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Irreducible proportionally modular numerical semigroups

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Abstract. We study the minimal generating systems of the irreducible numerical semigroups that can be represented as the set of solutions to a Diophantine inequality of the form $ax \mod b \leq cx$ with a, b and c positive integers. In particular we obtain a simple method to count all these numerical semigroups with a given Frobenius number.

1. Introduction

A numerical semigroup is a subset S of $\mathbb{N}$ (here $\mathbb{N}$ denotes the set of nonnegative integers) that is closed under addition, contains the zero and its complement to $\mathbb{N}$ is finite.

Following the terminology used in [7], a proportionally modular Diophantine inequality is an expression of the form $ax \mod b \leq cx$, where a, b and c are positive integers. The set $\mathrm{S}(a,b,c)$ of integer solutions to this inequality is a numerical semigroup. We say that a numerical semigroup is proportionally modular if it is the set of integer solutions to a proportionally modular Diophantine inequality.

For a subset A of $\mathbb{R}_0^+$, we denote by $\langle A\rangle$ the submonoid of $(\mathbb{R}_0^+,+)$ generated by A, that is, $\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N},\ a_1,\ldots,a_n \in A \text{ and } \lambda_1,\ldots,\lambda_n \in \mathbb{N}\}$. If $A = \{a_1,\ldots,a_n\}$, we simply write $\langle a_1,\ldots,a_n\rangle$ instead of $\langle A\rangle$. If A is not empty, then $\mathrm{S}(A) = \langle A\rangle \cap \mathbb{N}$ is a submonoid of $\mathbb{N}$. It is proved in [10] that the class of proportionally modular numerical semigroups agrees with the class of numerical semigroups $\mathrm{S}(I)$ with I a non-trivial bounded interval of $\mathbb{R}_0^+$.

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It is well known (see for instance [6]) that every numerical semigroup S is finitely generated, that is, there exists $\{n_1, \ldots, n_p\} \subseteq S$ such that $S = \langle n_1, \ldots, n_p \rangle$. The set $\{n_1, \ldots, n_p\}$ is called a generating system of S, and if no proper subset of $\{n_1, \ldots, n_p\}$ generates S, $\{n_1, \ldots, n_p\}$ is said to be a minimal generating system of S. Every numerical semigroup S admits a unique minimal generating system (see [6]) and any element in this set is called a minimal generator of S. The cardinality of the minimal generating system of S is the embedding dimension of S and is denoted by $\mathrm{e}(S)$. The least element in the minimal generating system of S is called the multiplicity of S and is denoted by $\mathrm{m}(S)$.

We say that a nonempty subset A of $\mathbb{N}$ is independent if no element in A can be expressed as a linear combination of the remaining elements in A with all the coefficients in $\mathbb{N}$.

The largest integer not belonging to S is its Frobenius number and we denote it by $\mathrm{g}(S)$ (see [4]). A numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups containing it properly (see [5]). A numerical semigroup S is symmetric (respectively, pseudo-symmetric), if S is an irreducible numerical semigroup with $\mathrm{g}(S)$ odd (respectively, with $\mathrm{g}(S)$ even). This kind of numerical semigroups has been widely studied in the literature (see [1] and the references included there).

The contents of this paper are organized as follows. Taking as starting point the representations supplied in [10] for symmetric and pseudo-symmetric proportionally modular numerical semigroups and the concept of Bézout sequence introduced in [9], we describe in Section 3 the minimal generating system of symmetric proportionally modular numerical semigroups. Analogously in Section 4 we study the minimal generating system of pseudo-symmetric proportionally modular numerical semigroups. Finally in Section 5 we give a simple procedure to obtain the number of irreducible proportionally modular numerical semigroups with a given Frobenius number. This method extends the one given in [11] for symmetric proportionally modular numerical semigroups.

2. Preliminaries and basic results

As we have mentioned in the previous section, proportionally modular numerical semigroups can be defined from nonempty closed intervals of positive real numbers and alternatively from proportionally modular Diophantine inequalities. The following result, which is a reformulation of [7, Corollary 9], shows that we can restrict to closed intervals.

