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Translation curves and their spheres in homogeneous geometries

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Abstract. In this paper five 3-dimensional homogeneous geometries **Sol**, **Nil**, $\mathbf{SL_2}\mathbb{R}$ and product geometries ($\mathbf{H^2}\times\mathbb{R}$, $\mathbf{S^2}\times\mathbb{R}$) are discussed. We introduce a seemingly new family of curves, called translation curves. These curves seem to be more natural in these geometries, than their geodesic lines. Visualization of the corresponding curves and spheres has also been elaborated.

1. Introduction

The eight homogeneous 3-geometries or Thurston spaces $\mathbf{E^3}$, $\mathbf{S^3}$, $\mathbf{H^3}$, $\mathbf{S^2}\times\mathbb{R}$, $\mathbf{H^2}\times\mathbb{R}$, $\widetilde{\mathbf{SL_2}\mathbb{R}}$, **Nil**, **Sol** seem to be actual for investigations analogous to those of classical differential geometry. The last three geometries are strange a little bit with some interesting new phenomena. Here these will be discussed.

Let (M,g) be a Riemannian manifold. If for any $x,y\in M$ there does exist an isometry $\Phi: M\to M$ such that $y=\Phi(x)$, then the Riemannian manifold is called homogeneous.

Translations in Riemannian 3-spaces can be introduced in a natural way. Consider a unit vector at an origin and a geodesic curve starting in the direction of this vector by arc-length parameter. Any other vector at the origin can be translated along the above geodesic curve and so geodesic translations as local isometries can be defined along geodesic curves. But a Riemann space is not

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necessarily homogeneous. In a homogeneous space (above) there are postulated isometries, mapping each point to any point. Specific translations, in $\mathbf{S^2} \times \mathbb{R}$, $\mathbf{H^2} \times \mathbb{R}$, **Nil**, **Sol** and $\mathbf{SL_2}\mathbb{R}$, can be introduced in a natural way, involving an invariant Riemann metric. However, its geodesic translations will be different from these specific translations in $\widetilde{\mathbf{SL_2}\mathbb{R}}$, **Nil** and **Sol**. Again, consider a unit vector at the origin. Translations, postulated at the beginning, carry this vector to any point by its tangent mapping. If a curve $t \mapsto (x(t), y(t), z(t))$ has just this translated vector as tangent vector in each point, then the curve is called *translation curve*. This assumption leads to a first order differential equation, thus the translation curves are simpler than the geodesics and differ from them, except of spaces of constant curvature $\mathbf{E^3}$, $\mathbf{S^3}$, $\mathbf{H^3}$, moreover in $\mathbf{S^2} \times \mathbb{R}$ and $\mathbf{H^2} \times \mathbb{R}$. Similarly to geodesic sphere and ball, translation sphere and ball can be defined and visualized in our projected figures as usual in their affine-projective model in $\mathbf{E^3}$ [M97], [Sz07], [Sz09].

2. Sol geometry as a typical affine example

Sol geometry can be obtained by giving a group structure to the real affine 3-space, a semi-direct product $\mathbb{R} \ltimes \mathbb{R}^2$ by

$$\begin{pmatrix} 1 & a & b & c \end{pmatrix} \begin{pmatrix} 1 & x & y & z \\ 0 & e^{-z} & 0 & 0 \\ 0 & 0 & e^{z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + ae^{-z} & y + be^{z} & z + c \end{pmatrix}$$

This is the right action by a translation $(1, x, y, z)$ on a point $(1, a, b, c)$ yielding also a point of Sol, expressed in homogeneous (affine-projective) coordinates for (x, y, z) and (a, b, c), respectively, after having chosen a fixed origin $O(1, 0, 0, 0)$.

Thus a translation T in **Sol** can be specified by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x & y & z \\ 0 & e^{-z} & 0 & 0 \\ 0 & 0 & e^{z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & y & z \end{pmatrix}. \tag{2.1}$$

The inverse of (2.1) by

$$\begin{pmatrix} 1 & x & y & z \end{pmatrix} \begin{pmatrix} 1 & -xe^{z} & -ye^{-z} & -z \\ 0 & e^{z} & 0 & 0 \\ 0 & 0 & e^{-z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \tag{2.2}$$

defines the pull back of coordinate differentials $(0, dx, dy, dz)$ in $1, x, y, z$ to that $(0, d\bar{x}, d\bar{y}, d\bar{z})$ in the origin $(1, 0, 0, 0)$, so that

$$\begin{pmatrix} 0 & dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & -xe^{z} & -ye^{-z} & -z \\ 0 & e^{z} & 0 & 0 \\ 0 & 0 & e^{-z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & d\bar{x} & d\bar{y} & d\bar{z} \end{pmatrix} \tag{2.3}$$

holds. The conventional positive definite are length square, fixed in the origin, will then be extended to that in any point $(1, x, y, z)$ by

$$(ds)^2 = (d\bar{x})^2 + (d\bar{y})^2 + (d\bar{z})^2 = (dx)^2 e^{2z} + (dy)^2 e^{-2z} + (dz)^2 =: du^i g_{ij} du^j \tag{2.4}$$

leading to the usual Riemannian metric g of **Sol** with

$$dx =: du^1, \quad dy =: du^2, \quad dz =: du^3.$$

Consider the former fundamental metric tensor

$$g_{ij} = \begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

As motivating example here, compute the Christoffel symbols (applying the Einstein conventions; $i, j, k, l \in \{1, 2, 3\}$ and $u^1 = x$, $u^2 = y$, $u^3 = z$):

$$\Gamma^k_{ij} = \frac{1}{2} \left(\frac{\partial g_{jl}}{\partial u^i} + \frac{\partial g_{li}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l} \right) g^{lk}, \tag{2.6}$$

where (g^{lk}) denotes the inverse matrix of (g_{ij}).

