Publ. Math. Debrecen
78/2 (2011), 297–316
DOI: 10.5486/PMD.2011.4631

Sopdes and nonlinear connections

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Abstract. The canonical k-tangent structure on $T^1_kQ = TQ\oplus \overset{k}{\ldots} \oplus TQ$ allows us to characterize nonlinear connections on T^1_kQ and to develop Günther's (k-symplectic) Lagrangian formalism. We study the relationship between nonlinear connections and second-order partial differential equations (sopdes), which appear in Günther's Lagrangian formalism.

1. Introduction

Lagrangian mechanics have been entirely geometrized in terms of symplectic geometry. In this approach there exists certain dynamical vector field on the tangent bundle of a manifold whose integral curves are the solutions of the Euler-Lagrange equations. This vector field is usually called *second-order differential equation* (sode to short) or *spray* (sometimes it is called *semispray* and the term spray is reserved to homogeneous second-order differential equations, see for instance, [1], [8]). Let us remember that a sode on TQ is a vector field on TQ such that $JS = C$, where J is the *almost tangent structure* or *vertical endomorphism* and C is the *canonical field* or *Liouville field*.

In [1], [2], [3], Grifone studies the relationship among sodes, nonlinear connections and the autonomous Lagrangian formalism. This study was extended to the non-autonomous case by M. de León and P. Rodrigues [8].

The natural generalization to Classical Field Theory of the concept of sode is called *second order partial differential equation* (sopde to short). This concept

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Mathematics Subject Classification: 53C05, 70S05.
Key words and phrases: sopde, nonlinear connection.

was introduced by GÜNTHER in [5] in order to develop his Lagrangian polysymplectic (k-symplectic) formalism. The "phase space" of this formalism is the tangent bundle of k^1-velocities T^1_kQ, that is, the Whitney sum of k-copies of the bundle TQ,

$$T^1_kQ := TQ \oplus \overset{k}{\ldots} \oplus TQ.$$

In this paper we study the relationship between nonlinear connections and arbitrary SOPDEs on T^1_kQ.

The structure of the paper is the following:

In Section 2 we describe briefly the bundle of k^1-velocities T^1_kQ of a manifold Q (see [9], [10]). After, following to GRIFONE [1], [2], [3] and SZILASI [14] we define a canonical short exact sequence

$$0 \longrightarrow T^1_kQ \times_Q T^1_kQ \xrightarrow{\mathbf{i}} T(T^1_kQ) \xrightarrow{\mathbf{j}} T^1_kQ \times_Q TQ \longrightarrow 0$$

which allows us to introduce in an alternative way the canonical geometric elements on T^1_kQ: *the Liouville vector field* and *the canonical k-tangent structure*, (see also [9]). The usual definition of these geometric elements can be found in [11], [12], [13].

In Section 3 we give two characterizations of the nonlinear connections on $\tau^k_Q : T^1_kQ \to Q$. In the first one we use the canonical short exact sequence constructed in Section 2 in an analogous way to that one in SZILASI's Handbook study [14] for the case $k = 1$; in this first characterization our theory is similar to the theory developed in [9]. In the second one we characterize nonlinear connections on $\tau^k_Q : T^1_kQ \to Q$ using the canonical k-tangent structure $(J^1, \ldots, J^k)$. In the particular case $k = 1$ we reobtain some results given by GRIFONE in [1], [2], [3].

Finally in Section 4 we recall the notion of SOPDEs (second order partial differential equations) and we study the relationship between SOPDEs and nonlinear connections on T^1_kQ.

Along the paper we have used the SZILASI's Handbook study [14] and GRIFONE's papers [1], [3] as principal reference.

All manifolds are real, paracompact, connected and C^∞. All maps are C^∞. Sum over crossed repeated indices is understood.

2. The canonical short exact sequence

In this section we describe briefly the bundle of k^1-velocities T^1_kQ of a manifold Q (see [9], [10]), that is, the Whitney sum of k-copies of the tangent bundle

TQ, which is the phase space where the k-symplectic Lagrangian formalism of classical field theories (GÜNTHER's formalism [5]) is developed. After, following GRIFONE [1], [2], [3] and SZILASI [14] we define a canonical short exact sequence which allows us to introduce the canonical geometric elements on T^1_kQ, which are necessary to develop the k-symplectic Lagrangian formalism: *the Liouville vector field and the canonical k-tangent structure.*

Moreover, the canonical short exact sequence introduced in this section will be used, in the following section, to characterize nonlinear connections on T^1_kQ.

The tangent bundle of k^1-velocities of a manifold

Let Q be a n-dimensional differential manifold, and let $\tau_Q : TQ \to Q$ be the tangent bundle of Q. Denote by T^1_kQ the Whitney sum $TQ\oplus \overset{k}{\dots} \oplus TQ$ of k copies of TQ, with projection $\tau^k_Q : T^1_kQ \to Q$, $\tau^k_Q(v_{1q}, \dots, v_{kq}) = q$. The fibre over $q \in Q$ is the nk-dimensional vector space $(T^1_kQ)_q = T_qQ\oplus \overset{k}{\dots} \oplus T_qQ$. Along this paper an element of T^1_kQ will be denoted by $\mathbf{v}_q = (v_{1q}, \dots, v_{kq})$.

The manifold $J^1_{\mathbf{0}}(\mathbb{R}^k, Q)$ of 1-jets of maps with source at $\mathbf{0} \in \mathbb{R}^k$ and projection map $\tau^k_Q : J^1_{\mathbf{0}}(\mathbb{R}^k, Q) \to Q$, $\tau^k_Q(j^1_{\mathbf{0},q}\sigma) = \sigma(\mathbf{0}) = q$, can be identified with T^1_kQ as follows:

$$J^1_{\mathbf{0}}(\mathbb{R}^k, Q) \equiv TQ\oplus \overset{k}{\dots} \oplus TQ, \qquad j^1_{\mathbf{0},q}\sigma \equiv (v_{1q}, \dots, v_{kq}),$$

where $q = \sigma(\mathbf{0})$, and $v_{Aq} = \sigma_*(\mathbf{0})\big(\frac{\partial}{\partial t^A}(\mathbf{0})\big)$. T^1_kQ is called the *bundle of k^1-velocities* of Q, see [10].

If (q^i) are local coordinates on $U \subset Q$, then the induced local coordinates (q^i, v^i_A) on $T^1_kU = (\tau^k_Q)^{-1}(U)$ are given by

$$q^i(\mathbf{v}_q) = q^i(v_{1q}, \dots, v_{kq}) = q^i(q), \qquad v^i_A(\mathbf{v}_q) = v^i_A(v_{1q}, \dots, v_{kq}) = v_{Aq}(q^i).$$

The vector bundle $(T^1_kQ \times_Q T^1_kQ, (\tau^k_Q)^*\tau^k_Q, T^1_kQ)$

Let us consider the fibre bundle $\tau^k_Q : T^1_kQ \to Q$ and the pull-back bundle of τ^k_Q by τ^k_Q, that is,

$$(T^1_kQ \times_Q T^1_kQ, (\tau^k_Q)^*\tau^k_Q, T^1_kQ),$$

where the total space is the fibre product

$$T^1_kQ \times_Q T^1_kQ = \{(\mathbf{v}_q, \mathbf{w}_q) \in T^1_kQ \times T^1_kQ \mid \tau^k_Q(\mathbf{v}_q) = \tau^k_Q(\mathbf{w}_q)\},$$

and $(\tau^k_Q)^*\tau^k_Q : T^1_kQ \times_Q T^1_kQ \to T^1_kQ$ is the canonical projection on the first factor, that is, $(\tau^k_Q)^*\tau^k_Q(\mathbf{v}_q, \mathbf{w}_q) = \mathbf{v}_q$.