Proposition 1.

(1) *Let a, b and c be positive integers such that $c < a < b$. Then* $\mathrm{S}\big(\big[\frac{b}{a}, \frac{b}{a-c}\big]\big) = \mathrm{S}(a, b, c)$.

(1) *Conversely, if a_1, b_1, a_2 and b_2 are positive integers such that $\frac{b_1}{a_1} < \frac{b_2}{a_2}$, then* $\mathrm{S}\big(\big[\frac{b_1}{a_1}, \frac{b_2}{a_2}\big]\big) = \mathrm{S}(a_1b_2, b_1b_2, a_1b_2 - a_2b_1)$.

An ordered sequence of rational numbers $\frac{b_1}{a_1} < \cdots < \frac{b_p}{a_p}$ is a *Bézout sequence* if $p, a_1, \ldots, a_p$, $b_1, \ldots, b_p$ are positive integers, $p \geq 2$ and $a_ib_{i+1} - a_{i+1}b_i = 1$ for every $i \in \{1, \ldots, p-1\}$. The fractions $\frac{b_1}{a_1}$ and $\frac{b_p}{a_p}$ are said to be the ends of the sequence. Note that $\gcd(b_i, a_i) = 1$ for every $i \in \{1, \ldots, p\}$. If $1 \leq r < s \leq p$ are positive integers, then clearly $\frac{b_r}{a_r} < \cdots < \frac{b_s}{a_s}$ is a Bézout sequence. We will say that $\frac{b_1}{a_1} < \cdots < \frac{b_p}{a_p}$ is an extension of $\frac{b_r}{a_r} < \cdots < \frac{b_s}{a_s}$.

Bézout sequences are closely related to proportionally modular numerical semigroups.

Lemma 2 ([9, Theorem 12])**.** *Assume that $\frac{b_1}{a_1} < \cdots < \frac{b_p}{a_p}$ is a Bézout sequence. Then*

$$\langle b_1, \ldots, b_p \rangle = \mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_p}{a_p}\right]\right).$$

If $\frac{b_1}{a_1} < \cdots < \frac{b_p}{a_p}$ is a Bézout sequence and $b'_1, \ldots, b'_p$ is an arrangement of $b_1, \ldots, b_p$, then we say that $\frac{b_1}{a_1} < \cdots < \frac{b_p}{a_p}$ is a Bézout sequence for $b'_1, \ldots, b'_p$.

A Bézout sequence $\frac{b_1}{a_1} < \cdots < \frac{b_p}{a_p}$ is proper if $a_ib_{i+h} - a_{i+h}b_i \geq 2$ for all $h \geq 2$ such that $i, i+h \in \{1, \ldots, p\}$.

The following result is a part of [3, Theorem 7].

Lemma 3. *Let a_1, a_2, b_1, b_2 be positive integers such that $\frac{b_1}{a_1} < \frac{b_2}{a_2}$ and* $\gcd(a_1, b_1) = \gcd(a_2, b_2) = 1$*. Then there exists a unique proper Bézout sequence with ends $\frac{b_1}{a_1}$ and $\frac{b_2}{a_2}$.*

Moreover in [3] is given an algorithm to compute the unique proper Bézout sequence with two given ends.

Lemma 4 ([9, Corollary 18])**.** *Let $\frac{b_1}{a_1} < \cdots < \frac{b_p}{a_p}$ be a proper Bézout sequence. Then $b_1, \ldots, b_p$ is a convex sequence, that is, there exists $h \in \{1, \ldots, p\}$ such that*

$$b_1 \geq \cdots \geq b_h \leq \cdots \leq b_p.$$

Two fractions $\frac{b_i}{a_i} < \frac{b_j}{a_j}$ are said to be *adjacent* if

$$\frac{b_j}{a_j + 1} < \frac{b_i}{a_i}, \text{ and either } a_i = 1 \text{ or } \frac{b_j}{a_j} < \frac{b_i}{a_i - 1}.$$

In view of Lemma 2, the next result is an equivalent version of [9, Theorem 20].