The usual Levi–Civita connection with Christoffel symbols (2.6) leads to the equation system of geodesics $u^i(t)$ with given initial conditions as follows

$$u^i_0 = u^i(0), \quad \dot{u}^i_0 = \dot{u}^i(0), \quad \ddot{u}^k + \Gamma^k_{ij} \dot{u}^i \dot{u}^j = 0. \tag{2.7}$$

Now, for geodesics in **Sol** geometry we get

$$\begin{aligned} \ddot{x} &= -2\dot{x}\dot{z} \\ \ddot{y} &= 2\dot{y}\dot{z} \\ \ddot{z} &= e^{2z}(\dot{x})^2 - e^{-2z}(\dot{y})^2 \end{aligned} \tag{2.8}$$

and the solution for initial conditions $(x(0), y(0), z(0)) = (0, 0, 0)$ and $(\dot{x}(0), \dot{y}(0), \dot{z}(0)) = (u, v, w)$ with assumption $1 = u^2+v^2+w^2$ (that means arc-length parametrization).

We summarize the solutions for **Sol**-geodesics from paper [BS07].

Case 1) $u \neq 0 \neq v$, $0 < w^2 = 1 - u^2 - v^2 < 1$;

$$dt = \frac{\operatorname{sign}(w) \cdot dz}{\sqrt{1 - u^2 e^{-2z} - v^2 e^{2z}}},$$

since $1 - u^2 e^{-2z} - v^2 e^{2z} > 0$ holds, this leads to non-elementary function $t \to z(t)$ with $z(0) = 0$, $\dot{z}(0) = w$. Then $x(t) = u \int_0^t e^{-2z(\tau)} d\tau$, $y(t) = v \int_0^t e^{2z(\tau)} d\tau$.

Case 2) $u \neq 0 \neq v$, $w = 0 = 1 - u^2 - v^2$;

$$(x(t), y(t), z(t)) = (ut, vt, 0).$$

Case 3) $u \neq 0 = v$, $0 < w^2 = 1 - u^2 < 1$;

$$x(t) = \frac{u \sinh t}{\cosh t + w \sinh t}, \quad y(t) = 0, \quad z(t) = \ln(\cosh t + w \sinh t).$$

Case 4) $u = 0 \neq v$, $0 < w^2 = 1 - u^2 < 1$;

$$x(t) = 0, \quad y(t) = \frac{v \sinh t}{\cosh t - w \sinh t}, \quad z(t) = -\ln(\cosh t - w \sinh t).$$

Case 5) $u = 0 = v$, $w = \pm 1$;

$$x(t) = 0, \quad y(t) = 0, \quad z(t) = \operatorname{sign}(w) t.$$

Remark. By numerical solution we can visualize **Sol**-geodesics as well [BS07], moreover we can compare them with translation curves in Section 3.

3. Translation curves and sphere in Sol geometry

In the following we introduce a new family of curves, characteristic to **Sol** geometry. Namely, for a given starting tangent vector at the origin $(1, 0, 0, 0)$

$$u = \dot{x}(0), \quad v = \dot{y}(0), \quad w = \dot{z}(0),$$

we define its image in $(1, x(t), y(t), z(t))$ by the inverse of formula (2.3) so that

$$\begin{pmatrix} 0 & u & v & w \end{pmatrix} \begin{pmatrix} 1 & x(t) & y(t) & z(t) \\ 0 & e^{-z(t)} & 0 & 0 \\ 0 & 0 & e^{z(t)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \dot{x}(t) & \dot{y}(t) & \dot{z}(t) \end{pmatrix}. \tag{3.1}$$

This yields a curve starting at the origin in direction (u, v, w) with differential equation for $(x(t), y(t), z(t))$

$$\begin{aligned}\dot{x}(t) &= ue^{-z(t)},\\ \dot{y}(t) &= ve^{z(t)},\\ \dot{z}(t) &= w.\end{aligned} \tag{3.2}$$

The solution is easy in the following steps:

$$\begin{aligned}z(t) &= wt,\\ y(t) &= \int_0^t ve^{w\tau}\, d\tau = \frac{v}{w}\left(e^{wt} - 1\right),\\ x(t) &= \int_0^t ue^{-w\tau}\, d\tau = -\frac{u}{w}\left(e^{-wt} - 1\right).\end{aligned} \tag{3.3}$$

We see that any curve (3.3) has the prescribed tangent vector, say of unit length $\sqrt{u^2 + v^2 + w^2} = 1$ also by formula (2.4).

Our new curves seem to be more natural in **Sol** geometry than the geodesic lines. Their name may be chosen as *translation curves.*

With unit velocity translation curves, i.e. with arc-length parameter, we can define by (3.3) the sphere of radius r with centre in the origin of usual longitude and altidude parameters φ and ϑ, respectively:

$$\begin{aligned}u &= \cos\vartheta\cos\varphi & -\pi \le \varphi \le \pi\\ v &= \cos\vartheta\sin\varphi & -\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2}\\ w &= \sin\vartheta;\end{aligned}$$

$$\begin{aligned}x(\vartheta, \varphi) &= \left(1 - e^{-r\sin\vartheta}\right)\cot\vartheta\cos\varphi\\ y(\vartheta, \varphi) &= \left(e^{r\sin\vartheta} - 1\right)\cot\vartheta\sin\varphi\\ z(\vartheta, \varphi) &= r\sin\vartheta.\end{aligned} \tag{3.4}$$

Then we can visualize the "altitude circle" of constant ϑ, varying φ; and the "longitude half circle" of constant φ, varying ϑ.

Summarizing the arguments above we get

Theorem 3.1. *Translation curves in* **Sol** *geometry, starting at the origin in direction* (u, v, w)*, are described in formulas* (3.2), (3.3). *The unit velocity translation curves ending in parameter* r *describe the translation spheres (and balls), centred in the origin, of radius* r *by equations* (3.4).

These concepts can be extended to any homogeneous geometry where translations can naturally be defined. E.g. in each 3-dimensional Thurston geometry in projective-spherical interpretation [M97] this will be discussed.

