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The properties of solutions of some linear differential equations

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Abstract. In this paper, we study the growth and the oscillation of certain first order nonhomogeneous linear differential polynomials generated by solutions of the differential equation

$$f'' + A_1(z)f' + A_0(z)f = F,$$

where $A_1(z)$, $A_0(z)$ ($\not\equiv 0$), F are entire functions of finite order.

1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory see [9], [13], [16]. In addition, we will use $\lambda(f)$ ($\lambda_2(f)$) and $\overline{\lambda}(f)$ ($\overline{\lambda}_2(f)$) to denote respectively the exponents (hyper-exponents) of convergence of the zero-sequence and the sequence of distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of f and $\rho_2(f)$ to denote the hyper-order of f.

To give the precise estimate of fixed points, we define:

Definition 1.1 ([6], [11], [15]). Let f be a meromorphic function and let $z_1, z_2, \dots$ ($|z_j| = r_j$, $0 < r_1 \leq r_2 \leq \dots$) be the sequence of the fixed points of f, each point being repeated only once. The exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}(f) = \inf\left\{\tau > 0 : \sum_{j=1}^{+\infty} |z_j|^{-\tau} < +\infty\right\}.$$

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Clearly,

$$\overline{\tau}(f)=\overline{\lambda}(f-z)=\overline{\lim_{r\to+\infty}}\frac{\log\overline{N}\left(r,\frac{1}{f-z}\right)}{\log r}, \tag{1.1}$$

where $\overline{N}\left(r,\frac{1}{f-z}\right)$ is the counting function of distinct fixed points of $f(z)$ in $\{z : |z|<r\}$.

Definition 1.2 ([6], [11], [15]). Let f be a meromorphic function. Then the hyper exponent of convergence of the sequence of distinct fixed points $\overline{\tau}_2(f)$ of $f(z)$ is defined by

$$\overline{\tau}_2(f)=\overline{\lambda}_2(f-z)=\overline{\lim_{r\to+\infty}}\frac{\log\log\overline{N}\left(r,\frac{1}{f-z}\right)}{\log r}. \tag{1.2}$$

Consider the second order linear differential equation

$$f''+A_1(z)f'+A_0(z)f=F, \tag{1.3}$$

where $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F are transcendental entire functions with finite order. It is well-known that all solutions of equation (1.3) are entire functions. Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [17]). However, there are few studies on the fixed points of solutions of differential equations. In [6], Z. X. Chen firstly studied the problem on the fixed points and hyper-order of solutions of second order linear differential equations with entire coefficients.

We know that a differential equation bears a relation to all derivatives of its solutions. Hence, linear differential polynomials generated by its solutions must have special nature because of the control of differential equations.

The main purpose of this paper is to study the growth and the oscillation of some differential polynomials generated by solutions of second order linear differential equation (1.3). We obtain some estimates of their hyper order and fixed points. Before we state our results, we denote by

$$\alpha_0=d_0'-d_1A_0,\quad \alpha_1=d_1'+d_0-d_1A_1,\quad h=d_1\alpha_0-d_0\alpha_1 \tag{1.4}$$

and

$$\psi=\frac{d_1(\varphi'-b'-d_1F)-\alpha_1(\varphi-b)}{h}, \tag{1.5}$$

where $A_1(z)$, $A_0(z)$, F, d_j $(j=0,1)$, b and φ are entire functions with finite order.

Theorem 1.3. *Let $A_1(z)$, $A_0(z) \not\equiv 0$, F be entire functions of finite order. Let $d_0(z)$, $d_1(z)$, $b(z)$ be entire functions such that at least one of $d_0(z)$, $d_1(z)$ does not vanish identically with $\rho(d_j) < \infty$ $(j = 0, 1)$, $\rho(b) < \infty$ and that $h \not\equiv 0$. Let φ be an entire function with finite order such that $\psi(z)$ is not a solution of (1.3). If f is an infinite order solution of (1.3) with $\rho_2(f) = \rho < +\infty$, then the differential polynomial $g_f = d_1 f' + d_0 f + b$ satisfies*