Lemma 5. *If $\frac{b_1}{a_1} < \cdots < \frac{b_p}{a_p}$ is a proper Bézout sequence with adjacent ends, then the numerical semigroup* $\mathrm{S}\big(\big[\frac{b_1}{a_1}, \frac{b_p}{a_p}\big]\big)$ *is minimally generated by the set* $\{b_1, \dots, b_p\}$.

Note that the assumption "adjacent ends" is necessary in the previous lemma. The Bézout sequence $\frac{2}{1} < \frac{3}{1} < \frac{4}{1}$ is proper but the set $\{2, 3, 4\}$ is not independent.

The following lemma is an easy consequence of [10, Lemma 2] and will be used several times in this paper.

Lemma 6. *Let I be a non-trivial bounded interval of $\mathbb{R}_0^+$ and let x be a positive integer. Then $x \in \mathrm{S}(I)$ if and only if there exists a positive integer y such that $\frac{x}{y} \in I$.*

Given a numerical semigroup S, we define the set of gaps of S as $\mathrm{H}(S) = \mathbb{N} \backslash S$. The cardinality of $\mathrm{H}(S)$ is an important invariant of S and is called the singularity degree or genus of S (see [1]).

Following the notation in [10], a numerical semigroup S is a half-line if there exists a positive integer m such that $S = \{0\} \cup \{x \in \mathbb{Z} \mid x \geq m\}$, and it is opened modular if either S is a half-line or there exist positive integers a and b such that $2 \leq a < b$ and $S = \mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$.

As we will see below, opened modular numerical semigroups are closely related to irreducible proportionally modular numerical semigroups. The next result records some properties of these semigroups.

Theorem 7 ([10, Theorem 11]). *Let $1 < a < b$ be integers. Then $\mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$ is a proportionally modular numerical semigroup with Frobenius number b and singularity degree $\frac{1}{2}(b - 1 + d + d')$, where $d = \gcd(a, b)$ and $d' = \gcd(a-1, b)$.*

Proposition 8 ([10, Proposition 8]). *Let $x \in \mathbb{N}$. Then $x \in \mathrm{S}\big(\big[\frac{b}{a}, \frac{b}{a-1}\big]\big)$ and $x \notin \mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$ if and only if $x \in \big\{\lambda\frac{b}{d} \mid \lambda \in \{1, \dots, d\}\big\} \cup \big\{\lambda\frac{b}{d'} \mid \lambda \in \{1, \dots, d'\}\big\}$, where d, d' are as in Theorem 7.*

A numerical semigroup S is symmetric if $x \in \mathbb{Z} \backslash S$ implies $\mathrm{g}(S) - x \in S$, and it is pseudo-symmetric if $\mathrm{g}(S)$ is even and the only integer such that $x \in \mathbb{Z} \setminus S$ and $\mathrm{g}(S) - x \notin S$ is $x = \frac{\mathrm{g}(S)}{2}$ (see [1] and [2]).

The starting point for our study of the minimal generating systems of irreducible proportionally modular numerical semigroups is the following classification theorem.

Theorem 9 ([10, Theorem 20]). *Let S be a proportionally modular numerical semigroup. Then:*

(1) *S is symmetric if and only if either $S = \mathbb{N}$, $S = \langle 2, 3\rangle$ or $S = \mathrm{S}\big(\left]\frac{b}{a}, \frac{b}{a-1}\right[\big)$ where a and b are integers such that $2 \le a < b$ and $\gcd(a, b) = \gcd(a-1, b) = 1$.*

(2) *S is pseudo-symmetric if and only if either $S = \langle 3, 4, 5\rangle$ or $S = \mathrm{S}\big(\left]\frac{b}{a}, \frac{b}{a-1}\right[\big)$ where a and b are integers such that $2 \le a < b$ and $\{\gcd(a, b), \gcd(a-1, b)\} = \{1, 2\}$.*

