, y_1, \dots, y_m) \prod_{i=1}^m f_i(y_i) d\mu(\vec{y}).$$

We say that T is bounded on $L^p(\mu)$ if the operators T_ϵ are bounded on $L^p(\mu)$ uniformly on $\epsilon > 0$, and T satisfies the weak type estimate if T_ϵ satisfy the same weak type estimate uniformly on $\epsilon > 0$.

The analysis of multilinear singular integrals has much of its origins in several works by Coifman and Meyer in the 70's, see for example [9]. In [10]–[13], an updated systematic treatment of multilinear singular integral operators was presented in light of some new developments. L. Grafakos and R. Torres established some boundedness of multilinear Calderón–Zygmund operators and its commutators on the products of Lebesgue spaces. More recently, the multilinear Calderón–Zygmund operator was studied on spaces with non-doubling measures.

In [16], Xu established the boundedness of the multilinear Calderón–Zygmund operator on $\mathbb{R}^d$, then Xu and Li obtained the boundedness of commutators generated by multilinear Calderón–Zygmund operator and RBMO(μ) defined as (3.1) and (4.1) below, see [17], [18] for a detailed description. For other works about the multilinear operators with nondoubling measures, see [19], [20].

The aim of this paper is to establish some estimates for the multilinear Calderón–Zygmund operator and its commutator on the Morrey space with measure only satisfying the growth condition (1.1).

Throughout this paper, by a cube $Q \subset \mathbb{R}^d$, we mean a closed cube in $\mathbb{R}^d$ with sides parallel to the axes and centered at some point of supp(μ), and we denote its side length by $\ell(Q)$ and its center by x_Q. For $\alpha > 0$, αQ will denote a cube concentric to Q with its sidelength $\alpha\ell(Q)$. The set of all cubes $Q \subset \mathbb{R}^d$ with positive $\mu-$measure will be denoted by $\Omega(\mu)$. We recall the definition of the Morrey space with non-doubling measure.

Definition 1.1. Let $k > 1$ and $1 \le q \le p < \infty$. We define the Morrey space $\mathcal{M}^p_q(k,\mu)$ as

$$\mathcal{M}^p_q(k,\mu) := \{f \in L^q_{loc}(\mu) \mid \|f \mid \mathcal{M}^p_q(k,\mu)\| < \infty\}, \tag{1.5}$$

where the norm $\|f \mid \mathcal{M}^p_q(k,\mu)\|$ is given by

$$\|f \mid \mathcal{M}^p_q(k,\mu)\| := \sup_{Q\in\Omega(\mu)} \mu(kQ)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f|^q d\mu\right)^{\frac{1}{q}}. \tag{1.6}$$

By using Hölder inequality to (1.6), it is easy to see that

$$L^p(\mu) = \mathcal{M}^p_p(k,\mu) \subset \mathcal{M}^p_{q_1}(k,\mu) \subset \mathcal{M}^p_{q_2}(k,\mu)$$

for $1 \le q_2 \le q_1 \le p < \infty$. The definition of space $\mathcal{M}^p_q(k,\mu)$ is independent of the constant $k > 1$. The norms for different choices of $k > 1$ are equivalent, see [14].

The Morrey space $\mathcal{M}^p_q(k,\mu)$ was introduced by Yoshihiro Sawano and Hitoshi Tanaka in [14], [15], where they investigated the bounded behavior of the singular integral operator, the fractional integral operator and their commutators.

Main theorems are stated in each section. Section 2 is devoted to the study of the multilinear Calderón–Zygmund operators. In Section 3, we focus on the boundedness of the commutator defined as (3.1). Finally, in Section 4, we investigate an other commutator defined as (4.1).

In what follows, we use the constant C with subscripts to indicate its dependence on the parameters in the subscripts. We denote simply by $A \lesssim B$ if there exists a constant $C > 0$ such that $A \le CB$; and $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. For a μ-measurable set E, χ_E denotes its characteristic function. For any $p \in [1,\infty]$, we denote by p' its conjugate index, namely, $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Boundedness of multilinear operator

Theorem 2.1. *Assume that T is an operator defined by (1.4) and satisfying conditions (1.2) and (1.3). Let $p_i \in (1,\infty)$ and $f_i \in L^{p_i}(\mu)$ for $i = 1,2,\dots,m$. If T is a bounded operator from $L^1(\mu)\times\cdots\times L^1(\mu)$ to $L^{1/m,\infty}(\mu)$, then*

$$\|T(f_1,\dots,f_m) \mid \mathcal{M}^p_q(k,\mu)\| \lesssim \|f_1 \mid \mathcal{M}^{p_1}_{q_1}(k,\mu)\| \times\cdots\times \|f_m \mid \mathcal{M}^{p_m}_{q_m}(k,\mu)\|,$$

where $1 < q_i \le p_i$ and $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p}$, $\sum_{i=1}^m \frac{1}{q_i} = \frac{1}{q}$.

We will use the following results in our proof.