4. Translation curves and spheres in Nil geometry

Let us consider **Nil** as another homogeneous 3-geometry. Werner HEISENBERG introduced his famous real matrix group $L(\mathbb{R})$ whose left (row-column) multiplication by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+xb+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix} \tag{4.1}$$

provided a new addition of points (translations) by

$$(x,y,z) * (a,b,c) = (a+x, b+y, c+xb+z). \tag{4.2}$$

Thus, our right translations can be written

$$\begin{pmatrix} 1 & p & q & r \end{pmatrix} \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+p & y+q & z+qx+r \end{pmatrix} \tag{4.3}$$

in homogeneous (affine-projective) interpretation.

Moreover, we have the "pull back transform"

$$\begin{pmatrix} 0 & dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & -x & -y & xy-z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & d\bar{x} & d\bar{y} & d\bar{z} \end{pmatrix} \tag{4.4}$$

for the basis differential forms at $(1,x,y,z)$ onto that at the origin, respectively. From this we obtain the infinitesimal arc-length-square at any point of **Nil** as follows

$$(ds)^2 = (d\bar{x})^2 + (d\bar{y})^2 + (d\bar{z})^2 = (dx)^2 + (dy)^2 + (-xdy+dz)^2. \tag{4.5}$$

Hence we get the symmetric metric tensor field g on **Nil** by components, furthermore its inverse:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+x^2 & -x \\ 0 & -x & 1 \end{pmatrix}, \qquad g^{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & 1+x^2 \end{pmatrix}.$$

The first author explicitly determined the geodesic curves (4.6), (4.7) in **Nil** geometry [M03].

Using (2.6) and (2.7) he obtained the system of geodesics to solve

$$\ddot{x} - (\dot{y})^2 x + \dot{y}\dot{z} = 0$$
$$\ddot{y} + \dot{x}\dot{y}x - \dot{x}\dot{z} = 0$$
$$\ddot{z} + \dot{x}\dot{y}(x^2-1) - \dot{x}\dot{z}x = 0$$

with

$$(x(0), y(0), z(0)) = (0,0,0) \quad \text{and} \quad (\dot{x}(0), \dot{y}(0), \dot{z}(0)) = (u, v, w)$$

as initial values.

(For simplicity – similar to the case of **Sol** – we can choose the origin as starting point, by homogeneity of **Nil**.)

In case $w \neq 0$ we get a helix-like solution for $(x(t), y(t), z(t))$ as follows

$$\begin{aligned} x(t) &= \frac{1}{w}\left[u \sin wt + v \cos wt - v\right], \\ y(t) &= \frac{1}{w}\left[-u \cos wt + v \sin wt + u\right], \\ z(t) &= wt + \frac{u^2+v^2}{2w}t - \frac{1}{2w^2}\big[(u^2-v^2)\sin wt \cos wt \\ &\quad + 2v^2 \sin wt + 2uv(\cos^2 wt + \cos wt - 2)\big]. \end{aligned} \tag{4.6}$$

If $w = 0$ than we have

$$(x(t), y(t), z(t)) = (ut, vt, \frac{uv}{2}t^2) \tag{4.7}$$

as a parabola on the hyperbolic paraboloid surface $2Z - XY = 0$.

The translation ((4.3) and its inverse in (4.4)) define the so-called translation curve with differential equation for $(x(t), y(t), z(t))$ analogously to (3.1):

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = vx + w \tag{4.8}$$

with

$$(x(0), y(0), z(0)) = (0, 0, 0) \quad \text{and} \quad (\dot{x}(0), \dot{y}(0), \dot{z}(0)) = (u, v, w).$$

The solution now is straightforward

$$x(t) = ut, \quad y(t) = vt, \quad z(t) = \frac{1}{2}uvt^2 + wt, \tag{4.9}$$

again more simple than the geodesics, coincidence occurs at $w = 0$.

Our **Nil** translation spheres can be defined again. The length of the tangent vectors are taken constant $1 = \sqrt{u^2 + v^2 + w^2}$. Thus with $u = \cos\vartheta\cos\varphi$, $v = \cos\vartheta\sin\varphi$, $w = \sin\vartheta$

$$\begin{aligned} x(\vartheta, \varphi) &= r\cos\vartheta\cos\varphi \\ y(\vartheta, \varphi) &= r\cos\vartheta\sin\varphi \\ z(\vartheta, \varphi) &= \frac{r^2}{2}\cos^2\vartheta\cos\varphi\sin\varphi + r\sin\vartheta \end{aligned} \tag{4.10}$$

just describe the translation sphere of radius r of centre in the origin. The φ- and ϑ-lines can be visualized again.

Theorem 4.1. *Translation curves and spheres (balls) in* **Nil** *geometry are summarized in formulas* (4.9) *and* (4.10), *respectively.*

5. Translation curves and spheres in $\widetilde{\mathbf{SL_2}\mathbb{R}}$

$\mathbf{SL_2}\mathbb{R}$ denotes the set of 2×2 real matrices $\left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right)$ now with unit determinant $ad - bc = 1$. That means, we have a 3-parametric group structure over reals. $\widetilde{\mathbf{SL_2}\mathbb{R}}$ denotes its universal covering space and group. This can be interpreted also as substitutions by linear fractions $z \to \frac{az+b}{cz+d}$ with complex variable $z \in \mathbb{C}^\infty$, i.e. extended with the infinity ∞ with usual operations. All these are the upper half plane model of the hyperbolic plane $\mathbf{H^2}$.

We can define the infinitesimal arc-length-square by the Lie algebra method. By the help of the projective model the definition can be formulated in another way initiated in [M97].