The map $\mathbf{i} : T^1_kQ \times_Q T^1_kQ \longrightarrow T(T^1_kQ)$

Let $V(T^1_kQ) = \left\langle \frac{\partial}{\partial v^i_A} \right\rangle_{1\leq i\leq n,\, 1\leq A\leq k}$ be denote the vertical subbundle of $\tau^k_Q : T^1_kQ \to Q$ and define the map

$$\mathbf{i} : T^1_kQ \times_Q T^1_kQ \longrightarrow V(T^1_kQ) \subset T(T^1_kQ)$$

by

$$\mathbf{i}(\mathbf{v}_q, \mathbf{w}_q) = \sum_{A=1}^{k} \frac{d}{ds}\Big|_0 (v_{1q}, \dots, v_{Aq} + s w_{Aq}, \dots, v_{kq}).$$

In coordinates, this map is given by

$$\mathbf{i}(\mathbf{v}_q, \mathbf{w}_q) = \sum_{A=1}^{k} w^i_A \frac{\partial}{\partial v^i_A}\Big|_{\mathbf{v}_q}. \tag{1}$$

Canonical vector fields on T^1_kQ

The canonical vector field (or Liouville vector field) $\Delta \in \mathfrak{X}(T^1_kQ)$ is defined by $\Delta(\mathbf{v}_q) = \mathbf{i}(\mathbf{v}_q, \mathbf{v}_q)$. This vector field is used (among others) to introduce the energy Lagrangian function in the k-symplectic Lagrangian formalism, see Section 4.2.

From (1) we obtain that its coordinate expression is

$$\Delta = \sum_{A=1}^{k} \sum_{i=1}^{n} v^i_A \frac{\partial}{\partial v^i_A}. \tag{2}$$

The canonical vector fields $\Delta_A \in \mathfrak{X}(T^1_kQ)$ are defined by

$$\Delta_A(\mathbf{v}_q) = \mathbf{i}(\mathbf{v}_q, (0, \dots, \overset{A}{v_{Aq}}, \dots, 0))$$

for all $A \in \{1, \dots, k\}$, and they are given in coordinates by

$$\Delta_A = \sum_{i=1}^{n} v^i_A \frac{\partial}{\partial v^i_A}. \tag{3}$$

The vector bundle $(T^1_kQ \times_Q TQ, (\tau^k_Q)^*\tau_Q, T^1_kQ)$

Let us consider now the fibre bundle $(\tau^k_Q)^*\tau_Q$, which is the pull-back of the tangent bundle TQ by τ^k_Q. This fibre is also called the *transverse fibre to* τ^k_Q. Its total bundle space is

$$T^1_kQ \times_Q TQ = \{(\mathbf{v}_q, u_q) \in T^1_kQ \times TQ \mid \tau^k_Q(\mathbf{v}_q) = \tau_Q(u_q)\},$$

and $(\tau^k_Q)^*\tau_Q : T^1_kQ \times_Q TQ \to T^1_kQ$ is the canonical projection

$$(\tau^k_Q)^*\tau_Q(\mathbf{v}_q, u_q) = \mathbf{v}_q.$$

The map $\mathbf{j} : T(T^1_kQ) \longrightarrow T^1_kQ \times_Q TQ$

Let $\tau_{T^1_kQ} : T(T^1_kQ) \to T^1_kQ$ be the tangent bundle of T^1_kQ and $T\tau^k_Q : T(T^1_kQ) \to TQ$ the tangent map of τ^k_Q. We define the map

$$\begin{aligned}\mathbf{j} := (\tau_{T^1_kQ}, T\tau^k_Q) : T(T^1_kQ) &\to T^1_kQ \times_Q TQ,\\ Z_{\mathbf{v}_q} &\to (\mathbf{v}_q, T_{\mathbf{v}_q}\tau^k_Q(Z_{\mathbf{v}_q})).\end{aligned}$$

In coordinates,

$$\mathbf{j}(Z_{\mathbf{v}_q}) = \mathbf{j}\left(Z^i \frac{\partial}{\partial q^i}\Big|_{\mathbf{v}_q} + Z^i_A \frac{\partial}{\partial v^i_A}\Big|_{\mathbf{v}_q}\right) = \left(\mathbf{v}_q, Z^i \frac{\partial}{\partial q^i}\Big|_q\right). \tag{4}$$

The map $\mathbf{j}$ is a surjective bundle homomorphism and the induced maps $\mathbf{j}_{\mathbf{v}_q} : T_{\mathbf{v}_q}(T^1_kQ) \to \{\mathbf{v}_q\} \times T_qQ$ are linear, for all $\mathbf{v}_q \in T^1_kQ$.

In SZILASI's Handbook study [14], page 1239, one can be found the definition of $\mathbf{j}$ for an arbitrary vector bundle (E, π, M). In our case $E = T^1_kQ$, $M = Q$ and $\pi = \tau^k_Q$.

The short exact sequence arising from τ^k_Q

Lemma 2.1. *The sequence*

$$\begin{tikzcd} 0 \arrow[r] & T^1_kQ \times_Q T^1_kQ \arrow[r, "\mathbf{i}"] \arrow[dr] & T(T^1_kQ) \arrow[r, "\mathbf{j}"] \arrow[d, no head] & T^1_kQ \times_Q TQ \arrow[r] \arrow[dl] & 0 \\ & & T^1_kQ & & \end{tikzcd} \tag{5}$$

is a short exact sequence of vector bundle maps, that we will be called the canonical short exact sequence *arising from τ^k_Q.*

PROOF. This result can be proved for a general vector bundle (E, π, M), see [14]. In any case the principal point of the proof is that $\mathbf{j} \circ \mathbf{i} = 0$, which is an immediate consequence of (1) and (4). $\square$

Canonical k-tangent structure on T^1_kQ

The canonical k-tangent structure is a certain family of k tensor fields of type $(1,1)$. This structure was introduced in [6], [9]. Next we will describe an alternative definition of this structure.