Lemma 2.1 (see e.g. [16])**.** *Suppose T as in (1.4) and satisfying conditions (1.2) and (1.3). Let $p_i \in (1,\infty)$ and $f_i \in L^{p_i}(\mu)$ for $i = 1,2,\dots,m$. If T is a bounded operator from $L^1(\mu)\times\cdots\times L^1(\mu)$ to $L^{1/m,\infty}(\mu)$, then*

$$\|T(f_1,\dots,f_m) \mid L^p(\mu)\| \lesssim \|f_1 \mid L^{p_1}(\mu)\| \times\cdots\times \|f_m \mid L^{p_m}(\mu)\|.$$

Lemma 2.2. *Let $1 < q < p < \infty$ and $f \in \mathcal{M}^p_q(k,\mu)$ for $x = x_Q$, we have*

$$\int_{\mathbb{R}^d\setminus 2Q} \frac{|f(z)|}{|z-x|^n} d\mu(z) \lesssim \ell(Q)^{-n/p}\|f \mid \mathcal{M}^p_q(k,\mu)\|.$$

Proof. We denote Q as $Q = Q(x,\ell(Q)) := \{y\in\mathbb{R}^d : |y-x| < \ell\}$. It is evident that

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus 2Q} \frac{|f(z)|}{|z-x|^n} d\mu(z) &\lesssim \int_{\mathbb{R}^d\setminus \frac{Q}{2}} \left(\int_{|z-x|}^\infty |f(z)| r^{-n-1} dr\right) d\mu(z)\\
&\lesssim \int_{\mathbb{R}^d\setminus \frac{Q}{2}} \left(\int_0^\infty |f(z)|\chi_{B(x,r)} r^{-n-1} dr\right) d\mu(z)\\
&\lesssim \int_{\frac{\ell}{2}}^\infty \left(\int_{B(x,r)\setminus \frac{Q}{2}} |f(z)| d\mu(z)\right) r^{-n-1} dr\\
&\lesssim \int_{\frac{\ell}{2}}^\infty \left[\left(\int_{B(x,r)} |f(z)|^q d\mu(z)\right)^{1/q} \mu(B(x,r))^{1/q'} r^{-n-1}\right] dr\\
&\lesssim \sup_{B(x,r)\in\Omega(\mu)} \mu(B(x,2r))^{1/p-1/q} \left(\int_{B(x,r)} |f(z)|^q d\mu(z)\right)^{1/q}\\
&\quad \times \int_{2\ell}^\infty \left[\mu(B(x,r))^{1/q'} \mu(B(x,2r))^{1/q-1/p} r^{-n-1}\right] dr\\
&\lesssim \ell(Q)^{-n/p}\|f \mid \mathcal{M}^p_q(k,\mu)\|.
\end{aligned}$$

Here we used Hölder inequality and condition (1.1). □

Proof of Theorem 2.1. For simplicity and without loss of generality, we restrict $m = 2$.

We only have to prove the case that $q_i < p_i$ for $i = 1, 2$. Firstly, we rewrite

$$\begin{aligned} T(f_1, f_2)(x) &= T\big(f_1\chi_{\frac{4}{3}Q}, f_2\chi_{\frac{4}{3}Q}\big)(x) + T\big(f_1\chi_{\frac{4}{3}Q}, f_2\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\big)(x) \\ &\quad + T\big(f_1\chi_{\frac{4}{3}Q}, f_2\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\big)(x) + T\big(f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\big)(x) \\ &= T_1(f_1, f_2)(x) + T_2(f_1, f_2)(x) + T_3(f_1, f_2)(x) + T_4(f_1, f_2)(x). \end{aligned}$$

For $T_1(f_1, f_2)$, by Lemma 2.1, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, we have

$$\begin{aligned} &\|T_1(f_1, f_2) \mid \mathcal{M}^p_q(k, \mu)\| \lesssim \sup_{Q\in\Omega(\mu)} \mu(2Q)^{1/p-1/q}\Big(\int_Q |T_1(f_1, f_2)(x)|^q d\mu(x)\Big)^{1/q} \\ &\lesssim \sup_{Q\in\Omega(\mu)} \mu(2Q)^{1/p-1/q}\|T_1(f_1, f_2) \mid L^q(\mu)\| \\ &\lesssim \sup_{Q\in\Omega(\mu)} \mu(2Q)^{1/p-1/q}\|f_1 \mid L^{q_1}(\mu)\|\,\|f_2 \mid L^{q_2}(\mu)\| \\ &\lesssim \sup_{Q\in\Omega(\mu)} \mu(2Q)^{1/p-1/q}\Big(\int_{\frac{4}{3}Q} |f_1(y_1)|^{q_1} d\mu(y_1)\Big)^{1/q_1}\Big(\int_{\frac{4}{3}Q} |f_2(y_2)|^{q_2} d\mu(y_2)\Big)^{1/q_2} \\ &\lesssim \|f_1 \mid \mathcal{M}^{p_1}_{q_1}(2, \mu)\| \times \|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2, \mu)\|. \qquad (2.1) \end{aligned}$$