An advantage of this method lies in the fact that we get a unified geometrical model of this space in a one-sheeted hyperboloid [M97] analogous to the Cayley–Klein model of the hyperbolic space $\mathbf{H^3}$. Imagine our hyperboloid solid with a projective Cartesian coordinate simplex in $\mathbb{E}^3$: the origin and the ideal points of the axes and with a unit point (Figure 1). This will model $\widetilde{\mathbf{SL_2}\mathbb{R}}$ in our projective interpretation, if we choose an appropriate subgroup G as isometry group. Consider first the one parameter "screw" collineation group (say):

$$S(\varphi): \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \to \begin{pmatrix} \cos\varphi & \sin\varphi & & \\ -\sin\varphi & \cos\varphi & & \\ & & \cos\varphi & -\sin\varphi \\ & & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{5.1}$$

acting in projective space $\mathcal{P}^3$ for $-\frac{\pi}{2} < \varphi \leq \frac{\pi}{2}$ (mod π), then in projective sphere $\mathcal{PS}^3$ for $-\pi < \varphi \leq \pi$. This leaves invariant the following polarity by quadratic form and its hyperboloid solid

$$\mathcal{H}: \langle x, x \rangle := -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 < 0$$

of signature $(-,-,+,+)$. The $S(\varphi)$-orbits are mutually skew lines. The universal covering space $\widetilde{\mathcal{H}}$ of our hyperboloid $\mathcal{H}$, constructed by extending the fixed point free action of $S(\varphi)$ for any $\varphi \in R$, models our space $\widetilde{\mathcal{H}} = \widetilde{\mathbf{SL_2}\mathbb{R}}$.

The isometry group G of our space will be defined by the collineations leaving invariant our line bundle and our $(-,-,+,+)$-polarity, both. Thus we shall obtain the non-commutative subgroup of translations.

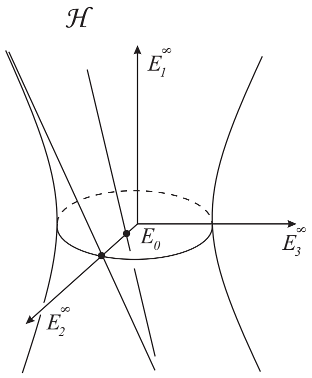

Figure 1. The one-parted hyperboloid model of $\widetilde{\mathbf{SL_2}\mathbb{R}}$ of skew line fibres growing in points of a hyperbolic base plane $\mathbf{H^2}$

Then by [M97] any translation, preserving the fibres by $S(\varphi)$ in (5.1), can be described by a matrix

$$(t_i^j) = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^2 & x^3 & x^0 & x^1 \\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix} \tag{5.2}$$

that means the origin $(0,0,0) \sim (1,0,0,0)$ goes into $(x,y,z) \sim (x^0, x^1, x^2, x^3)$ by $x = \frac{x^1}{x^0}$, $y = \frac{x^2}{x^0}$, $z = \frac{x^3}{x^0}$ if $x^0 \neq 0$, i.e. in the projective sphere $\mathcal{PS}^3$ with inverse

$$(t_i^j)^{-1} = \begin{pmatrix} x^0 & -x^1 & -x^2 & -x^3 \\ x^1 & x^0 & -x^3 & x^2 \\ -x^2 & -x^3 & x^0 & -x^1 \\ -x^3 & x^2 & x^1 & x^0 \end{pmatrix}$$

if $-x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 < 0$, e.g. fixed up to a positive factor, where $-x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 = 0$ is the boundary surface in $\mathbb{R}^4$.

Correspondence on the base of (5.2)

$$x^3 + x^0 =: a, \quad x^1 + x^2 =: b, \quad -x^1 + x^2 =: c, \quad -x^3 + x^0 =: d$$

as a coordinate transformation, implies just an isomorphism of corresponding translations

$$(x^0, x^1, x^2, x^3) \to \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \quad bc - ad < 0;$$

as at the beginning of this Section 5; $T^{-1}ST = S$ i.e. $ST = TS$ as a main motivation, where

$$(z^0, z^1) \to (z^0, z^1) \begin{pmatrix} d & b \\ c & a \end{pmatrix} \quad \Leftrightarrow \quad \frac{z^1}{z^0} =: z \to \frac{az+b}{cz+d}, \quad z \in \mathbb{C}^\infty$$

is a projective mapping of the complex projective line in homogeneous coordinates $(z^0 = 0 \leftrightarrow z = \infty)$.

Our problem now is to define translation curves $t \to (x^0(t), x^1(t), x^2(t), x^3(t))$ by

$$\begin{pmatrix} q & u & v & w \end{pmatrix} \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^2 & x^3 & x^0 & x^1 \\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix} = \begin{pmatrix} \dot{x}^0 & \dot{x}^1 & \dot{x}^2 & \dot{x}^3 \end{pmatrix}, \tag{5.3}$$

mapping the initial tangent vector (q, u, v, w) at the origin as starting point for $t = 0$ into the tangent vector at the curve point $(x^0(t), x^1(t), x^2(t), x^3(t))$ for any $0 < t$ by the translation matrix above, where $(q,u,v,w)=(\dot{x}^0(0), \dot{x}^1(0), \dot{x}^2(0), \dot{x}^3(0))$ is the given initial velocity of the curve $(x^0(t), x^1(t), x^2(t), x^3(t))$ to be determined.

Thus by (5.3) we get

$$\begin{aligned}\begin{pmatrix} x^0(t) & x^1(t) & x^2(t) & x^3(t) \end{pmatrix} \cdot \begin{pmatrix} q & u & v & w \\ -u & q & -w & v \\ v & -w & q & -u \\ w & v & u & q \end{pmatrix} \\ = \begin{pmatrix} \dot{x}^0(t) & \dot{x}^1(t) & \dot{x}^2(t) & \dot{x}^3(t) \end{pmatrix}\end{aligned} \tag{5.4}$$

in the hyperboloid model

$$-x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 < 0.$$

This is determined up to a positive constant factor

$$\begin{pmatrix} x^0 & x^1 & x^2 & x^3 \end{pmatrix} \sim \begin{pmatrix} cx^0 & cx^1 & cx^2 & cx^3 \end{pmatrix}, \quad c \in \mathbb{R}^+$$

i.e.

$$\left(1, \frac{x^1}{x^0} = x, \frac{x^2}{x^0} = y, \frac{x^3}{x^0} = z, \right)$$

will serve the usual Euclidean coordinates (x, y, z) at the end.