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty, \tag{1.6}$$

$$\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) = \rho. \tag{1.7}$$

Theorem 1.4. *Let $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, $F \not\equiv 0$ be entire functions of finite order such that all solutions of equation (1.3) are of infinite order. Let $d_0(z)$, $d_1(z)$, $b(z)$ be entire functions such that at least one of $d_0(z)$, $d_1(z)$ does not vanish identically with $\rho(d_j) < \infty$ $(j = 0, 1)$, $\rho(b) < \infty$ and that $h \not\equiv 0$. Let φ be a finite order entire function. If f is a solution of equation (1.3) with $\rho_2(f) = \rho < +\infty$, then the differential polynomial $g_f = d_1 f' + d_0 f + b$ satisfies (1.6) and (1.7).*

Applying Theorem 1.4 for $\varphi(z) = z$, we obtain the following result.

Corollary 1.5. *Under the assumptions of Theorem 1.4, if f is a solution of equation (1.3) with $\rho_2(f) = \rho < +\infty$, then the differential polynomial $g_f = d_1 f' + d_0 f + b$ has infinitely many fixed points and satisfies $\overline{\tau}(g_f) = \tau(g_f) = \rho(g_f) = \rho(f) = \infty$, $\overline{\tau}_2(g_f) = \tau_2(g_f) = \rho_2(g_f) = \rho_2(f) = \rho$.*

In the following, we obtain a result which is an example of Theorem 1.4.

Theorem 1.6. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where a_i, b_i $(i = 0, 1, \dots, n)$ are complex numbers, $a_n b_n \neq 0$ such that $\arg a_n \neq \arg b_n$ or $a_n = c b_n$ $(0 < c < 1)$. Let $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ and $F \not\equiv 0$ be entire functions with $\max\{\rho(A_j)$ $(j = 0, 1)$, $\rho(F)\} < n$. Let $d_0(z)$, $d_1(z)$, $b(z)$ be entire functions such that at least one of $d_0(z)$, $d_1(z)$ does not vanish identically with $\rho(d_j) < n$ $(j = 0, 1), \rho(b) < n$, and let $\varphi(z)$ be an entire function with finite order. If f is a solution of the equation*

$$f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f = F, \tag{1.8}$$

then the differential polynomial $g_f = d_1 f' + d_0 f + b$ satisfies

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty, \tag{1.9}$$

$$\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) = n. \tag{1.10}$$

In particular, if f is a solution of equation (1.8), then the differential polynomial $g_f = d_1 f' + d_0 f + b$ has infinitely many fixed points and satisfies $\overline{\tau}(g_f) = \tau(g_f) = \rho(g_f) = \rho(f) = \infty$, $\overline{\tau}_2(g_f) = \tau_2(g_f) = \rho_2(g_f) = \rho_2(f) = n$.

2. Auxiliary lemmas

Our proofs depend mainly upon the following lemmas. Before starting these lemmas, we recall the concepts of linear and logarithmic measure. For $E \subset [0, +\infty)$, we define the linear measure of a set E by $m(E) = \int_0^{+\infty} \chi_E(t)dt$ and the logarithmic measure of a set $F \subset [1, +\infty)$ by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt$, where χ_H is the characteristic function of a set H.

Lemma 2.1 ([7]). *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exist a set $E_1 \subset (1, +\infty)$ of finite logarithmic measure and a constant $B > 0$ that depends only on α and (m, n) (m, n positive integers with $m < n$) such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left| \frac{f^{(n)}(z)}{f^{(m)}(z)} \right| \leq B \left[\frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f) \right]^{n-m}. \tag{2.1}$$

Lemma 2.2 ([5]). *Let $f(z)$ be an entire function of order $\rho(f) = \alpha < +\infty$. Then for any given $\varepsilon > 0$, there exists a set $E_2 \subset [1, +\infty)$ that has finite linear measure and finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$, we have*

$$\exp\left\{-r^{\alpha+\varepsilon}\right\} \leq |f(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{2.2}$$