We introduce the maps $\mathbf{k_A}$ from $T^1_kQ \times_Q TQ$ to $T^1_kQ \times_Q T^1_kQ$ as follows

$$\begin{array}{rcl} \mathbf{k_A} : T^1_kQ \times_Q TQ & \longrightarrow & T^1_kQ \times_Q T^1_kQ \\ (\mathbf{v}_q, u_q) & \to & (\mathbf{v}_q, (0,\dots,0,\overset{A}{u_q},0,\dots,0)) \end{array} \qquad 1 \leq A \leq k.$$

The composition $J^A = \mathbf{i} \circ \mathbf{k_A} \circ \mathbf{j}$ is a tensor field on T^1_kQ of type $(1,1)$ displayed by the following diagram:

$$\begin{tikzcd} T(T^1_kQ) \arrow[r, "\mathbf{j}"] \arrow[rrr, bend left, "J^A"] & T^1_kQ \times_Q TQ \arrow[r, "\mathbf{k_A}"] & T^1_kQ \times_Q T^1_kQ \arrow[r, "\mathbf{i}"] & T(T^1_kQ). \end{tikzcd}$$

In coordinates,

$$Z^i \frac{\partial}{\partial q^i}\Big|_{\mathbf{v}_q} + Z^i_A \frac{\partial}{\partial v^i_A}\Big|_{\mathbf{v}_q} \longrightarrow \Big(\mathbf{v}_q, Z^i \frac{\partial}{\partial q^i}\Big|_q\Big) \longrightarrow \Big(\mathbf{v}_q, (0,\dots, \overset{A}{Z^i \frac{\partial}{\partial q^i}\Big|_q}, \dots, 0)\Big) \longrightarrow Z^i \frac{\partial}{\partial v^i_A}\Big|_{\mathbf{v}_q},$$

or equivalently

$$J^A = \frac{\partial}{\partial v^i_A} \otimes dq^i. \tag{6}$$

The set $(J^1,\dots,J^k)$ is called the *canonical k-tangent structure* on T^1_kQ, see [6], [11], [13]. Along this paper we will use this structure to characterize nonlinear connections on $\tau^k_Q : T^1_kQ \to Q$.

3. Nonlinear connections on $\tau^k_Q : T^1_kQ \to Q$

Let us remember that an *Ehresmann connection or nonlinear connection* on $\tau^k_Q : T^1_kQ \to Q$ is a differentiable subbundle $H(T^1_kQ)$ of $T(T^1_kQ)$, called the *horizontal subbundle* of the connection, which is complementary to the vertical subbundle $V(T^1_kQ)$, that is, $T(T^1_kQ) = H(T^1_kQ) \oplus V(T^1_kQ)$.

In this section we give two characterizations of the nonlinear connections on $\tau^k_Q : T^1_kQ \to Q$. In the first one we use the canonical short exact sequence constructed in the above section in an analogous way to that one in Szilasi's Handbook study [14] for the case $k=1$. This first characterization also appears in [9]. After we characterize nonlinear connections on $\tau^k_Q : T^1_kQ \to Q$ using the k-tangent structure $(J^1,\dots,J^k)$. In the particular case $k=1$ this second result was obtained by Grifone [1], [2], [3].

3.1. The horizontal maps.

Definition 3.1. A right splitting of the short exact sequence

$$0 \longrightarrow T^1_kQ \times_Q T^1_kQ \xrightarrow{\ \mathbf{i}\ } T(T^1_kQ) \xrightarrow{\ \mathbf{j}\ } T^1_kQ \times_Q TQ \longrightarrow 0\ ,$$

is called a *horizontal map for* τ^k_Q. This map is a T^1_kQ-morphism $\mathcal{H} : T^1_kQ \times_Q TQ \longrightarrow T(T^1_kQ)$ of vector bundles (i.e.q the morphism over the base is $id_{T^1_kQ}$) satisfying

$$\mathbf{j} \circ \mathcal{H} = 1_{T^1_kQ\times_Q TQ}.$$

Next it will be shown that to give a horizontal map for τ^k_Q is equivalent to give a nonlinear connection on $\tau^k_Q : T^1_kQ \to Q$.

Proposition 1. *The horizontal map* $\mathcal{H} : T^1_kQ \times_Q TQ \longrightarrow T(T^1_kQ)$ *is locally given by*

$$\mathcal{H}(\mathbf{v}_q, u_q) = u^i\left(\frac{\partial}{\partial q^i}\Big|_{\mathbf{v}_q} - N^j_{Ai}(\mathbf{v}_q)\frac{\partial}{\partial v^j_A}\Big|_{\mathbf{v}_q}\right) \tag{7}$$

where $\mathbf{v}_q \in T^1_kQ$, $u_q \in TQ$ *and the functions* N^j_{Ai} *are called the components of the connection defined by* $\mathcal{H}$.

PROOF. We write

$$\mathcal{H}(\mathbf{v}_q, u_q) = H^i(\mathbf{v}_q, u_q)\frac{\partial}{\partial q^i}\Big|_{\mathbf{v}_q} - N^i_A(\mathbf{v}_q, u_q)\frac{\partial}{\partial v^i_A}\Big|_{\mathbf{v}_q}$$

for some functions on H^i, N^i_A defined only locally on $T^1_kQ \times_Q TQ$.

Since $\mathbf{j} \circ \mathcal{H} = 1_{T^1_kQ\times_Q TQ}$, from (4), we obtain

$$\mathcal{H}(\mathbf{v}_q, u_q) = u^i\frac{\partial}{\partial q^i}\Big|_{\mathbf{v}_q} - N^i_A(\mathbf{v}_q, u_q)\frac{\partial}{\partial v^i_A}\Big|_{\mathbf{v}_q}. \tag{8}$$

On the other hand, the induced maps

$$\mathcal{H}_{\mathbf{v}_q} : (T^1_kQ \times_Q TQ)_{\mathbf{v}_q} \cong \{\mathbf{v}_q\} \times T_qQ \to T_{\mathbf{v}_q}(T^1_kQ)$$

are linear for all $\mathbf{v}_q \in T^1_kQ$, so from (8) we obtain that

$$\begin{aligned}\mathcal{H}(\mathbf{v}_q, u_q) &= \mathcal{H}\left(\mathbf{v}_q, u^i\frac{\partial}{\partial q^i}\Big|_q\right) = u^i\mathcal{H}\left(\mathbf{v}_q, \frac{\partial}{\partial q^i}\Big|_{\mathbf{v}_q}\right)\\ &= u^i\left(\frac{\partial}{\partial q^i}\Big|_{\mathbf{v}_q} - N^j_A(\mathbf{v}_q, \frac{\partial}{\partial q^i}\Big|_q)\frac{\partial}{\partial v^j_A}\Big|_{\mathbf{v}_q}\right).\end{aligned} \tag{9}$$

Now defining the functions N^j_{Ai} on the domain of an induced chart of T^1_kQ by

$$N^j_{Ai}(\mathbf{v}_q) = N^j_A\left(\mathbf{v}_q, \frac{\partial}{\partial q^i}\Big|_q\right), \quad 1 \leq i,\ j \leq n,\ 1 \leq A \leq k$$

we obtain (7). □

To each horizontal map $\mathcal{H} : T^1_kQ \times_Q TQ \to T(T^1_kQ)$ we associate a horizontal and a vertical projector as follows:

(1) *The horizontal projector* is given by $\mathbf{h} := \mathcal{H} \circ \mathbf{j} : T(T^1_kQ) \to T(T^1_kQ)$.