Well, we obtain a first order differential equation system with constant coefficients. The basic idea is well-known that we use the method of matrix exponential on the base of canonical Jordan form of matrix A.

Denote by (A) the constant coefficient matrix in (5.4). By the theory we define

$$\exp{(A\tau)} := E + A\tau + \frac{A^2}{2!}\tau^2 + \dots + \frac{A^n}{n!}\tau^2 + \dots$$

as usual,

$$\begin{aligned}\frac{d}{dt}\exp{(A\tau)} &:= A\left(E + A\tau + \dots + \frac{A^{n-1}}{(n-1)!}\tau^{n-1} + \dots\right) = A \cdot \exp{(A\tau)} \\ &:= \exp{(A\tau)} \cdot A\end{aligned}$$

by matrix entries as absolute convergent power series of τ.

Thus

$$\big(x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)\big) = \big(x^0(0), x^1(0), x^2(0), x^3(0)\big) \exp(A\tau) \tag{5.5}$$

will just be our translation curve, since by derivation

$$\big(\dot{x}^0(\tau), \dot{x}^1(\tau), \dot{x}^2(\tau), \dot{x}^3(\tau)\big) = \big(x^0(0), x^1(0), x^2(0), x^3(0)\big) \exp(A\tau) \cdot A$$

and we see by (5.5) that (5.4) is satisfied, with correct initial values.

Assume first that A has eigenvalues (may be equal) ${}^0\lambda$, ${}^1\lambda$, ${}^2\lambda$, ${}^3\lambda$ with linearly independent eigenvectors $\mathbf{s}_0$, $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$, respectively. That means $(\mathbf{s}_i A) = {}^i\lambda \mathbf{s}_i$. Take any $\mathbf{y} = y^i \mathbf{e}_i$, and say it holds

$$\mathbf{y}A = \begin{pmatrix} y^0 & y^1 & y^2 & y^3 \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} A = (y^i a_i^j \mathbf{e}_j), \tag{5.6}$$

i.e. $e_i A = a_i^j e_j = (A)\mathbf{e}_i$ defines the matrix $(A) = (a_i^j)$ of the linear mapping A in basis $(\mathbf{e}_i)$ by our convention.

Continuing (5.6), we have

$$\begin{aligned} \begin{pmatrix} y^0 & y^1 & y^2 & y^3 \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} A &= \begin{pmatrix} y^0 & y^1 & y^2 & y^3 \end{pmatrix} (S^{-1}) \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \end{pmatrix} A \\ &= \begin{pmatrix} y^0 & y^1 & y^2 & y^3 \end{pmatrix} (S^{-1}) \begin{pmatrix} {}^0\lambda & & & \\ & {}^1\lambda & & \\ & & {}^2\lambda & \\ & & & {}^3\lambda \end{pmatrix} \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \end{pmatrix} \\ &= \begin{pmatrix} y^0 & y^1 & y^2 & y^3 \end{pmatrix} (S^{-1}) \begin{pmatrix} {}^0\lambda & & & \\ & {}^1\lambda & & \\ & & {}^2\lambda & \\ & & & {}^3\lambda \end{pmatrix} (S) \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \end{aligned}$$

By comparison we get

$$(A) = (S^{-1}) \begin{pmatrix} {}^0\lambda & & & \\ & {}^1\lambda & & \\ & & {}^2\lambda & \\ & & & {}^3\lambda \end{pmatrix} (S) \tag{5.7}$$

thus

$$\exp(A\tau) = \left(S^{-1}\right) \begin{pmatrix} \exp({}^0\lambda\tau) & & & \\ & \exp({}^1\lambda\tau) & & \\ & & \exp({}^2\lambda\tau) & \\ & & & \exp({}^3\lambda\tau) \end{pmatrix} (S) \tag{5.8}$$

by this convention just provides us with the complete solution, if (A) has four independent eigenvectors.

Compute the eigenvalues! The characteristic equation is of the form

$$\det \begin{pmatrix} q-\lambda & u & v & w \\ -u & q-\lambda & -w & v \\ v & -w & q-\lambda & -u \\ w & v & u & q-\lambda \end{pmatrix} = (q-\lambda)^4 + 2(uu - vv - ww)(q-\lambda)^2$$
$$+ (uu - vv - ww)^2 = 0.$$

Therefore $-uu + vv + ww \neq 0$ will be important for the initial velocity.

Solving this equation for λ we have

$$\lambda = q \pm \sqrt{-uu + vv + ww} \Rightarrow {}^0\lambda = {}^1\lambda; \quad {}^2\lambda = {}^3\lambda \tag{5.9}$$

pairwise equal real eigenvalues, if $-u^2+v^2+w^2 > 0$. Denote $a := \sqrt{-u^2 + v^2 + w^2}$. We can find four independent eigenvectors, indeed, in a standard way. Then we get the solution (5.7), and we can explicitly write the equations of translation curves:

$$x^0(\tau) = e^{q\tau}\cosh(a\tau),$$
$$x^1(\tau) = e^{q\tau}\frac{u}{a}\sinh(a\tau),$$
$$x^2(\tau) = e^{q\tau}\frac{v}{a}\sinh(a\tau),$$
$$x^3(\tau) = e^{q\tau}\frac{w}{a}\sinh(a\tau),$$

that satisfy (5.3) and (5.4), in fact, for $a = \sqrt{-u^2 + v^2 + w^2}$. From above equations we obtain

$$x(\tau) = \frac{u}{a}\tanh(a\tau),$$
$$y(\tau) = \frac{v}{a}\tanh(a\tau),$$
$$z(\tau) = \frac{w}{a}\tanh(a\tau), \quad \tau \in \mathbb{R}, \tag{5.10}$$

i.e. straight lines in our hyperboloid model.

Second, we assume that $-u^2+v^2+w^2<0$. The eigenvalues are in this case:

$$\lambda = q \pm i\sqrt{u^2-v^2-w^2} \Rightarrow {}^0\lambda = {}^1\lambda; \quad {}^2\lambda = {}^3\lambda.$$

A similar method can be used to compute the matrix function as above. The computation is carried out analogously to that of the preceding case with the change that we write ia instead of a.