Lemma 2.3 ([12], pp. 253–255). *Let $P(z) = \sum_{i=0}^{n} a_i z^i$, where n is a positive integer and $a_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0, 2\pi)$. For any given ε $(0 < \varepsilon < \pi/4n)$, we introduce $2n$ closed angles*

$$S_j : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon \leq \theta \leq -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon \quad (j = 0, 1, \ldots, 2n-1). \tag{2.3}$$

Then there exists a positive number $R = R(\varepsilon)$ such that for $|z| = r > R$,

$$Re P(z) > \alpha_n r^n (1 - \varepsilon) \sin(n\varepsilon), \tag{2.4}$$

if $z = re^{i\theta} \in S_j$, when j is even; while

$$Re P(z) < -\alpha_n r^n (1 - \varepsilon) \sin(n\varepsilon), \tag{2.5}$$

if $z = re^{i\theta} \in S_j$, when j is odd.

Lemma 2.4 ([4]). *Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution with $\rho(f) = +\infty$ of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F, \tag{2.6}$$

then $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$.

Lemma 2.5 ([1]). *Let $A_0, A_1, \dots, A_{k-1}$, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of equation (2.6) with $\rho(f) = +\infty$ and $\rho_2(f) = \rho < +\infty$, then f satisfies $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho$.*

Lemma 2.6. *Suppose that $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F are entire functions of finite order. Let $d_0(z)$, $d_1(z)$, $b(z)$ be entire functions such that at least one of $d_0(z)$, $d_1(z)$ does not vanish identically with $\rho(d_j) < \infty$ $(j = 0, 1), \rho(b) < \infty$ and that $h \not\equiv 0$, where h is defined in (1.4). If f is an infinite order solution of (1.3) with $\rho_2(f) = \rho < +\infty$, then the differential polynomial*

$$g_f = d_1 f' + d_0 f + b \tag{2.7}$$

satisfies

$$\rho\left(g_f\right) = \rho(f) = \infty, \quad \rho_2\left(g_f\right) = \rho_2(f) = \rho. \tag{2.8}$$

Proof. Suppose that f is a solution of equation (1.3) with $\rho(f) = +\infty$ and $\rho_2(f) = \rho < +\infty$. First we suppose that $d_1 \not\equiv 0$. Differentiating both sides of equation (2.7) and replacing f'' with $f'' = F - A_1 f' - A_0 f$, we obtain

$$g_f' - b' - d_1 F = \left(d_1' + d_0 - d_1 A_1\right) f' + \left(d_0' - d_1 A_0\right) f. \tag{2.9}$$

Then by (1.4), (2.7) and (2.9), we have

$$d_1 f' + d_0 f = g_f - b, \tag{2.10}$$

$$\alpha_1 f' + \alpha_0 f = g_f' - b' - d_1 F. \tag{2.11}$$

Set

$$h = d_1 \alpha_0 - d_0 \alpha_1 = d_1\left(d_0' - d_1 A_0\right) - d_0\left(d_1' + d_0 - d_1 A_1\right). \tag{2.12}$$

By $h \not\equiv 0$ and (2.10)–(2.12), we obtain

$$f = \frac{d_1\left(g_f' - b' - d_1 F\right) - \alpha_1\left(g_f - b\right)}{h}. \tag{2.13}$$

If $\rho(g_f) < \infty$, then by (2.13), we get $\rho(f) < \infty$ and this is a contradiction. Hence $\rho(g_f) = \infty$.

Finally, if $d_1 \equiv 0$, $d_0 \not\equiv 0$, then we have $g_f = d_0 f + b$ and by $\rho(d_0) < \infty$, $\rho(b) < \infty$, then we get $\rho(g_f) = \infty$.