From (4) we deduce that the local expression of $\mathbf{h}$ is

$$\mathbf{h} = \left(\frac{\partial}{\partial q^i} - N^j_{Ai}\frac{\partial}{\partial v^j_A}\right) \otimes dq^i, \tag{10}$$

and we have $\mathbf{h}^2 = \mathbf{h}, \operatorname{Ker}\mathbf{h} = V(T^1_kQ)$ and

$$\operatorname{Im}\mathbf{h} = \left\langle \frac{\partial}{\partial q^i} - N^j_{Ai}\frac{\partial}{\partial v^j_A}\right\rangle_{i=1,\dots,n}.$$

(2) *The vertical projector* is given by $\mathbf{v} := 1_{T(T^1_kQ)} - \mathbf{h}$ and it satisfies

$$\mathbf{v}^2 = \mathbf{v}, \qquad \operatorname{Ker}\mathbf{v} = \operatorname{Im}\mathbf{h}, \qquad \operatorname{Im}\mathbf{v} = V(T^1_kQ).$$

From (10) we obtain

$$\mathbf{v} = \frac{\partial}{\partial v^j_A} \otimes (dv^j_A + N^j_{Ai}dq^i). \tag{11}$$

Since $\mathbf{v} := 1_{T(T^1_kQ)} - \mathbf{h}$ and $\mathbf{h}^2 = \mathbf{h}$ we obtain that $\mathbf{vh} = \mathbf{hv} = 0$.

The following Lemma is well known.

Lemma 3.1. *Let M be an arbitrary manifold and Γ an almost product structure, i.e., Γ is a tensor field of type $(1,1)$ such that $\Gamma^2 = 1_{TM}$. If we put*

$$\mathbf{h} = \frac{1}{2}(1_{TM} + \Gamma), \qquad \mathbf{v} = \frac{1}{2}(1_{TM} - \Gamma)$$

then

$$\mathbf{h}^2 = \mathbf{h} \quad \mathbf{hv} = \mathbf{vh} = 0 \quad \mathbf{v}^2 = \mathbf{v}. \tag{12}$$

Conversely if $\mathbf{h}$ and $\mathbf{v}$ are two tensor fields of type $(1,1)$ and they satisfy (12) then $\Gamma = \mathbf{h} - \mathbf{v}$ is an almost product structure, and we have $TM = \operatorname{Im}\mathbf{h} \oplus \operatorname{Im}\mathbf{v}$.

Then, in our case $M = T^1_kQ$ we have

$$T(T^1_kQ) = \operatorname{Im}\mathbf{h} \oplus \operatorname{Im}\mathbf{v} = \operatorname{Im}\mathbf{h} \oplus V(T^1_kQ)$$

Thus $\operatorname{Im}\mathbf{h}$ is the nonlinear connection associated to $\mathcal{H}$.

We have seen that to each horizontal map $\mathcal{H}$ corresponds a horizontal projector $\mathbf{h}$ which defines a nonlinear connection on T^1_kQ. The converse of this is given in Lemma 1, page 1249 Szilasi [14], for an arbitrary vector bundle; in our case one obtains.

Lemma 3.2. *If $\mathbf{h} \in \mathcal{T}^1_1(T^1_kQ)$ is a horizontal projector for τ^k_Q, that is $\mathbf{h}^2 = \mathbf{h}$ and* $\operatorname{Ker}\mathbf{h} = V(T^1_kQ)$*, then there exists an unique horizontal map $\mathcal{H} : T^1_kQ \times_Q TQ \longrightarrow T(T^1_kQ)$ such that $\mathcal{H} \circ \mathbf{j} = \mathbf{h}$.* □

Let $X \in \mathfrak{X}(Q)$ be a vector field on Q. Then the *horizontal lift* X^h of X to $\mathfrak{X}(T^1_kQ)$ is defined by

$$X^h(\mathbf{v}_q) := \mathcal{H}(\mathbf{v}_q, X(q)) = X^i\left(\frac{\partial}{\partial q^i}\Big|_{\mathbf{v}_q} - N^j_{Ai}(\mathbf{v}_q)\frac{\partial}{\partial v^j_A}\Big|_{\mathbf{v}_q}\right), \tag{13}$$

where $X = X^i\frac{\partial}{\partial q^i}$.

The *curvature* $\Omega : \mathfrak{X}(T^1_kQ) \times \mathfrak{X}(T^1_kQ) \to \mathfrak{X}(T^1_kQ)$ of the horizontal map $\mathcal{H}$ is defined as $\Omega = -\frac{1}{2}[\mathbf{h}, \mathbf{h}]$ and it is locally given by

$$\Omega = \frac{1}{2}\left(\frac{\partial N^j_{Ak}}{\partial q^i} - \frac{\partial N^j_{Ai}}{\partial q^k} + N^m_{Bk}\frac{\partial N^j_{Ai}}{\partial v^m_B} - N^m_{Bi}\frac{\partial N^j_{Ak}}{\partial v^m_B}\right)\frac{\partial}{\partial v^j_A} \otimes dq^i \wedge dq^k. \tag{14}$$

3.2. Nonlinear connections and canonical k-tangent structure on T^1_kQ. In this section we characterize nonlinear connections on T^1_kQ using the canonical k-tangent structure $(J^1, \dots, J^k)$.

Proposition 2. *Let Γ be a tensor field of type $(1,1)$ on T^1_kQ satisfying*

$$J^A \circ \Gamma = J^A \quad \textit{and} \quad \Gamma \circ J^A = -J^A, \qquad 1 \leq A \leq k. \tag{15}$$

Then Γ is an almost product structure, that is, $\Gamma^2 = 1_{T(T^1_kQ)}$.

Proof. For each vector field Z on T^1_kQ we have $J^A(\Gamma Z) = J^A(Z)$, $1 \leq A \leq k$. Then $J^A(\Gamma(Z) - Z) = 0$, that is, the vector field $\Gamma(Z) - Z$ is vertical, hence it can be written as follows:

$$\Gamma(Z) - Z = \sum_{B=1}^{k} J^B(W_B),$$

where $W_1, \ldots, W_k$ are vector fields on T^1_kQ. Finally we obtain

$$\begin{aligned}\Gamma^2(Z) = \Gamma(\Gamma(Z)) &= \Gamma(Z + \sum_{B=1}^{k} J^B(W_B)) = \Gamma(Z) + \sum_{B=1}^{k} \Gamma(J^B(W_B)) \\ &= \Gamma(Z) - \sum_{B=1}^{k} J^B(W_B) = Z. \end{aligned}$$ □

From (15) we deduce that Γ is locally given by

$$\Gamma = \left(\frac{\partial}{\partial q^i} + \Gamma^j_{Ai} \frac{\partial}{\partial v^j_A} \right) \otimes dq^i - \frac{\partial}{\partial v^i_A} \otimes dv^i_A, \tag{16}$$

where Γ^j_{Ai} are functions defined in a neighbourhood of T^1_kQ called *the components of* Γ.