The equations of translation curve in this second case are

$$\begin{aligned} x^0(\tau) &= e^{q\tau}\cos(a\tau),\\ x^1(\tau) &= e^{q\tau}\frac{u}{a}\sin(a\tau),\\ x^2(\tau) &= e^{q\tau}\frac{v}{a}\sin(a\tau),\\ x^3(\tau) &= e^{q\tau}\frac{w}{a}\sin(a\tau), \end{aligned}$$

hence

$$\begin{aligned} x(\tau) &= \frac{u}{a}\tan(a\tau),\\ y(\tau) &= \frac{v}{a}\tan(a\tau),\\ z(\tau) &= \frac{w}{a}\tan(a\tau), \quad \tau\in\mathbb{R}, \end{aligned} \tag{5.11}$$

straight lines, again.

Third, let us suppose that $-u^2+v^2+w^2=0$. In this case the eigenvalues are equal to q, and we get only three independent eigenvectors. Therefore we turn back to the original equation (5.4).

The unique solution is straight line, simply of the form

$$x(\tau)=u\tau, \quad y(\tau)=v\tau, \quad z(\tau)=w\tau \tag{5.12}$$

as we can see by substitution.

With unit velocity we can define the translation sphere of radius r with centre in the origin:

$$\begin{aligned} u &= \sin\alpha, & -\frac{\pi}{2}<\alpha<\frac{\pi}{2}\\ v &= \cos\alpha\cos\theta, & -\pi<\theta\le\pi\\ w &= \cos\alpha\sin\theta. \end{aligned} \tag{5.13}$$

For the upper half translation sphere $0\le\alpha\le\frac{\pi}{2}$ can be assumed.

In the first case $a=\sqrt{-u^2+v^2+w^2}=\sqrt{-\sin^2\alpha+\cos^2\alpha}=\sqrt{\cos(2\alpha)}>0$, i.e. $0\le\alpha<\frac{\pi}{4}$; then a part of the half translation sphere of radius r is as follows

$$\begin{aligned}
x &= \frac{\sin\alpha}{\sqrt{\cos(2\alpha)}}\tanh(r\cdot\sqrt{\cos(2\alpha)}),\\
y &= \frac{\cos\alpha\cos\theta}{\sqrt{\cos(2\alpha)}}\tanh(r\cdot\sqrt{\cos(2\alpha)}),\\
z &= \frac{\cos\alpha\sin\theta}{\sqrt{\cos(2\alpha)}}\tanh(r\cdot\sqrt{\cos(2\alpha)}).
\end{aligned}\tag{5.14}$$

In the second case $a=\sqrt{u^2-v^2-w^2}=\sqrt{\sin^2\alpha-\cos^2\alpha}=\sqrt{-\cos(2\alpha)}>0$, i.e. $\frac{\pi}{4}\le\alpha<\frac{\pi}{2}$; then the upper part of translation sphere of radius r will be

$$\begin{aligned}
x &= \frac{\sin\alpha}{\sqrt{-\cos(2\alpha)}}\tan(r\cdot\sqrt{-\cos(2\alpha)}),\\
y &= \frac{\cos\alpha\cos\theta}{\sqrt{-\cos(2\alpha)}}\tan(r\cdot\sqrt{-\cos(2\alpha)}),\\
z &= \frac{\cos\alpha\sin\theta}{\sqrt{-\cos(2\alpha)}}\tan(r\cdot\sqrt{-\cos(2\alpha)}).
\end{aligned}\tag{5.15}$$

The case $0<r<\frac{\pi}{2}$ can be visualized in $\mathbb{E}^3$, $\frac{\pi}{2}\le r$ has no usual meaning in $\mathbb{E}^3$.

In the third case $a=\sqrt{-u^2+v^2+w^2}=0$ in the asymptotic direction of hyperboloid ("light direction"), we get

$$\alpha=\frac{\pi}{4},\quad u=\frac{\sqrt{2}}{2},\quad v=\frac{\sqrt{2}}{2}\cos\theta,\quad w=\frac{\sqrt{2}}{2}\sin\theta,\quad \text{and so}$$

$$x=\frac{\sqrt{2}}{2}r,\quad y=\frac{\sqrt{2}}{2}\cos\theta\cdot r,\quad z=\frac{\sqrt{2}}{2}\sin\theta\cdot r.\tag{5.16}$$

The continuity of the "meridian curve" at $\alpha=\frac{\pi}{4}$ is convincing for geometrical meaning. The cases $\frac{\pi}{2}\le r$ have to be imagined in the universal covering interpretation in $\widetilde{\mathcal{H}}=\widetilde{\mathbf{SL_2}\mathbb{R}}$ with fibre parameter $\frac{\pi}{2}\le\varphi$ at formula (5.1).

Theorem 5.1. *Translation curves and spheres in the one-parted hyperboloid model of* $\widetilde{\mathbf{SL_2}\mathbb{R}}$ *geometry are characterized in more steps by formulas* (5.10)–(5.12) *and* (5.13)–(5.16) *respectively.*

The geodesic curves in $\widetilde{\mathbf{SL_2}\mathbb{R}}$ are not straight lines in our hyperboloid model as paper [DESS09] shows by solving the usual non-linear second order ordinary differential equation system in an explicit way. All these visualizations are shown and animated in the conference paper [MSz09].

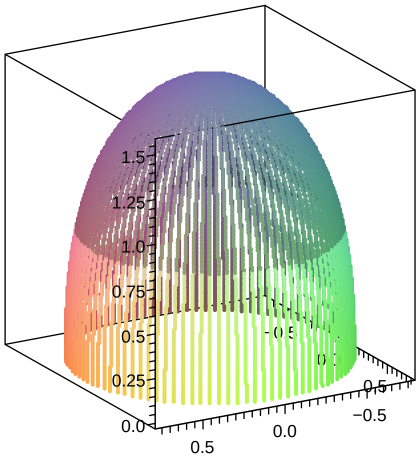

Figure 2. Translation half-sphere in $\widetilde{\mathbf{SL_2}\mathbb{R}}$

6. The simple geometries $\mathbb{E}^3$, $\mathbf{S}^3$, $\mathbf{H}^3$, $\mathbf{H}^2\times\mathbb{R}$, $\mathbf{S}^2\times\mathbb{R}$

In these geometries above the geodesic curves and translation curves coincide, if the initial point and starting unit velocity are the same.