Now, we prove that $\rho_2(g_f) = \rho_2(f) = \rho$. By (2.7), we get $\rho_2(g_f) \le \rho_2(f)$ and by (2.13) we have $\rho_2(f) \le \rho_2(g_f)$. This yield $\rho_2(g_f) = \rho_2(f) = \rho$. □

Remark 2.7. In Lemma 2.6, if we don't have the condition $h \not\equiv 0$, then the differential polynomial can be of finite order. For example, if $d_0' - d_1 A_0 \equiv 0$

and $d_1' + d_0 - d_1 A_1 \equiv 0$, then $h \equiv 0$ and $g_f' - b' - d_1 F \equiv 0$. It follows that $\rho(g_f) = \rho(g_f') = \rho(b' + d_1 F) < +\infty$.

Lemma 2.8 ([3]). *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where a_i, b_i $(i = 0, 1, \dots, n)$ are complex numbers, $a_n b_n \neq 0$ such that $\arg a_n \neq \arg b_n$ or $a_n = c b_n$ $(0 < c < 1)$. Let $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ and $F \not\equiv 0$ be entire functions with $\max\{\rho(A_j)\ (j = 0, 1),\ \rho(F)\} < n$. Then every solution f of equation (1.8) has infinite order.*

Lemma 2.9 ([8]). *Let $\varphi : [0, +\infty) \to \mathbb{R}$ and $\psi : [0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $\varphi(r) \leq \psi(r)$ for all $r \notin E_3 \cup [0, 1]$, where $E_3 \subset (1, +\infty)$ is a set of finite logarithmic measure. Let $\gamma > 1$ be a given constant. Then there exists an $r_1 = r_1(\gamma) > 0$ such that $\varphi(r) \leq \psi(\gamma r)$ for all $r > r_1$.*

By using Wiman–Valiron theory [5], [10], [14], we easily obtain the following result which we omit the proof.

Lemma 2.10. *Suppose that $k \geq 2$ and $A_0, A_1, \dots, A_{k-1}$, $F \not\equiv 0$ are entire functions of finite order. If f is a solution of equation (2.6) then $\rho_2(f) \leq \max\{\rho(A_j) : j = 0, \dots, k-1, \rho(F)\} = \sigma$.*

Lemma 2.11. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where a_i, b_i $(i = 0, 1, \dots, n)$ are complex numbers, $a_n b_n \neq 0$ such that $\arg a_n \neq \arg b_n$ or $a_n = c b_n$ $(0 < c < 1)$. Let $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ and $F \not\equiv 0$ be entire functions with $\max\{\rho(A_j)\ (j = 0, 1),\ \rho(F)\} < n$. Then every solution f of equation (1.8) satisfies $\rho(f) = \infty$ and $\rho_2(f) = n$.*

Proof. Assume that f is a solution of (1.8). Then by Lemma 2.8, we have $\rho(f) = \infty$. Now we prove that $\rho_2(f) = n$. Suppose first that $\arg a_n \neq \arg b_n$. By Lemma 2.3, there exist real numbers $b > 0$, $R_1 > 0$ and $\theta_1 < \theta_2$ such that for all $r > R_1$ and $\theta_1 \leq \theta \leq \theta_2$, we have

$$Re P\left(re^{i\theta}\right) < 0, \quad Re Q\left(re^{i\theta}\right) > br^n. \tag{2.14}$$

Set $\max\{\rho(A_j)\ (j = 0, 1),\ \rho(F)\} = \beta < n$. Then by Lemma 2.2, there exists a set $E_2 \subset [1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$, for any given ε $(0 < \varepsilon < n - \beta)$, we have

$$\exp\left\{-r^{\beta+\varepsilon}\right\} \leq |A_j(z)| \leq \exp\left\{r^{\beta+\varepsilon}\right\}\ (j = 0, 1), \quad |F(z)| \leq \exp\{r^{\beta+\varepsilon}\}. \tag{2.15}$$

By Lemma 2.1, there exist a set $E_1 \subset (1, +\infty)$ of finite logarithmic measure and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T(2r, f)\right]^3 \quad (j = 1, 2). \tag{2.16}$$