3. The minimal generating system of a symmetric proportionally modular numerical semigroup

In view of Theorem 9 any symmetric proportionally modular numerical semigroup other than $\mathbb{N}$ and $\langle 2, 3\rangle$ is of the form $\mathrm{S}\big(\left]\frac{b}{a}, \frac{b}{a-1}\right[\big)$ where a and b are integers such that $2 \le a < b$ and $\gcd(a, b) = \gcd(a - 1, b) = 1$. Thus throughout this section a and b will represent two integers such that $2 \le a < b$ and $\gcd(a, b) = \gcd(a - 1, b) = 1$.

Lemma 10. *Let $\frac{b}{a} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p} < \frac{b}{a-1}$ be a proper Bézout sequence. Then:*

(1) *$\frac{b}{a} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p}$ is a proper Bézout sequence with adjacent ends.*

(2) *$\mathrm{S}\big(\left[\frac{b}{a}, \frac{b}{a-1}\right]\big)$ is minimally generated by $\{b, n_1, \ldots, n_p\}$.*

(3) *If $p = 1$, then $n_1 = 2$.*

(4) *If $p \ge 2$, then $\{2b, b + n_1, \ldots, b + n_p\} \subseteq \langle n_1, \ldots, n_p\rangle$.*

Proof. Let $S = \mathrm{S}\big(\left]\frac{b}{a}, \frac{b}{a-1}\right[\big)$ and $\overline{S} = \mathrm{S}\big(\left[\frac{b}{a}, \frac{b}{a-1}\right]\big)$.

(1) Clearly the Bézout sequence $\frac{b}{a} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p}$ is proper. We show that its ends are adjacent. As by hypothesis we know that $\frac{n_p}{c_p} < \frac{b}{a-1}$, we only have to check that $\frac{n_p}{c_p+1} < \frac{b}{a}$.

The inequality $\frac{n_p}{c_p} < \frac{b}{a-1}$ implies that $an_p < bc_p + n_p$. Since by hypothesis the Bézout sequence $\frac{b}{a} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p} < \frac{b}{a-1}$ is proper, by Lemma 4 we get that $n_p \le b$. Hence $an_p < bc_p + b$, that is, $\frac{n_p}{c_p+1} < \frac{b}{a}$.

(2) As we are assuming that $\frac{b}{a} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p} < \frac{b}{a-1}$ is a Bézout sequence, Lemma 2 assures that $\overline{S} = \langle b, n_1, \ldots, n_p\rangle$. By assertion (1) we know that $\frac{b}{a} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p}$ is a proper Bézout sequence with adjacent ends. By applying Lemma 5 to this Bézout sequence we get that $\overline{S}$ is minimally generated by $\{b, n_1, \ldots, n_p\}$.

(3) If $\frac{b}{a} < \frac{n_1}{c_1} < \frac{b}{a-1}$ is a Bézout sequence, then $an_1 - bc_1 = 1$ and $bc_1 - (a-1)n_1 = 1$, and this leads to $n_1 = 2$.

(4) The assumption $\gcd(a,b) = \gcd(a-1,b) = 1$ and Proposition 8 lead to $S = \overline{S} \setminus \{b\}$. By part (1) in Theorem 9 and Theorem 7 we have that S is a symmetric numerical semigroup with Frobenius number b.

By (2) we have that n_1 and b are different minimal generators of $\overline{S}$. This implies that $n_1 - b \notin \overline{S}$ and so $n_1 - b \notin S$. From this we deduce that $2b - n_1 \in S \subseteq \overline{S}$. As $\overline{S} = \langle b, n_1, \dots, n_p \rangle$, we get that $2b - n_1 = \lambda b + a_1 n_1 + \cdots + a_p n_p$ for some $\lambda, a_1, \dots, a_p \in \mathbb{N}$. Clearly, Lemma 6 implies that $\langle n_1, \dots, n_p \rangle \subseteq S$ and by Theorem 7 we have that $b \notin S$. We deduce from this that $\lambda = 0$ and so $2b \in \langle n_1, \dots, n_p \rangle$.