Proposition 3. *To give a nonlinear connection N on $\tau^k_Q : T^1_kQ \to Q$ is equivalent to give a tensor field Γ of type $(1,1)$ satisfying* (15).

Proof. Let N be a nonlinear connection on $\tau^k_Q : T^1_kQ \to Q$ with horizontal projector $\mathbf{h}$. Then $\Gamma = 2\mathbf{h} - 1_{T(T^1_kQ)}$ satisfies (15). In fact, one obtains:

$$J^A \circ \Gamma = 2(J^A \circ \mathbf{h}) - J^A = 2J^A - J^A = J^A$$

where we have used that $J^A \circ \mathbf{h} = J^A$. On the other hand, since $\mathbf{h} \circ J^A = 0$ we

$$\Gamma \circ J^A = 2(\mathbf{h} \circ J^A) - J^A = -J^A.$$

Conversely, given Γ satisfying (15) from the above proposition we obtain that $\Gamma^2 = 1_{T(T^1_kQ)}$, then from Lemma 3.1 we deduce that there exists a horizontal projector $\mathbf{h} = \frac{1}{2}(1_{T(T^1_kQ)} + \Gamma)$, with local expression

$$\mathbf{h} = \frac{1}{2}(1_{T(T^1_kQ)} + \Gamma) = \left(\frac{\partial}{\partial q^i} + \frac{1}{2}\Gamma^j_{Ai} \frac{\partial}{\partial v^j_A} \right) \otimes dq^i,$$

which defines a nonlinear connection N_Γ. Moreover the components of the nonlinear connection N_Γ are given by

$$(N_\Gamma)^j_{Ai} = -\frac{1}{2}\Gamma^j_{Ai}.$$ □

4. k-vector fields. Second order partial differential equations (SOPDEs)

Second order differential equations, usually called SODEs play an important role in the geometric description of Lagrangian mechanics.

In this section we introduce SOPDEs (second order partial differential equations) which are a generalization of the concept of SODE. We study the relationship between SOPDEs and nonlinear connections on T^1_kQ and we also indicate the role of SOPDEs in Lagrangian classical field theories. Let us note that the role of SOPDE'S in the k-symplectic [5], [11], [13] and k-cosymplectic [7] Lagrangian formalisms of classical field theories is very important and similar to the role of second-order differential equations, SODE's, in Lagrangian mechanics.

Definition 4.1. Let M be a manifold and $\tau^k_M : T^1_kM \longrightarrow M$ the bundle of k^1-velocities. A *k-vector field* on M is a section $\xi : M \longrightarrow T^1_kM$ of the projection τ^k_M.

Since T^1_kM is the Whitney sum $TM\oplus \overset{k}{\ldots} \oplus TM$ of k copies of TM, we deduce that a k-vector field ξ defines a family of k vector fields $\{\xi_1,\dots,\xi_k\}$ on M by projecting ξ onto every factor. For this reason we will denote a k-vector field ξ by $(\xi_1,\dots,\xi_k)$.

Definition 4.2. An *integral section* of a k-vector field $\xi = (\xi_1,\dots,\xi_k)$ passing through a point $x \in M$ is a map $\phi : U_{\mathbf{0}} \subset \mathbb{R}^k \to M$, defined on some neighbourhood $U_{\mathbf{0}}$ of $\mathbf{0} \in \mathbb{R}^k$, such that

$$\phi(\mathbf{0}) = x, \quad \phi_*(\mathbf{t})\left(\frac{\partial}{\partial t^A}\Big|_{\mathbf{t}}\right) = \xi_A(\phi(\mathbf{t})) \quad \text{for every } \mathbf{t} \in U_{\mathbf{0}}, \tag{17}$$

or equivalently, ϕ satisfies $\xi \circ \phi = \phi^{(1)}$, where $\phi^{(1)}$ is the first prolongation of ϕ defined by

$$\begin{aligned} \phi^{(1)} : U_{\mathbf{0}} \subset \mathbb{R}^k &\longrightarrow T^1_kM \\ \mathbf{t} &\longrightarrow \phi^{(1)}(\mathbf{t}) = j^1_{\mathbf{0}}\phi_{\mathbf{t}}, \quad \phi_{\mathbf{t}}(\mathbf{t}) = \phi(\bar{\mathbf{t}} + \mathbf{t}), \end{aligned}$$

for every $\mathbf{t}, \bar{\mathbf{t}} \in \mathbb{R}^k$ such that $\bar{\mathbf{t}} + \mathbf{t} \in U_{\mathbf{0}}$.

A k-vector field $\xi = (\xi_1,\dots,\xi_k)$ on M is said to be *integrable* if there is an integral section passing through each point of M.

In local coordinates one obtains

$$\phi^{(1)}(t^1,\dots,t^k) = \left(\phi^i(t^1,\dots,t^k), \frac{\partial \phi^i}{\partial t^A}(t^1,\dots,t^k)\right). \tag{18}$$

Let us observe that in the case $k = 1$, an integral section is an integral curve and the first prolongation is the tangent lift from a curve on M to TM.

Next we will introduce the notion of SOPDE, which is a class of k-vector fields on T^1_kQ. We shall see that the integral sections of SOPDEs are first prolongations $\phi^{(1)}$ of maps $\phi : \mathbb{R}^k \to Q$.

If $F : M \to N$ is a differentiable map between the manifolds M and N, then $T^1_kF : T^1_kM \to T^1_kN$ is defined by $T^1_kF(v_{1q}, \dots, v_{kq}) = (F_*(q)(v_{1q}), \dots, F_*(q)v_{kq})$, or equivalently $T^1_kF(j^1_0\sigma) = j^1_0(F \circ \sigma)$.

Definition 4.3. A k-vector field $\xi = (\xi_1, \dots, \xi_k)$ on T^1_kQ is a *second order partial differential equation (*SOPDE*)* if it is also a section of the projection $T^1_k(\tau^k_Q) : T^1_k(T^1_kQ) \to T^1_kQ$; that is,

$$T^1_k(\tau^k_Q) \circ \xi = 1_{T^1_kQ}. \tag{19}$$

Let us observe that $\xi_A \in \mathfrak{X}(T^1_kQ)$ and (19) means

$$(\tau^k_Q)_*(\mathbf{v}_q)(\xi_A(\mathbf{v}_q)) = v_{A_q} \quad A = 1, \dots, k.$$

where $\mathbf{v}_q = (v_{1q}, \dots, v_{kq})$.

Let (q^i) be a local coordinate system on $U \subset Q$ and (q^i, v^i_A) the induced local coordinate system on T^1_kU. From (19), a direct computation shows that the local expression of a SOPDE $\xi = (\xi_1, \dots, \xi_k)$ is

$$\xi_A(q^i, v^i_A) = v^i_A \frac{\partial}{\partial q^i} + (\xi_A)^i_B \frac{\partial}{\partial v^i_B}, \quad 1 \le A \le k. \tag{20}$$

where $(\xi_A)^i_B \in \mathcal{C}^\infty(T^1_kU)$.