It follows from (1.8) that

$$\left|A_0(z)e^{Q(z)}\right| \leq \left|\frac{f''(z)}{f(z)}\right| + \left|A_1(z)e^{P(z)}\right| \left|\frac{f'(z)}{f(z)}\right| + \left|\frac{F(z)}{f(z)}\right|. \tag{2.17}$$

Since $\rho(f) = +\infty$, we may assume that $M(r, f) \geq 1$. Hence

$$\left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)|}{M(r,f)} \leq |F(z)| \leq \exp\left\{r^{\beta+\varepsilon}\right\}, \tag{2.18}$$

where $|z| = r$ and $|f(z)| = M(r, f)$. Thus, by (2.14)–(2.18), we get for $z = re^{i\theta}$, $r > R_1$, $\theta_1 \leq \theta \leq \theta_2$, $|z| = r \notin [0, 1] \cup E_1 \cup E_2$, at which $|f(z)| = M(r, f)$

$$\begin{aligned}\exp\left\{-r^{\beta+\varepsilon}\right\}\exp\left\{br^n\right\} &\leq \left(1 + \exp\left\{r^{\beta+\varepsilon}\right\}\right)B[T(2r,f)]^3 + \exp\left\{r^{\beta+\varepsilon}\right\}\\ &\leq 3B\exp\left\{r^{\beta+\varepsilon}\right\}[T(2r,f)]^3. \end{aligned} \tag{2.19}$$

Hence by $\beta + \varepsilon < n$, Lemma 2.9 and (2.19) we get

$$\rho_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r} \geq n. \tag{2.20}$$

Using Lemma 2.10, we obtain $\rho_2(f) = n$.

Suppose now $a_n = cb_n$ $(0 < c < 1)$. Since $\deg Q > \deg(P - cQ)$, by Lemma 2.3, there exist real numbers $d > 0$, λ, $R_2 > 0$ and $\theta_3 < \theta_4$ such that for all $r > R_2$ and $\theta_3 \leq \theta \leq \theta_4$, we have

$$ReQ(re^{i\theta}) > dr^n, \quad Re\big(P(re^{i\theta}) - cQ(re^{i\theta})\big) < \lambda. \tag{2.21}$$

It follows from (1.8) that

$$\begin{aligned}\left|A_0(z)e^{(1-c)Q(z)}\right| \leq \left|e^{-cQ(z)}\right| \left|\frac{f''(z)}{f(z)}\right| + \left|A_1(z)e^{P(z)-cQ(z)}\right| \left|\frac{f'(z)}{f(z)}\right|\\ + \left|e^{-cQ(z)}\right| \left|\frac{F(z)}{f(z)}\right|. \end{aligned} \tag{2.22}$$

Hence by (2.15), (2.16), (2.18), (2.21) and (2.22), we get for $z = re^{i\theta}, r > R_2$, $\theta_3 \leq \theta \leq \theta_4$, $|z| = r \notin [0, 1] \cup E_1 \cup E_2$, at which $|f(z)| = M(r, f)$

$$\begin{aligned}&\exp\left\{-r^{\beta+\varepsilon}\right\}\exp\left\{(1-c)dr^n\right\}\\ &\leq \left[\exp\{-cdr^n\} + \exp\{r^{\beta+\varepsilon}\}\exp\{\lambda\}\right]B[T(2r,f)]^3\\ &+ \exp\{-cdr^n\}\exp\{r^{\beta+\varepsilon}\} \leq K\exp\{r^{\beta+\varepsilon}\}[T(2r,f)]^3, \end{aligned} \tag{2.23}$$

where $K > 0$ is some constant. Thus, by $\beta + \varepsilon < n$, $0 < c < 1$, Lemma 2.9 and (2.23), we obtain

$$\rho_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r} \geq n. \tag{2.24}$$