Now we fix $i \in \{1, \dots, p\}$ and choose $j \in \{1, \dots, p\} \setminus \{i\}$. Then n_i and n_j are different minimal generators of $\overline{S}$. This implies that $n_j - n_i \notin \overline{S}$ and so $n_j - n_i \notin S$. From this we get that $b + n_i - n_j \in S \subseteq \overline{S}$, that is, $b + n_i - n_j = \lambda b + a_1 n_1 + \cdots + a_p n_p$ for some $\lambda, a_1, \dots, a_p \in \mathbb{N}$. Since $\{b, n_1, \dots, n_p\}$ is the minimal generating system of $\overline{S}$, we obtain that $\lambda = 0$ and hence $b + n_i \in \langle n_1, \dots, n_p \rangle$. □

Theorem 11. *Let $\frac{b}{a} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p} < \frac{b}{a-1}$ be a proper Bézout sequence.*

(1) *If $p = 1$, then $\{2, b+2\}$ is the minimal generating system of $\mathrm{S}\big(\,\big]\frac{b}{a}, \frac{b}{a-1}\big[\,\big)$.*

(2) *If $p \geq 2$, then $\{n_1, \dots, n_p\}$ is the minimal generating system of $\mathrm{S}\big(\,\big]\frac{b}{a}, \frac{b}{a-1}\big[\,\big)$.*

Proof. Let $S = \mathrm{S}\big(\,\big]\frac{b}{a}, \frac{b}{a-1}\big[\,\big)$ and $\overline{S} = \mathrm{S}\big(\big[\frac{b}{a}, \frac{b}{a-1}\big]\big)$.

(1) Suppose that $p = 1$. By parts (2) and (3) in Lemma 10 we know that $\overline{S}$ is minimally generated by $\{2, b\}$. As by Proposition 8 we have that $S = \overline{S} \setminus \{b\}$, we deduce that $S = \langle 2, b+2 \rangle$.

(2) Suppose that $p \geq 2$. We know from assertion (2) in Lemma 10 that $\overline{S}$ is minimally generated by $\{b, n_1, \dots, n_p\}$, and from Proposition 8 that $S = \overline{S} \setminus \{b\}$. By taking into account part (4) in Lemma 10, it is easy to prove that S is minimally generated by $\{n_1, \dots, n_p\}$. □

Remark 12. Given integers $2 \leq a < b$ such that $\gcd(a,b) = \gcd(a-1,b) = 1$, and in view of Lemma 3, we can use [3, Algorithm 5] to get the only proper Bézout sequence with ends $\frac{b}{a}$ and $\frac{b}{a-1}$. Hence by applying Theorem 11 we obtain at once the minimal generating system of the symmetric proportionally modular numerical semigroup $\mathrm{S}\big(\,\big]\frac{b}{a}, \frac{b}{a-1}\big[\,\big)$.

Remark 13. Let $\frac{b}{a} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p} < \frac{b}{a-1}$ be a proper Bézout sequence with $p \geq 2$. By Theorem 11 and Lemma 2 we have $\mathrm{S}\big(\,\big]\frac{b}{a}, \frac{b}{a-1}\big[\,\big) = \mathrm{S}\big(\big[\frac{n_1}{c_1}, \frac{n_p}{c_p}\big]\big)$. By applying Proposition 1 we get

$$\mathrm{S}\left(\left]\frac{b}{a},\frac{b}{a-1}\right[\right)=\{x\in\mathbb{N}\mid c_1n_px \mod n_1n_p\le(c_1n_p-n_1c_p)x\}.$$

We close this section with an example.