For T_2, we make to use condition (1.2) and Lemma 2.2, then

$$\begin{aligned} &\Big(\int_Q |T_2(f_1, f_2)(x)|^q d\mu(x)\Big)^{1/q} \\ &\lesssim \Big(\int_Q \Big[\int_{\frac{4}{3}Q}\int_{\mathbb{R}^d\setminus\frac{4}{3}Q} \frac{|f_1(y_1)f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}} d\mu(y_2)d\mu(y_1)\Big]^q d\mu(x)\Big)^{1/q} \\ &\lesssim \Big(\int_Q \Big[\int_{\frac{4}{3}Q} |f_1(y_1)|d\mu(y_1) \times \int_{\mathbb{R}^d\setminus\frac{4}{3}Q} \frac{|f_2(y_2)|}{|x-y_2|^{2n}} d\mu(y_2)\Big]^q d\mu(x)\Big)^{1/q} \\ &\lesssim \Big(\int_Q \Big[\int_{\frac{4}{3}Q} |f_1(y_1)|d\mu(y_1) \times \ell(\frac{4}{3}Q)^{-n}\int_{\mathbb{R}^d\setminus\frac{4}{3}Q} \frac{|f_2(y_2)|}{|x-y_2|^{n}} d\mu(y_2)\Big]^q d\mu(x)\Big)^{1/q} \\ &\lesssim \ell(\frac{4}{3}Q)^{-n}\Big(\int_Q \Big[\Big(\int_{\frac{4}{3}Q} |f_1(y_1)|^{q_1} d\mu(y_1)\Big)^{1/q_1} \mu(\frac{4}{3}Q)^{1/q_1'} \\ &\quad \times \int_{\mathbb{R}^d\setminus\frac{4}{3}Q} \frac{|f_2(y_2)|}{|x-y_2|^{n}} d\mu(y_2)\Big]^q d\mu(x)\Big)^{1/q} \\ &\lesssim \ell(\frac{4}{3}Q)^{-n}\Big(\int_Q \Big[\Big(\int_{\frac{4}{3}Q} |f_1(y_1)|^{q_1} d\mu(y_1)\Big)^{1/q_1} \mu(\frac{4}{3}Q)^{1/q_1'} \end{aligned}$$

$$\times \ell\left(\frac{4}{3}Q\right)^{-n/p_2} \|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2,\mu)\|\Bigg]^q d\mu(x)\Bigg)^{1/q}$$

$$\lesssim \ell\left(\frac{4}{3}Q\right)^{-n+n/q_1'-n/p_2} \mu(Q)^{1/q}\left(\int_{\frac{4}{3}Q}|f_1(y_1)|^{q_1}d\mu(y_1)\right)^{1/q_1}\|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2,\mu)\|$$

$$\lesssim \ell(Q)^{n/q_2-n/p_2}\left(\int_{\frac{4}{3}Q}|f_1(y_1)|^{q_1}d\mu(y_1)\right)^{1/q_1}\|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2,\mu)\|.$$

Thus we have

$$\begin{aligned}
\|T_2(f_1,f_2) \mid \mathcal{M}^p_q(k,\mu)\| &\lesssim \sup_{Q\in\Omega(\mu)}\mu(2Q)^{1/p-1/q}\left(\int_Q |T_2(f_1,f_2)(x)|^q d\mu(x)\right)^{1/q}\\
&\lesssim \sup_{Q\in\Omega(\mu)}\mu(2Q)^{1/p-1/q}\ell(Q)^{n/q_2-n/p_2}\|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2,\mu)\|\\
&\quad\times\left(\int_{\frac{4}{3}Q}|f_1(y_1)|^{q_1}d\mu(y_1)\right)^{1/q_1}\\
&\lesssim \mu(2Q)^{1/p_2-1/q_2}\ell(Q)^{n/q_2-n/p_2}\|f_1 \mid \mathcal{M}^{p_1}_{q_1}(2,\mu)\|\,\|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2,\mu)\|\\
&\lesssim \|f_1 \mid \mathcal{M}^{p_1}_{q_1}(2,\mu)\|\,\|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2,\mu)\|. \qquad (2.2)
\end{aligned}$$

On the other hand, via an argument similar to the estimate for $T_2(f_1,f_2)$ gives that

$$\|T_3(f_1,f_2) \mid \mathcal{M}^p_q(k,\mu)\| \lesssim \|f_1 \mid \mathcal{M}^{p_1}_{q_1}(2,\mu)\|\,\|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2,\mu)\|. \qquad (2.3)$$

Finally, by the Minkowski inequality, condition (1.2) and the fact that $|y-x| > \frac{1}{4}\ell\big(\frac{4}{3}Q\big)$ for each $x\in Q, y\in\mathbb{R}^d\backslash\frac{4}{3}Q$, we have