Using Lemma 2.10, we get $\rho_2(f) = n$. □

Lemma 2.12 ([2]). *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where a_i, b_i $(i = 0, 1, \ldots, n)$ are complex numbers, $a_n b_n \neq 0$ such that $\arg a_n \neq \arg b_n$ or $a_n = cb_n$ $(0 < c < 1)$. We denote index set by $\Lambda_1 = \{0, P\}$. If $H_j (j \in \Lambda_1)$ and $H_Q \not\equiv 0$ are all meromorphic functions of orders that are less than n, setting $\Psi_1(z) = \sum_{j \in \Lambda_1} H_j(z) e^j$, then $\Psi_1(z) + H_Q e^Q \not\equiv 0$.*

3. Proof of Theorem 1.3.

Suppose that f is a solution of equation (1.3) with $\rho(f) = \infty$ and $\rho_2(f) = \rho < +\infty$. Set $w(z) = d_1 f' + d_0 f + b - \varphi$. Since $\rho(\varphi) < \infty$, then by Lemma 2.6 we have $\rho(w) = \rho(g_f) = \rho(f) = \infty$ and $\rho_2(w) = \rho_2(g_f) = \rho_2(f) = \rho$. In order to prove $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho$, we need to prove only $\overline{\lambda}(w = \lambda(w) = \infty$ and $\overline{\lambda}_2(w) = \lambda_2(w) = \rho$. By $g_f = w + \varphi$, we get from (2.13)

$$f = \frac{d_1 w' - \alpha_1 w}{h} + \psi, \tag{3.1}$$

where α_1, h, ψ are defined in (1.4)–(1.5). Substituting (3.1) into equation (1.3), we obtain

$$\frac{d_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = F - \big(\psi'' + A_1(z)\psi' + A_0(z)\psi\big) = A, \tag{3.2}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho(\phi_j) < \infty$ $(j = 0, 1, 2)$. Since $\psi(z)$ is not a solution of (1.3), it follows that $A \not\equiv 0$. Then by Lemma 2.4 and Lemma 2.5, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, $\overline{\lambda}_2(w) = \lambda_2(w) = \rho_2(w) = \rho$, i.e., $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) = \rho$.

Remark 3.1. From the proof of Theorem 1.3, we see that the condition $\psi(z)$ is not a solution of equation (1.3) is necessary.

4. Proof of Theorem 1.4.

By the hypotheses of Theorem 1.4 all solutions of equation (1.3) are of infinite order. From (1.5), we see that $\psi(z)$ is a meromorphic function of finite order, then $\psi(z)$ is not a solution of (1.3). By Theorem 1.3, we obtain Theorem 1.4.

5. Proof of Theorem 1.6.

Suppose that f is a solution of equation (1.8). Then, by Lemma 2.11 we have $\rho(f) = \infty$ and $\rho_2(f) = n$. First we suppose that $d_1 \not\equiv 0$. Set

$$\alpha_0 = d_0' - d_1 A_0 e^Q, \quad \alpha_1 = d_1' + d_0 - d_1 A_1 e^P, \tag{5.1}$$

$$h = d_1\alpha_0 - d_0\alpha_1 = d_1\big(d_0' - d_1 A_0 e^Q\big) - d_0\big(d_1' + d_0 - d_1 A_1 e^P\big). \tag{5.2}$$

By (5.2) we can write $h = \Psi_1(z) - d_1^2 A_0 e^Q$, where $\Psi_1(z)$ is defined as in Lemma 2.12. By $d_1 \not\equiv 0$, $A_0 \not\equiv 0$ and Lemma 2.12, we see that $h \not\equiv 0$. By Theorem 1.4 the differential polynomial $g_f = d_1 f' + d_0 f + b$ satisfies $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) = n$.

Now suppose $d_1 \equiv 0$, $d_0 \not\equiv 0$. Then $h = -d_0^2 \not\equiv 0$ and by Theorem 1.4 we get $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) = n$.

Setting now $\varphi(z) = z$, we obtain that $\overline{\tau}(g_f) = \tau(g_f) = \rho(g_f) = \rho(f) = \infty$ and $\overline{\tau}_2(g_f) = \tau_2(g_f) = \rho_2(g_f) = \rho_2(f) = n$.

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