Namely, the geodesics are straight lines in spaces $\mathbb{E}^3$, $\mathbf{S}^3$, $\mathbf{H}^3$ of constant zero, positive, negative curvature, respectively, also by their projective models [M97].

Translation curves will be also straight lines, although the orbits of a usual translation in spherical space $\mathbf{S}^3$ and in hyperbolic space $\mathbf{H}^3$ are not straight lines in general.

E.g. in $\mathbf{S}^2$ a translation is nothing but a rotation about opposite poles where the corresponding equator line (main circle) is the translation curve and geodesic curve at the same time. To arbitrary two points of $\mathbf{S}^2$ there exists such a translation (rotation); for opposite two points not uniquely.

In $\mathbf{H}^2$ a translation along a line is well defined as a product of two reflections in two appropriate lines perpendicular to the given line. The orbits are equidistants (hypercycles at $\mathbf{H}^2$, circles at $\mathbf{S}^2$) to the given straight line. The situation is analogous in $\mathbf{S}^3$ and $\mathbf{H}^3$. For $\mathbb{E}^2$ and $\mathbb{E}^3$ the questions are easy.

We consider now the projective models of $\mathbf{S}^2\times\mathbb{R}$ and $\mathbf{H}^2\times\mathbb{R}$ suggested in [M97].

a) For $\mathbf{S}^2\times\mathbb{R}$ we consider the projective space $\mathcal{P}^3(\mathbf{V}^4(\mathbb{R}), V_4)$ with dual basis pair $\{\mathbf{e}_i\} \subset \mathbf{V}^4$, $\{e^i\} \subset V_{\mathbf{4}}$ with $(\mathbf{e}_i, e^i) = \delta_i^j$ (Kronecker symbol, $i, j = 0, 1, 2, 3$), and with canonical degenerate (symmetric, linear) polarity

$$V_4 \underset{*}{\rightarrow} \mathbf{V}^4 : e^0_* = \mathbf{0}, \quad e^1_* = \mathbf{e_1}, \quad e^2_* = \mathbf{e_2}, \quad e^3_* = \mathbf{e_3} \tag{6.1}$$

of signature $(0, +, +, +)$.

We exclude from $\mathcal{P}^3$ the ideal points of the plane (e^0), furthermore the point $(\mathbf{e}_0)$ as origin.

We just give the "geographic" parametrization of $\mathbf{S^2} \times \mathbb{R}$ in $\mathbf{V}^4$

$$\begin{gathered} x^0 = 1, \quad x^1 = e^r \cos\varphi \cos\vartheta, \quad x^2 = e^r \sin\varphi \cos\vartheta, \quad x^3 = e^r \sin\vartheta \\ -\pi < \varphi \le \pi, \quad -\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2}, \quad r \in \mathbb{R} \end{gathered} \tag{6.2}$$

where (φ, ϑ) are the usual (longitude, altitude) coordinates of $\mathbf{S}^2$, r is the $\mathbb{R}$-component in the direct product $\mathbf{S^2} \times \mathbb{R}$. With $x = \frac{x^1}{x^0}$, $y = \frac{x^2}{x^0}$, $z = \frac{x^3}{x^0}$, $r = 0$ describes the unit sphere in (6.2), $r = \infty$ would be the ideal plane (e^0) at infinity, $r = -\infty$ would be the origin $(\mathbf{e}_0)$ in limit in $\mathbb{E}^3$ model. Central similarity with factor e^t means the translation by $\mathbb{R}$-component t. Indeed, such a similarity commutes with any isometry of $\mathbf{S}^2$, especially with $\mathbf{S}^2$ translation (rotation)

$$\begin{aligned} &(\varphi, \vartheta) \to (\varphi + \overline{\varphi}, \vartheta + \overline{0}) \quad \text{mapping} \quad (0,0) \to (\overline{\varphi}, \overline{0}) \\ &\text{along the equator line (circle)} \quad \vartheta = 0, \\ &\text{say, with unit velocity} \quad \dot{\varphi}(0) = 1; \quad \overline{\varphi} = 1 \cdot t \quad \text{by time } t. \end{aligned} \tag{6.3}$$

The usual arc-length-square in $\mathbf{S^2} \times \mathbb{R}$ is by [M97]

$$(ds)^2 = (d\varphi)^2 \cos^2\vartheta + (d\vartheta)^2 + (dr)^2. \tag{6.4}$$

This leads by (2.6), (2.7), (6.4) to geodesics of $\mathbf{S^2} \times \mathbb{R}$ by

$$\ddot{\varphi} - 2\tan\vartheta \dot{\varphi}\dot{\vartheta} = 0, \quad \ddot{\vartheta} + \sin\vartheta\cos\vartheta\dot{\varphi}^2 = 0, \quad \ddot{r} = 0. \tag{6.5}$$

The solution for starting point

$$\big(\varphi(0), \vartheta(0), r(0)\big) = (0, 0, 0), \quad \text{and} \quad \big(\dot{\varphi}(0), \dot{\vartheta}(0), \dot{r}(0)\big) = (\cos\theta, 0, \sin\theta)$$

i.e. with unit velocity, arc-length parameter s, will be

$$\varphi = \cos\theta \cdot s, \quad \vartheta = 0, \quad r = \sin\theta \cdot s. \tag{6.6}$$

The same will be for translation curve on the base of (6.3).