$$\begin{aligned}
&\left(\int_Q |T_4(f_1,f_2)(x)|^q d\mu(x)\right)^{1/q}\\
&\quad\lesssim\left(\int_Q\left[\int_{\mathbb{R}^d\backslash\frac{4}{3}Q}\int_{\mathbb{R}^d\backslash\frac{4}{3}Q}\frac{|f_1(y_1)f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}d\mu(y_2)d\mu(y_1)\right]^q d\mu(x)\right)^{1/q}\\
&\quad\lesssim\left(\int_Q\left[\int_{\mathbb{R}^d\backslash\frac{4}{3}Q}\frac{|f_1(y_1)|}{|x-y_1|^n}d\mu(y_1)\times\int_{\mathbb{R}^d\backslash\frac{4}{3}Q}\frac{|f_2(y_2)|}{|x-y_2|^n}d\mu(y_2)\right]^q d\mu(x)\right)^{1/q}\\
&\quad\lesssim\left(\int_Q\left[\ell\left(\frac{4}{3}Q\right)^{-n/p_1}\|f_1 \mid \mathcal{M}^{p_1}_{q_1}(2,\mu)\|\right.\right.\\
&\qquad\left.\left.\times\,\ell\left(\frac{4}{3}Q\right)^{-n/p_2}\|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2,\mu)\|\right]^q d\mu(x)\right)^{1/q}
\end{aligned}$$

$$\lesssim \ell\left(\frac{4}{3}Q\right)^{-n/p}\mu(Q)^{1/q}\|f_1 \mid \mathcal{M}^{p_1}_{q_1}(2,\mu)\|\,\|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2,\mu)\|$$
$$\lesssim \ell(Q)^{n/q-n/p}\|f_1 \mid \mathcal{M}^{p_1}_{q_1}(2,\mu)\|\,\|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2,\mu)\|.$$

From the estimate above, it is easy to deduce that

$$\|T_4(f_1,f_2) \mid \mathcal{M}^p_q(k,\mu)\| \lesssim \sup_{Q\in\Omega(\mu)} \mu(2Q)^{1/p-1/q}\left(\int_Q |T_4(f_1,f_2)(x)|^q d\mu(x)\right)^{1/q}$$
$$\lesssim \sup_{Q\in\Omega(\mu)} \mu(2Q)^{1/p-1/q}\ell(Q)^{n/q-n/p}\|f_1 \mid \mathcal{M}^{p_1}_{q_1}(2,\mu)\|\,\|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2,\mu)\|$$
$$\lesssim \|f_1 \mid \mathcal{M}^{p_1}_{q_1}(2,\mu)\|\,\|f_2 \mid \mathcal{M}^{p_2}_{q_2}(2,\mu)\|. \tag{2.4}$$

Thus combining these inequalities from (2.1) to (2.4), we finally get the result of Theorem 2.1. □

3. Estimate for commutators

In this section, we will consider commutators of the multilinear Calderón–Zygmund operator T defined as

$$T_{\vec{b}}(\vec{f})(x) = \int \sum_{i=1}^{m}(b_i(x)-b_i(y_i))K(x,\vec{y})f_1(y_1)\dots f_m(y_m)d\mu(\vec{y}), \tag{3.1}$$

where $\vec{b}=(b_1,b_2,\dots,b_m)$ be a vector valued function with each component b_j is a locally integrable function on $\mathbb{R}^d$, $f_1,\dots,f_m$ are $C^\infty(\mu)$ functions with compact support and $x \notin \cap_{j=1}^m \text{supp} f_j$.

When $m=1$, $T(\vec{f})$ and $T_{\vec{b}}$ recapture the singular integral operator T and T_b studied by X. Tolsa in [2].

Now, we recall some notations and definitions. Let α and β_d be positive constants such that $\alpha>1$ and $\beta_d>\alpha^n$. For a cube Q, we say that Q is (α,β_d)-doubling if $\mu(\alpha Q)\le\beta_d\mu(Q)$, where αQ denotes the cube concentric with Q and having side length $\alpha\ell(Q)$. For two cubes $Q\subset R$, set

$$S_{Q,R} = 1+\sum_{k=1}^{N_{Q,R}}\frac{\mu(2^kQ)}{\ell(2^kQ)},$$

where $N_{Q,R}$ is the first positive integer k such that $\ell(2^kQ)\ge\ell(R)$.

Definition 3.1. Let $\rho > 1$ and b be a μ-locally integrable function on $\mathbb{R}^d$. We say b belongs to the space RBMO(μ) if there is a constant $B > 0$ such that

$$\sup_Q \frac{1}{\mu(\rho Q)} \int_Q |b(x) - m_{\widetilde{Q}}(b)| d\mu(x) \leq B < \infty, \tag{3.2}$$

and if $Q \subset R$ are doubling cubes,

$$|m_Q(b) - m_R(b)| \leq BS_{Q,R}, \tag{3.3}$$

where the supremum is taken over all cubes centered at some point of supp(μ), $\widetilde{Q}$ is the smallest (α, β_d)-doubling cube of the form $2^k Q$ with $k \in \mathbb{N} \cup \{0\}$, and $m_{\widetilde{Q}}(b)$ is the mean value of b on $\widetilde{Q}$, namely,

$$m_{\widetilde{Q}}(b) = \frac{1}{\mu(\widetilde{Q})} \int_{\widetilde{Q}} b(x) d\mu(x).$$

The minimal constant B appeared in (3.2) and (3.3) is the RBMO(μ) norm of b and is denoted by $\|b\|_*$.