If $\varphi : \mathbb{R}^k \to T^1_kQ$ is an integral section of a SOPDE $(\xi_1, \dots, \xi_k)$ locally given by $\varphi(\mathbf{t}) = \big(\varphi^i(\mathbf{t}), \varphi^i_B(\mathbf{t})\big)$ then $\xi_A(\varphi(\mathbf{t})) = \varphi_*(\mathbf{t})[\partial/\partial t^A(\mathbf{t})]$ and thus

$$\frac{\partial \varphi^i}{\partial t^A}(\mathbf{t}) = v^i_A(\varphi(\mathbf{t})) = \varphi^i_A(\mathbf{t}), \qquad \frac{\partial \varphi^i_B}{\partial t^A}(\mathbf{t}) = (\xi_A)^i_B(\varphi(\mathbf{t})). \tag{21}$$

From (18) and (21) we obtain:

Proposition 4. *Let $\xi = (\xi_1, \dots, \xi_k)$ be an integrable* SOPDE *on T^1_kQ. If φ is an integral section of ξ then $\varphi = \phi^{(1)}$, where $\phi^{(1)}$ is the first prolongation of the map $\phi = \tau^k_Q \circ \varphi : \mathbb{R}^k \xrightarrow{\varphi} T^1_kQ \xrightarrow{\tau^k_Q} Q$ and it is a solution to the system*

$$\frac{\partial^2 \phi^i}{\partial t^A \partial t^B}(\mathbf{t}) = (\xi_A)^i_B(\phi^{(1)}(\mathbf{t})) = (\xi_A)^i_B(\phi^i(\mathbf{t}), \frac{\partial \phi^i}{\partial t^C}(\mathbf{t})). \tag{22}$$

Conversely, if $\phi : \mathbb{R}^k \to Q$ is any map satisfying (22), then $\phi^{(1)}$ is an integral section of $\xi = (\xi_1, \dots, \xi_k)$. □

For an integrable SOPDE we have $(\xi_A)^i_B = (\xi_B)^i_A$.

The following characterization of SOPDEs can be given using the canonical k-tangent structure of T^1_kQ (see (3), (6) and (20)):

Proposition 5. *A k-vector field $\xi = (\xi_1, \dots, \xi_k)$ on T^1_kQ is a SOPDE if, and only if, $S^A(\Gamma_A) = \Delta_A$, for all $A \in \{1, \dots, k\}$.* □

Example 1. Let us consider the following SOPDE (ξ_1, ξ_2) on $T^1_2\mathbb{R}$, with coordinates (q, v_1, v_2), given by

$$\xi_1 = v_1\frac{\partial}{\partial q} - \frac{k}{\lambda^2}v_1\frac{\partial}{\partial v_1} - \frac{k}{\lambda^2}v_2\frac{\partial}{\partial v_2}$$

$$\xi_2 = v_2\frac{\partial}{\partial q} - \frac{k}{\lambda^2}v_2\frac{\partial}{\partial v_1} + \frac{1}{k}v_1\frac{\partial}{\partial v_2} \tag{23}$$

Let $\phi : (t, x) \in \mathbb{R}^2 \to \mathbb{R}$ be a map. If $\phi^{(1)} : \mathbb{R}^2 \to T^1_2\mathbb{R}$ is an integral section of (ξ_1, ξ_2) then from (22) we obtain

$$-\frac{k}{\lambda^2}\frac{\partial \phi}{\partial t} = \frac{\partial^2\phi}{\partial t^2} \tag{24}$$

$$-\frac{k}{\lambda^2}\frac{\partial \phi}{\partial x} = \frac{\partial^2\phi}{\partial t\partial x} \tag{25}$$

$$\frac{1}{k}\frac{\partial \phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} \tag{26}$$

Equation (26) is the one-dimensional heat equation where k is the thermal diffusivity and the solutions $\phi(t, x)$ represents the temperature at the point x of a rod at time t.

Any integral section of this SOPDE is the first prolongation of a solution of the heat equation. The general solution of (26) is

$$\phi(t, x) = e^{-\frac{\kappa}{\lambda^2}t}\left[C\cos\left(\frac{x}{\lambda}\right) + D\sin\left(\frac{x}{\lambda}\right)\right] = Ae^{-\frac{\kappa}{\lambda^2}t}\sin\left(\frac{x}{\lambda} + \delta\right)$$

where λ, C and D are arbitrary constants and $A = \sqrt{C^2 + D^2}$, $\tan\delta = \frac{C}{D}$. Thus any solution of (26) is solution of (24) and (25).

4.1. Relationship between SOPDEs and nonlinear connections. In this section we prove that each nonlinear connection defines a second order partial differential equation (SOPDE) on T^1_kQ and conversely, given a SOPDE ξ on T^1_kQ a nonlinear connection N_ξ on $\tau^k_Q : T^1_kQ \to Q$ can be defined.

SOPDE associated to a nonlinear connection

Let us consider a nonlinear connection on $\tau^k_Q : T^1_kQ \to Q$ with horizontal map $\mathcal{H} : T^1_kQ \times_Q TQ \to T(T^1_kQ)$. For each $A = 1, \dots, k$ we define $\xi^A_{\mathcal{H}} \in \mathfrak{X}(T^1_kQ)$

as follows

$$\xi^A_{\mathcal{H}}(\mathbf{v}_q) = \mathcal{H}(\mathbf{v}_q, v_{A_\mathbf{q}}) \quad \text{where} \quad \mathbf{v_q} = (v_{1_q}, \dots, v_{k_q}) \in T^1_k Q.$$

From (7) we obtain that the SOPDE $\xi_{\mathcal{H}} = (\xi^1_{\mathcal{H}}, \dots, \xi^k_{\mathcal{H}})$ associated to $\mathcal{H}$ is

$$\xi^A_{\mathcal{H}}(\mathbf{v}_q) = v^i_A \left(\frac{\partial}{\partial q^i}\Big|_{\mathbf{v}_q} - N^k_{Bi} \frac{\partial}{\partial v^k_B}\Big|_{\mathbf{v}_q} \right). \tag{27}$$

Nonlinear connection associated to a SOPDE

Theorem 4.1. *To each* SOPDE ξ *on* $T^1_k Q$ *a nonlinear connection* N_ξ *may be associated, with horizontal projector*

$$\mathbf{h}_\xi = \frac{1}{k+1}\Big(1_{T(T^1_k Q)} - \sum_{A=1}^{k} \mathcal{L}_{\xi_A} J^A \Big). \tag{28}$$

PROOF. Let $\xi = (\xi_1, \dots, \xi_k)$ be a SOPDE on $T^1_k Q$ locally given by

$$\xi_A = v^i_A \frac{\partial}{\partial q^i} + (\xi_A)^j_B \frac{\partial}{\partial v^j_B}, \quad A = 1, \dots, k.$$