As a summary we formulate

Theorem 6.1. *The geodesic line and translation curve with equal initial values in the projective model* (6.2) *of* $\mathbf{S^2}\times\mathbb{R}$ *will be the same*

$$x^0=1,\quad x^1=e^{\sin\theta\cdot s}\cos(\cos\theta\cdot s),\quad x^2=e^{\sin\theta\cdot s}\sin(\cos\theta\cdot s),\quad x^3=0$$

with initial values

$$\begin{aligned}&x^0=1,\quad x^1(0)=1,\quad x^2(0)=0,\quad x^3(0)=0\qquad\text{and}\\&\dot{x}^0=0,\quad \dot{x}^1(0)=\sin\theta,\quad \dot{x}^2(0)=\cos\theta,\quad \dot{x}^3(0)=0.\end{aligned}\tag{6.7}$$

The geodesic and translation spheres of radius R*, with longitude* ϕ *and altitude* θ *and with centre, now in Euclidean coordinates* $\frac{x^1}{x^0}=x$, $\frac{x^2}{x^0}=y$, $\frac{x^3}{x^0}=z$,

$$\begin{aligned}&x(0)=1,\quad y(0)=0,\quad z(0)=0,\qquad\text{will be}\\&x=e^{R\sin\theta}\cos(R\cos\theta),\\&y=e^{R\sin\theta}\sin(R\cos\theta)\cos\phi,\\&z=e^{R\sin\theta}\sin(R\cos\theta)\sin\phi.\end{aligned}\tag{6.8}$$

b) For $\mathbf{H^2}\times\mathbb{R}$ our $\mathcal{P}^3(\mathbf{V}^4(\mathbb{R}),V_4)$, as before, will have the degenerate polarity

$$V_4\to\mathbf{V}^4:e_*^0=\mathbf{0},\quad e_*^1=-\mathbf{e_1},\quad e_*^2=\mathbf{e_2},\quad e_*^3=\mathbf{e_3}\tag{6.9}$$

of signature $(0,-,+,+)$. This leads to the so-called cone model of $\mathbf{H^2}\times\mathbb{R}$, now in hypercyclic (equidistance) parametrization (different from [M97] where polar coordinates of $\mathbf{H}^2$ were applied)

$$\begin{aligned}&x^0=1,\quad x^1=e^r\cosh\vartheta\cosh\varphi,\quad x^2=e^r\cosh\vartheta\sinh\varphi,\quad x^3=e^r\sinh\vartheta\\&-\infty<\varphi<\infty,\quad -\infty<\vartheta<\infty,\quad r\in\mathbb{R}\end{aligned}\tag{6.10}$$

the points of ideal line (e^0) and the origin $(\mathbf{e}_0)$ are excluded from $\mathcal{P}^3$. This is analogous to (6.2), (6.3); namely, a $\mathbf{H}^2$ translation will be described by

$$(\varphi,\vartheta)\to(\varphi+\overline{\varphi},\vartheta+\overline{0})\quad\text{mapping}\quad(0,0)\to(\overline{\varphi},\overline{0}).\tag{6.11}$$

With $x=\frac{x^1}{x^0}$, $y=\frac{x^2}{x^0}$, $z=\frac{x^3}{x^0}$, $r=0$ describes the upper part of unit two-parted (two-sheeted) hyperboloid in the positive cone $\mathcal{C}^+$ in $\mathbb{E}^3$ model. The central similarity with factor e^t means the translation by $\mathbb{R}$-component t, commuting with any isometry of any $\mathbf{H}^2$-level.

The corresponding arc-length-square in $\mathbf{H^2} \times \mathbb{R}$ is also well-known

$$(ds)^2 = (d\varphi)^2 \cosh^2 \vartheta + (d\vartheta)^2 + (dr)^2. \tag{6.12}$$

This leads by (2.6), (2.7), (6.12) to geodesics of $\mathbf{H^2} \times \mathbb{R}$ by

$$\ddot{\varphi} - 2 \tanh \vartheta \dot{\varphi} \dot{\vartheta} = 0, \quad \ddot{\vartheta} + \sinh \vartheta \cosh \vartheta \dot{\varphi}^2 = 0, \quad \ddot{r} = 0. \tag{6.13}$$

The solution for starting point $(\varphi(0), \vartheta(0), r(0)) = (0, 0, 0)$ and unit velocity $(\dot{\varphi}(0), \dot{\vartheta}(0), \dot{r}(0)) = (\cos\theta, 0, \sin\theta)$, i.e. with arc-length parameter s will be

$$\varphi = \cos\theta \cdot s, \quad \vartheta = 0, \quad r = \sin\theta \cdot s. \tag{6.14}$$

The same will be for translation curve on the base of (6.11).

Summarizing the above arguments we formulate

Theorem 6.2. *The geodesic line and translation curve with equal initial values in the projective model* (6.10) *of* $\mathbf{H^2} \times \mathbb{R}$ *will be the same*

$$x^0 = 1, \quad x^1 = e^{\sin\theta \cdot s} \cosh(\cos\theta \cdot s), \quad x^2 = e^{\sin\theta \cdot s} \sinh(\cos\theta \cdot s), \quad x^3 = 0$$

with initial values

$$\begin{aligned} &x^0 = 1, \quad x^1(0) = 1, \quad x^2(0) = 0, \quad x^3(0) = 0 \qquad \textit{and} \\ &\dot{x}^0 = 0, \quad \dot{x}^1(0) = \sin\theta, \quad \dot{x}^2(0) = \cos\theta, \quad \dot{x}^3(0) = 0. \end{aligned} \tag{6.15}$$

The geodesic and translation spheres of radius R, with longitude ϕ *and altitude* θ *and with centre, now in Euclidean coordinates* $\frac{x^1}{x^0} = x$, $\frac{x^2}{x^0} = y$, $\frac{x^3}{x^0} = z$,

$$\begin{aligned} x(0) = 1, \quad y(0) = 0, \quad z(0) = 0, \qquad &\textit{will be} \\ x = e^{R\sin\theta} \cosh(R\cos\theta), \\ y = e^{R\sin\theta} \sinh(R\cos\theta)\cos\phi, \\ z = e^{R\sin\theta} \sinh(R\cos\theta)\sin\phi. \end{aligned} \tag{6.16}$$

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