The space RBMO(μ) was introduced by X. Tolsa [2]. It is showed that the definition of RBMO(μ) does not depend on the choices of numbers ρ, α and β_d provided that $\rho > 1$, $\alpha > 1$ and $\beta_d > \alpha^n$. In the proof of our theorem, we will choose $\rho = \alpha = 2$ and $\beta_d > 2^{d+1}$. Also, it can be seen that one obtains an equivalent definition for the space RBMO(μ) if instead of cubes centered at point $\operatorname{supp} \mu$ by all the cubes in $\mathbb{R}^d$. Furthermore, RBMO(μ) is small enough to fulfil the properties enjoyed by the classical BMO space introduced by John and Nirenberg; see Sections 2–3 of [2] for details.

We will use the following Lemmas for the commutator.

Lemma 3.1 (see e.g. [18]). *Suppose $T_{\vec{b}}$ as in (3.1) and T satisfying conditions (1.2) and (1.3). Let $\vec{b} = (b_1, b_2, \ldots, b_m)$ and $b_i \in$ RBMO(μ), $f_i \in L^{p_i}(\mu)$ for $i = 1, 2, \ldots, m$. If T is bounded from $L^1(\mu) \times \cdots \times L^1(\mu)$ to $L^{1/m,\infty}(\mu)$, then*

$$\|T_{\vec{b}}(\vec{f}) \mid L^p(\mu)\| \lesssim \left(\sum_{i=1}^m \|b_i\|_*\right) \prod_{i=1}^m \|f_i \mid L^{p_i}(\mu)\|,$$

where $p_i > 1$ and $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p}$.

Lemma 3.2. *Suppose that $1 < q \leq p < \infty$, for all $f \in \mathcal{M}_q^p(k,\mu)$, $b \in$ RBMO(μ), $Q \in \Omega(\mu)$ and $x \in Q$, then*

$$\int_{\mathbb{R}^d \setminus 2Q} \frac{|m_{\widetilde{Q}}(b) - b(y)|\,|f(y)|}{|x-y|^n} d\mu(y) \lesssim \ell(Q)^{-n/p} \|b\|_* \|f \mid \mathcal{M}_q^p(k,\mu)\|.$$

Proof. By an elementary calculation and Fubini's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d\setminus 2Q} \frac{|m_{\widetilde{Q}}(b)-b(y)|\,|f(y)|}{|x-y|^n}d\mu(y)\\
&\lesssim \int_{\mathbb{R}^d\setminus B(x,\ell(Q)/2)} \left(\int_{|x-y|}^{\infty} |m_{\widetilde{Q}}(b)-b(y)|\,|f(y)|l^{-n-1}dl\right)d\mu(y)\\
&\lesssim \int_{\mathbb{R}^d\setminus B(x,\ell(Q)/2)} \left(\int_{0}^{\infty} \chi_{B(x,l)}|m_{\widetilde{Q}}(b)-b(y)|\,|f(y)|l^{-n-1}dl\right)d\mu(y)\\
&\lesssim \int_0^\infty \left(l^{-n-1}\int_{B(x,l)\setminus B(x,\ell(Q)/2)} |m_{\widetilde{Q}}(b)-b(y)|\,|f(y)|d\mu(y)\right)dl\\
&\lesssim \int_{\ell(Q)/2}^\infty l^{-n-1}\left(\int_{B(x,l)}|m_{\widetilde{Q}}(b)-b(y)|^{q'}d\mu(y)\right)^{1/q'}\left(\int_{B(x,l)}|f(y)|^q d\mu(y)\right)^{1/q}dl\\
&\lesssim \|f\mid \mathcal{M}_q^p(k,\mu)\|\int_{\ell(Q)/2}^\infty l^{-1-n/p}\left(l^{-n}\int_{B(x,l)}|m_{\widetilde{Q}}(b)-b(y)|^{q'}d\mu(y)\right)^{1/q'}dl.
\end{aligned}$$

Let k be the least integer satisfying $2^kQ\supset B(x,l)$. Then, by the growth condition (1.1), we get

$$\begin{aligned}
&\left(l^{-n}\int_{B(x,l)}|m_{\widetilde{Q}}(b)-b(y)|^{q'}d\mu(y)\right)^{1/q'}\lesssim\left(\frac{1}{\mu(\frac{3}{2}2^kQ)}\int_{2^kQ}|m_{\widetilde{Q}}(b)-b(y)|^{q'}d\mu(y)\right)^{1/q'}\\
&\lesssim\left\{\left(\frac{1}{\mu(\frac{3}{2}2^kQ)}\int_{2^kQ}|m_{\widetilde{2^kQ}}(b)-b(y)|^{q'}d\mu(y)\right)^{1/q'}+|m_{\widetilde{2^kQ}}(b)-m_{\widetilde{Q}}(b)|\right\}\\
&\lesssim S_{\widetilde{Q},\widetilde{2^kQ}}\|b\|_*.
\end{aligned}$$