Since $\mathcal{L}_{\xi_A} J^A(Z) = [\xi_A, J^A Z] - J^A[\xi_A, Z]$ for all vector field Z on $T^1_k Q$, we obtain

$$\sum_{A=1}^{k} \mathcal{L}_{\xi_A} S^A = -\left(k \frac{\partial}{\partial q^i} + \sum_{A=1}^{k} \frac{\partial (\xi_A)^j_B}{\partial v^i_A} \frac{\partial}{\partial v^j_B} \right) \otimes dq^i + \frac{\partial}{\partial v^i_B} \otimes dv^i_B.$$

Then a straightforward computation in local coordinates shows that $\mathbf{h}_\xi$ is locally given by

$$\mathbf{h}_\xi = \left(\frac{\partial}{\partial q^j} + \frac{1}{k+1} \sum_{A=1}^{k} \frac{\partial (\xi_A)^i_B}{\partial v^j_A} \frac{\partial}{\partial v^i_B} \right) \otimes dq^j, \tag{29}$$

and satisfies

$$\mathbf{h}^2_\xi = \mathbf{h}_\xi \quad \text{and} \quad \operatorname{Ker} \mathbf{h}_\xi = V(T^1_k Q).$$

So defining $\mathbf{v}_\xi = 1_{T(T^1_k Q)} - \mathbf{h}_\xi$ we obtain, see Lemma 3.1, that $T(T^1_k Q) = \operatorname{Im} \mathbf{h}_\xi \oplus V(T^1_k Q)$. □

In the case $k = 1$, the horizontal projector $\mathbf{h}_\xi$ given in (28), coincides with the projector given by GRIFONE [1], [3] and by SZILASI [14]. From (10) and (29) we deduce that the components of the connection N_ξ are given by

$$(N_\xi)^i_{Bj} = -\frac{1}{k+1}\sum_{A=1}^{k}\frac{\partial(\xi_A)^i_B}{\partial v^j_A}. \tag{30}$$

We can associate to each SOPDE ξ the almost product structure $\Gamma_\xi = 2\mathbf{h}_\xi - 1_{T(T^1_kQ)}$, locally given by

$$\Gamma_\xi = \frac{1}{k+1}\left((1-k)1_{T(T^1_kQ)} - 2\sum_{A=1}^{k}\mathcal{L}_{\xi_A}J^A)\right)$$

In the case $k = 1$, this tensor field is $\Gamma_\xi = -\mathcal{L}_\xi J$, where J is the canonical tangent structure on TQ. The nonlinear connection associated to this structure was introduced by GRIFONE in Proposition I.41 of [1] and Proposition 1.3 of [3].

A turned out that there is a correspondence such that to each nonlinear connection on T^1_kQ a SOPDE ξ is associated and conversely, given a SOPDE on T^1_kQ there exists a nonlinear connection associated to this SOPDE. Is this correspondence a bijection? In general the answer to this question is negative. In fact:

(1) Let ξ be a SOPDE and N_ξ be the nonlinear connection associated to ξ. We denote by $\mathcal{H}_\xi$ the horizontal map associated to N_ξ. From (27) and (30) we deduce that $\xi = \xi_{\mathcal{H}_\xi}$ if and only if

$$(\xi_A)^j_B = \frac{1}{k+1}\sum_{C=1}^{k}\frac{\partial(\xi_C)^j_B}{\partial v^i_C}v^i_A,\ 1 \le A, B \le k,\ 1 \le i \le n.$$

When $k = 1$ we obtain $\xi_{\mathcal{H}_\xi} = \xi$ if and only if $\frac{1}{2}\frac{\partial \xi^k}{\partial v^i}\ v^i = \xi^k$ which means that the functions ξ^k are positive-homogeneous of degree 2 (see [3]).

(2) Let us consider now a nonlinear connection N defined from a horizontal map $\mathcal{H}$, the SOPDE $\xi_\mathcal{H}$ associated to this connection and the connection $N_{\xi_\mathcal{H}}$ associated to the SOPDE $\xi_\mathcal{H}$.

From (27) and (30) we obtain that $N = N_{\xi_\mathcal{H}}$ if and only if

$$N^j_{B\,i} = v^l_A\frac{\partial N^j_{Bl}}{\partial v^i_A}, \qquad 1 \le i, j \le n,\ 1 \le B \le k.$$

4.2. SOPDEs in Classical Field Theory. In this subsection, we recall the Lagrangian formalism developed by GÜNTHER [5], see also [11]. Here we show the role of SOPDEs and its integral sections in the Lagrangian Field Theory.

Let $L : T^1_kQ \to \mathbb{R}$ be a Lagrangian, that is a function $L(\phi^i, \partial\phi^i/\partial t^A)$ that depend on the components of the field and on its first partial derivatives. This Lagrangian is called autonomous in the sense that not depends on the time-space variables (t^A).

The *generalized Euler-Lagrange equations* for L are:

$$\sum_{A=1}^{k} \frac{\partial}{\partial t^A}\Big|_t \left(\frac{\partial L}{\partial v^i_A}\Big|_{\psi(\mathbf{t})} \right) = \frac{\partial L}{\partial q^i}\Big|_{\psi(\mathbf{t})}, \quad v^i_A(\psi(\mathbf{t})) = \frac{\partial \psi^i}{\partial t^A} \tag{31}$$

whose solutions are maps $\psi : \mathbb{R}^k \to T^1_kQ$ with $\psi(\mathbf{t}) = (\psi^i(\mathbf{t}), \psi^i_A(\mathbf{t}))$. Let us observe that $\psi(\mathbf{t}) = \phi^{(1)}(\mathbf{t})$, for $\phi = \tau^k_Q \circ \psi$. Using the canonical k-tangent structure, one introduces a family of 1-forms θ^A_L on T^1_kQ, and a family of 2-forms ω^A_L on T^1_kQ , as follows

$$\theta^A_L = dL \circ J^A, \quad \omega^A_L = -\mathrm{d}\theta^A_L, \quad 1 \leq A \leq k. \tag{32}$$

In natural local coordinates we have

$$\theta^A_L = \frac{\partial L}{\partial v^i_A}\, dq^i, \quad \omega^A_L = \frac{\partial^2 L}{\partial q^j \partial v^i_A} \mathrm{d}q^i \wedge \mathrm{d}q^j + \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} \mathrm{d}q^i \wedge \mathrm{d}v^j_B. \tag{33}$$

We also introduce the *energy function* $E_L = \Delta(L) - L \in \mathrm{C}^\infty(T^1_kQ)$, whose local expression is

$$E_L = v^i_A \frac{\partial L}{\partial v^i_A} - L. \tag{34}$$

Definition 4.4. The Lagrangian $L : T^1_kQ \longrightarrow \mathbb{R}$ is said to be *regular* if the matrix $\left(\frac{\partial^2 L}{\partial v^i_A \partial v^j_B}\right)$ is not singular at every point of T^1_kQ.

Let $(\xi_1, \ldots, \xi_k)$ be a k-vector field on T^1_kQ locally given by

$$\xi_A = (\xi_A)^i \frac{\partial}{\partial q^i} + (\xi_A)^i_B \frac{\partial}{\partial v^i_B}.$$

Then from (33) and (34) we deduce that $(\xi_1, \ldots, \xi_k)$ is a solution to the equation

$$\sum_{A=1}^{k} i_{\xi_A}\, \omega^A_L = \mathrm{d}E_L \tag{35}$$

if, and only if, $(\xi_A)^i$ and $(\xi_A)^i_B$ satisfy the system of equations

$$\left(\frac{\partial^2 L}{\partial q^i \partial v^j_A} - \frac{\partial^2 L}{\partial q^j \partial v^i_A}\right)(\xi_A)^j - \frac{\partial^2 L}{\partial v^i_A \partial v^j_B}(\xi_A)^j_B = v^j_A \frac{\partial^2 L}{\partial q^i \partial v^j_A} - \frac{\partial L}{\partial q^i},$$
$$\frac{\partial^2 L}{\partial v^j_B \partial v^i_A}(\xi_A)^i = \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} v^i_A.$$

If the Lagrangian is regular, the above equations are equivalent to the equations

$$\frac{\partial^2 L}{\partial q^j \partial v^i_A} v^j_A + \frac{\partial^2 L}{\partial v^i_A \partial v^j_B}(\xi_A)^j_B = \frac{\partial L}{\partial q^i} \tag{36}$$
$$(\xi_A)^i = v^i_A, \quad 1 \leq i \leq n,\ 1 \leq A \leq k. \tag{37}$$

Thus, if L is a regular Lagrangian, we deduce:

- If $(\xi_1, \ldots, \xi_k)$ is a solution of (35) then it is a SOPDE, (see (37)).
- Since $(\xi_1, \ldots, \xi_k)$ is a SOPDE, from Proposition 4 we know that, if it is integrable, then its integral sections are first prolongations $\phi^{(1)} : \mathbb{R}^k \to T^1_k Q$ of maps $\phi : \mathbb{R}^k \to Q$, and from (36) we deduce that ϕ is a solution to the Euler–Lagrange equations (31).
- Equation (36) leads us to define local solutions to (35) in a neighbourhood of each point of $T^1_k Q$ and, using a partition of unity, global solutions to (35).
- In the case $k = 1$, equation (35) reduces to $\imath_\xi \omega_L = dE_L$, which is the Euler-Lagrange equation in mechanics.

Example 2. Let $L : T^1_3\mathbb{R} \to \mathbb{R}$ be a Lagrangian given by

$$L : T^1_3\mathbb{R} \to \mathbb{R}, \quad L(q, v_1, v_2, v_3) = \frac{1}{2}\big(v_1^2 - c^2(v_2^2 + v_3^2)\big). \tag{38}$$

Let us suppose that (ξ_1, ξ_2, ξ_3) is a solution of the equation (35):

$$\sum_{A=1}^{3} i_{\xi_A}\, \omega_L^A = \mathrm{d}E_L.$$

Since L is regular we know that (ξ_1, ξ_2, ξ_3) is a SOPDE satisfying (36). Then each ξ_A is locally given by

$$\xi_A = v_A \frac{\partial}{\partial q} + (\xi_A)_1 \frac{\partial}{\partial v_1} + (\xi_A)_2 \frac{\partial}{\partial v_2} + (\xi_A)_3 \frac{\partial}{\partial v_3}, \quad 1 \leq A \leq 3.$$

From (36) we have

$$0 = \frac{\partial^2 L}{\partial v_A \partial v_B}(\xi_A)_B = (\xi_1)_1 - c^2[(\xi_2)_2 + (\xi_3)_3]. \tag{39}$$

From (22) and (39) we obtain that if $\phi^{(1)}(\mathbf{t})$ is an integral section of the 3-vector field (ξ_1, ξ_2, ξ_3) then $\phi : \mathbb{R}^k \to Q$ satisfies the equation

$$0 = \frac{\partial^2 \phi}{\partial (t^1)^2} - c^2 \left(\frac{\partial^2 \phi}{\partial (t^2)^2} + \frac{\partial^2 \phi}{\partial (t^3)^2} \right)$$

which is the 2-*dimensional wave equation*.

4.3. Linearizable SOPDEs. In this section we introduce the definition of linearizable SOPDE and we establish a necessary condition so that a SOPDE is linearizable.

Definition 4.5. A SOPDE $\xi = (\xi_1, \dots, \xi_k)$ on $T^1_k Q$ is said to be **linearizable** if in a neighbourhood of each point on $T^1_k Q$, its components $(\xi_A)^j_B$ can be written as follows

$$(\xi_A)^j_B = \left(\mathcal{A}^j_{AB}\right)^C_m v^m_C + \left(\mathcal{B}^j_{AB}\right)_m q^m + \mathcal{C}^j_{AB} \tag{40}$$

with $\left(\mathcal{A}^j_{AB}\right)^C_m$, $\left(\mathcal{B}^j_{AB}\right)_m$, $\mathcal{C}^j_{AB} \in \mathbb{R}$.

Proposition 6. *If ξ is linearizable then the curvature of the nonlinear connection $\mathcal{H}_\xi$ vanishes.*

PROOF. Since ξ is linearizable, from (30) and (40) we obtain that the components of the nonlinear connection $\mathcal{H}_\xi$ are

$$(N_\xi)^\imath_{Bj} = -\frac{1}{k+1} \sum_{A=1}^{k} \left(\mathcal{A}^\imath_{AB}\right)^A_j .$$

Now from (14) we deduce that the curvature Ω vanishes. □

In the particular case of a linearizable SOPDE, Proposition 4 can be formulated as follows.

Proposition 7. *Let $\xi = (\xi_1, \dots, \xi_k)$ be a linearizable and integrable* SOPDE. *If the first prolongation of $\phi : \mathbb{R}^k \to Q$ is an integral section of $\xi = (\xi_1, \dots, \xi_k)$ then we have*

$$\frac{\partial^2 \phi^j}{\partial t^A \partial t^B}(\mathbf{t}) = (\xi_A)^j_B(\phi^{(1)}(\mathbf{t})) = \left(\mathcal{A}^j_{AB}\right)^C_m \frac{\partial \phi^m}{\partial t^C} + \left(\mathcal{B}^j_{AB}\right)_m \phi^m(\mathbf{t}) + \mathcal{C}^j_{AB} \tag{41}$$

Conversely, if $\phi : \mathbb{R}^k \to Q$ is a map satisfying (41), *then $\phi^{(1)}$ is an integral section of ξ.*

Example 3. From (23) and (40) we deduce that the SOPDE (23) is linearizable.

ACKNOWLEDGMENTS. We acknowledge the partial financial support of *Ministerio de Ciencia e Innovación*, Project MTM2008-03606-E/ and Project MTM2008-00689/MTM. We are grateful to PROF. IOAN BUCATARU ("Al.I.Cuza" University) for his suggestions and comments on this work.

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(Received June 15, 2009; revised August 25, 2010)