It follows from John–Nirenberg inequality as

$$\sup_{Q\in\Omega(\mu)}\left(\frac{1}{\mu(\rho Q)}\int_Q|m_{\widetilde{Q}}(b)-b(y)|^r d\mu(y)\right)^{1/r} + \sup_{Q\subset R,\Omega(2,\mu)}\frac{|m_Q(b)-m_R(b)|}{S_{Q,R}}\lesssim\|b\|_*,$$

and the fact proved in [2,Lemma 2.1] that,

$$S_{\widetilde{Q},\widetilde{2^kQ}}\le C(1+k)\lesssim\left(1+\log\frac{l}{\ell(Q)/2}\right).$$

Thus we obtain

$$\int_{\mathbb{R}^d\setminus 2Q}\frac{|m_{\widetilde{Q}}(b)-b(y)f(y)|}{|x-y|^n}d\mu(y)$$

$$\lesssim \|b\|_* \|f \mid \mathcal{M}^p_q(k,\mu)\| \int_{\ell(Q)/2}^{\infty} l^{-n/p-1} \left(1 + \log \frac{l}{\ell(Q)/2}\right) dl$$
$$\lesssim \ell(Q)^{-n/p} \|b\|_* \|f \mid \mathcal{M}^p_q(k,\mu)\|.$$

This is precisely the assertion of Lemma 3.2. □

Theorem 3.1. *Suppose $T_{\vec{b}}$ as in* (3.1) *and T satisfying conditions* (1.2) *and* (1.3). *Let $\vec{b} = (b_1, b_2, \ldots, b_m)$ and $b_i \in \mathrm{RBMO}(\mu)$, $f_i \in L^{p_i}(\mu)$ for $i = 1, 2, \ldots, m$. If T is bounded from $L^1(\mu) \times \cdots \times L^1(\mu)$ to $L^{1/m,\infty}(\mu)$, then*

$$\|T_{\vec{b}}(\vec{f}) \mid \mathcal{M}^p_q(k,\mu)\| \lesssim \left(\sum_{i=1}^m \|b_i\|_*\right) \|f_1 \mid \mathcal{M}^{p_1}_{q_1}(k,\mu)\| \times \cdots \times \|f_m \mid \mathcal{M}^{p_m}_{q_m}(k,\mu)\|,$$

where $1 < q_i \le p_i$ and $\sum_{i=1}^m 1/p_i = 1/p$, $\sum_{i=1}^m 1/q_i = 1/q$.

Proof. For simplicity and without loss of generality, we restrict $m = 2$. Firstly, we rewrite

$$|T_{\vec{b}}(f_1, f_2)(x)| \le \left|T_{\vec{b}}\big(f_1\chi_{\frac{4}{3}Q}, f_2\chi_{\frac{4}{3}Q}\big)(x)\right| + \left|T_{\vec{b}}\big(f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\frac{4}{3}Q}\big)(x)\right|$$
$$+ \left|T_{\vec{b}}\big(f_1\chi_{\frac{4}{3}Q}, f_2\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\big)(x)\right| + \left|T_{\vec{b}}\big(f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\big)(x)\right| = I + II + III + IV.$$

To estimate I, by Lemma 2.2, we conclude that

$$\|I \mid \mathcal{M}^p_q(k,\mu)\| \lesssim \sup_{Q\in\Omega(\mu)} \mu(2Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \left|T_{\vec{b}}\big(f_1\chi_{\frac{4}{3}Q}, f_2\chi_{\frac{4}{3}Q}\big)\right|^q d\mu\right)^{\frac{1}{q}}$$
$$\lesssim \sup_{Q\in\Omega(\mu)} \mu(2Q)^{\frac{1}{p}-\frac{1}{q}} \left\|T_{\vec{b}}\big(f_1\chi_{\frac{4}{3}Q}, f_2\chi_{\frac{4}{3}Q}\big)\right\|_q$$
$$\lesssim \|\vec{b}\|_* \sup_{Q\in\Omega(\mu)} \mu(2Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\frac{4}{3}Q} |f_1(x)|^{q_1} d\mu(x)\right)^{1/q_1} \left(\int_{\frac{4}{3}Q} |f_2(x)|^{q_2} d\mu(x)\right)^{1/q_2}$$
$$\lesssim \|\vec{b}\|_* \|f_1 \mid \mathcal{M}^{p_1}_{q_1}(k,\mu)\| \, \|f_2 | \mathcal{M}^{p_2}_{q_2}(k,\mu)\|. \tag{3.4}$$

Then we estimate term II, by expanding $b(x) - b(y) = b(x) - m_{\widetilde{Q}}(b) + m_{\widetilde{Q}}(b) - b(y)$, we yield

$$\left|T_{\vec{b}}\big(f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\frac{4}{3}Q}\big)(x)\right| \le \left|(b_1(x) - m_{\widetilde{Q}}(b_1))T\big(f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\frac{4}{3}Q}\big)(x)\right|$$
$$+ \left|(b_2(x) - m_{\widetilde{Q}}(b_2))T\big(f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\frac{4}{3}Q}\big)(x)\right|$$
$$+ \left|T\big((m_{\widetilde{Q}}(b_1) - b_1(y_1))f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\frac{4}{3}Q}\big)(x)\right|$$
$$+ \left|T\big(f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, (m_{\widetilde{Q}}(b_2) - b_2(y_2))f_2\chi_{\frac{4}{3}Q}\big)(x)\right| = II_1 + II_2 + II_3 + II_4.$$

By Lemma 2.1, Hölder inequality and condition (1.1), we have

$$
\begin{aligned}
II_1 &\le |(b_1(x)-m_{\widetilde{Q}}(b_1))|\,\big|T\big(f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\frac{4}{3}Q}\big)(x)\big| \\
&\lesssim |(b_1(x)-m_{\widetilde{Q}}(b_1))| \int_{\mathbb{R}^d\setminus\frac{4}{3}Q}\int_{\frac{4}{3}Q} \frac{|f_1(y_1)f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}d\mu(y_2)d\mu(y_1) \\
&\lesssim |(b_1(x)-m_{\widetilde{Q}}(b_1))| \int_{\frac{4}{3}Q}|f_2(y_2)|d\mu(y_2)\times \ell\left(\frac{4}{3}Q\right)^{-n}\int_{\mathbb{R}^d\setminus\frac{4}{3}Q}\frac{|f_1(y_1)|}{|x-y_1|^n}d\mu(y_1) \\
&\lesssim \ell(Q)^{-n/p}|(b_1(x)-m_{\widetilde{Q}}(b_1))|\,\|f_1\mid\mathcal{M}^{p_1}_{q_1}(k,\mu)\|\,\|f_2\mid\mathcal{M}^{p_2}_{q_2}(k,\mu)\|.
\end{aligned}
$$

By (3.2), it is easily to check that

$$
\begin{aligned}
\|II_1\mid\mathcal{M}^p_q(k,\mu)\| &\lesssim \sup_{Q\in\Omega(\mu)}\mu(2Q)^{-\frac{1}{q}}\left(\int_Q|(b_1(x)-m_{\widetilde{Q}}(b_1))|^q d\mu\right)^{\frac{1}{q}} \\
&\quad\times\|f_1\mid\mathcal{M}^{p_1}_{q_1}(k,\mu)\|\,\|f_2\mid\mathcal{M}^{p_2}_{q_2}(k,\mu)\| \\
&\lesssim \|b_1\|_*\|f_1\mid\mathcal{M}^{p_1}_{q_1}(k,\mu)\|\,\|f_2\mid\mathcal{M}^{p_2}_{q_2}(k,\mu)\|. \qquad (3.5)
\end{aligned}
$$

The same proof remains valid for II_2,

$$
\|II_2\mid\mathcal{M}^p_q(k,\mu)\|\lesssim\|b_2\|_*\|f_1\mid\mathcal{M}^{p_1}_{q_1}(k,\mu)\|\,\|f_2\mid\mathcal{M}^{p_2}_{q_2}(k,\mu)\|. \qquad (3.6)
$$

For II_3, by Lemma 3.2 and the estimate for II_1, we have

$$
\begin{aligned}
II_3 &\lesssim \big|T\big((b_1-m_{\widetilde{Q}}(b_1))f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\frac{4}{3}Q}\big)(x)\big| \\
&\lesssim \int_{\frac{4}{3}Q}|f_2(y_2)|d\mu(y_2)\times\ell\left(\frac{4}{3}Q\right)^{-n}\int_{\mathbb{R}^d\setminus\frac{4}{3}Q}\frac{|(b_1(y_1)-m_{\widetilde{Q}}(b_1))f_1(y_1)|}{|x-y_1|^n}d\mu(y_1) \\
&\lesssim \ell(Q)^{-n/p}\|b_1\|_*\|f_1\mid\mathcal{M}^{p_1}_{q_1}(k,\mu)\|\,\|f_2\mid\mathcal{M}^{p_2}_{q_2}(k,\mu)\|.
\end{aligned}
$$

Then, it is clear that

$$
\|II_3\mid\mathcal{M}^p_q(k,\mu)\|\lesssim\|b_1\|_*\|f_1\mid\mathcal{M}^{p_1}_{q_1}(k,\mu)\|\,\|f_2\mid\mathcal{M}^{p_2}_{q_2}(k,\mu)\|. \qquad (3.7)
$$

The estimate above carries more, we get

$$
\|II_4\mid\mathcal{M}^p_q(k,\mu)\|\lesssim\|b_2\|_*\|f_1\mid\mathcal{M}^{p_1}_{q_1}(k,\mu)\|\,\|f_2\mid\mathcal{M}^{p_2}_{q_2}(k,\mu)\|. \qquad (3.8)
$$

Combining from with, we deduce that

$$
\|II\mid\mathcal{M}^p_q(k,\mu)\|\lesssim(\|b_1\|_*+\|b_2\|_*)\|f_1\mid\mathcal{M}^{p_1}_{q_1}(k,\mu)\|\,\|f_2\mid\mathcal{M}^{p_2}_{q_2}(k,\mu)\|. \qquad (3.9)
$$

By an argument similar to the above, we can obtain

$$\|III \mid \mathcal{M}_q^p(k,\mu)\| \lesssim (\|b_1\|_* + \|b_2\|_*)\|f_1 \mid \mathcal{M}_{q_1}^{p_1}(k,\mu)\| \, \|f_2|\mathcal{M}_{q_2}^{p_2}(k,\mu)\|. \tag{3.10}$$

Finally, we estimate IV,

$$\begin{aligned} \big|T_{\vec{b}}\big(f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\big)(x)\big| &\le \big|(b_1(x) - m_{\widetilde{Q}}(b_1))T\big(f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\big)(x)\big| \\ &+ \big|(b_2(x) - m_{\widetilde{Q}}(b_2))T\big(f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\big)(x)\big| \\ &+ \big|T\big((m_{\widetilde{Q}}(b_1) - b_1)f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, f_2\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\big)(x)\big| \\ &+ \big|T\big(f_1\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}, (m_{\widetilde{Q}}(b_2) - b_2)f_2\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\big)(x)\big| = IV_1 + IV_2 + IV_3 + IV_4. \end{aligned}$$

By Lemma 2.1, Lemma 3.2 and condition (1.2), we deduce that

$$\|IV_1 \mid \mathcal{M}_q^p(k,\mu)\| \lesssim \|b_1\|_*\|f_1 \mid \mathcal{M}_{q_1}^{p_1}(k,\mu)\| \, \|f_2 \mid \mathcal{M}_{q_2}^{p_2}(k,\mu)\|. \tag{3.11}$$

$$\|IV_2 \mid \mathcal{M}_q^p(k,\mu)\| \lesssim \|b_2\|_*\|f_1 \mid \mathcal{M}_{q_1}^{p_1}(k,\mu)\| \, \|f_2 \mid \mathcal{M}_{q_2}^{p_2}(k,\mu)\|. \tag{3.12}$$

$$\|IV_3 \mid \mathcal{M}_q^p(k,\mu)\| \lesssim \|b_1\|_*\|f_1 \mid \mathcal{M}_{q_1}^{p_1}(k,\mu)\| \, \|f_2 \mid \mathcal{M}_{q_2}^{p_2}(k,\mu)\|. \tag{3.13}$$

$$\|IV_4 \mid \mathcal{M}_q^p(k,\mu)\| \lesssim \|b_2\|_*\|f_1 \mid \mathcal{M}_{q_1}^{p_1}(k,\mu)\| \, \|f_2 \mid \mathcal{M}_{q_2}^{p_2}(k,\mu)\|. \tag{3.14}$$

Thus we get

$$\|IV \mid \mathcal{M}_q^p(k,\mu)\| \lesssim \|\vec{b}\|_*\|f_1 \mid \mathcal{M}_{q_1}^{p_1}(k,\mu)\| \, \|f_2 \mid \mathcal{M}_{q_2}^{p_2}(k,\mu)\|. \tag{3.15}$$

Combining with (3.4), (3.9), (3.10) and (3.15), which completed the proof of Theorem 3.1. □

4. Appendix

In this section, we will consider another type commutator defined as

$$T_{\vec{b}}^{\star}(\vec{f})(x) = \int \prod_{i=1}^{m} (b_i(x) - b_i(y_i))K(x,\vec{y})f_1(y_1)\dots f_m(y_m)d\mu(\vec{y}). \tag{4.1}$$

For $T_{\vec{b}}^{\star}$, Xu proved the $L^p(\mu)$-boundedness as follows.

Lemma 4.1 (see e.g. [17])**.** *Suppose $T_{\vec{b}}$ as in (4.1) and T satisfying conditions (1.2) and (1.3). Let $\vec{b} = (b_1, b_2, \dots, b_m)$ and $b_i \in \mathrm{RBMO}(\mu)$, $f_i \in L^{p_i}(\mu)$ for $i = 1, 2, \dots, m$. If T is bounded from $L^1(\mu) \times \cdots \times L^1(\mu)$ to $L^{1/m,\infty}(\mu)$, then*

$$\|T_{\vec{b}}^{\star}(\vec{f}) \mid L^p(\mu)\| \lesssim \prod_{i=1}^{m} \|b_i\|_*\|f_i||L^{p_i}(\mu)\|,$$

where $1 < p_i$ and $\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{p}$.

The following result is true for the commutator defined by (4.1).

Theorem 4.1. *Suppose* $T_{\vec{b}}$ *as in* (4.1) *and* T *satisfying conditions* (1.2) *and* (1.3). *Let* $\vec{b} = (b_1, b_2, \dots, b_m)$ *and* $b_i \in \mathrm{RBMO}(\mu)$, $f_i \in L^{p_i}(\mu)$ *for* $i = 1, 2, \dots, m$. *If* T *is bounded from* $L^1(\mu) \times \cdots \times L^1(\mu)$ *to* $L^{1/m,\infty}(\mu)$, *then*

$$\|T_{\vec{b}}^{\star}(\vec{f}) \mid \mathcal{M}_q^p(k,\mu)\| \lesssim \prod_{i=1}^{m} \|b_i\|_* \|f_i \mid \mathcal{M}_{q_i}^{p_i}(k,\mu)\|,$$

where $1 < q_i \le p_i$ *and* $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p}$, $\sum_{i=1}^m \frac{1}{q_i} = \frac{1}{q}$.

The proof of Theorem 4.1 is just linguistic iterations with a slight modification of the proof of Theorem 3.1. We leave the details to the reader.

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