Example 14. Let $b=5$, $a=2$ and consider the numerical semigroup $S=\mathrm{S}\big(\big]\frac{b}{a},\frac{b}{a-1}\big[\big)$. As $\gcd(a,b)=\gcd(a-1,b)=1$, by Theorems 7 and 9 we have that S is a symmetric proportionally modular numerical semigroup with Frobenius number 5.

It can be easily checked that $\frac{5}{2}<\frac{3}{1}<\frac{4}{1}<\frac{5}{1}$ is a proper Bézout sequence. Hence, by Theorem 11 we get that $S=\langle 3,4\rangle$. Finally, by Remark 13 we have

$$S=\mathrm{S}\left(\left[\frac{3}{1},\frac{4}{1}\right]\right)=\{x\in\mathbb{N}\mid 4x \mod 12\le x\}=\mathrm{S}(4,12,1).$$

4. The minimal generating system of a pseudo-symmetric proportionally modular numerical semigroup

By Theorem 9 any pseudo-symmetric proportionally modular numerical semigroup other than $\langle 3,4,5\rangle$ is of the form $\mathrm{S}\big(\big]\frac{b}{a},\frac{b}{a-1}\big[\big)$, where a and b are integers such that $2\le a<b$ and $\{\gcd(a,b),\gcd(a-1,b)\}=\{1,2\}$. In this section we use this representation to determine the minimal generating systems of pseudo-symmetric proportionally modular numerical semigroups.

Let a and b be integers such that $2\le a<b$. We note that $2\le b+1-a<b$, $\gcd(a,b)=\gcd(b-a,b)$ and $\gcd(a-1,b)=\gcd(b+1-a,b)$.

Lemma 15. *If a and b are integers such that $2\le a<b$, then $\mathrm{S}\big(\big]\frac{b}{a},\frac{b}{a-1}\big[\big)=\mathrm{S}\big(\big]\frac{b}{b+1-a},\frac{b}{b-a}\big[\big)$.*

Proof. By Lemma 6, it is enough to observe that $\frac{b}{a}<\frac{x}{y}<\frac{b}{a-1}$ if and only if $\frac{b}{b+1-a}<\frac{x}{x-y}<\frac{b}{b-a}$. □

As a consequence of Lemma 15 and Theorem 9 we obtain the following proposition.

Proposition 16. *Let S be a numerical semigroup. Then S is proportionally modular and pseudo-symmetric if and only if either $S=\langle 3,4,5\rangle$ or $S=\mathrm{S}\big(\big]\frac{b}{a},\frac{b}{a-1}\big[\big)$ where a and b are integers such that $2\le a<b$, $\gcd(a,b)=2$ and $\gcd(a-1,b)=1$.*

In view of Proposition 16, in the rest of this section a and b will represent integers such that $2 \leq a < b$, $\gcd(a,b) = 2$ and $\gcd(a-1,b) = 1$.

Lemma 17. *If* $\frac{b/2}{a/2} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p} < \frac{b}{a-1}$ *is a proper Bézout sequence, then:*

(1) $\frac{b/2}{a/2} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p}$ *is a proper Bézout sequence with adjacent ends.*

(2) $\mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-1}\right]\right)$ *is minimally generated by* $\left\{\frac{b}{2}, n_1, \ldots, n_p\right\}$.

(3) $\frac{b/2}{a/2} < \frac{\frac{b}{2}+n_1}{\frac{a}{2}+c_1} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p} < \frac{b}{a-1}$ *is a Bézout sequence.*

(4) $\langle n_1, \ldots, n_p, \frac{b}{2} + n_1\rangle \subseteq \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$.

(5) $3\frac{b}{2} = (a_1+1)n_1 + a_2n_2 + \cdots + a_pn_p$ *for some* $a_1, a_2, \ldots, a_p \in \mathbb{N}$.

(6) $2b \in \langle n_1, \ldots, n_p, \frac{b}{2} + n_1\rangle$.

(7) *If* $p = 1$, *then* $n_1 = 3$.

(8) *If* $p \geq 2$, *then* $b + n_p \in \langle n_1, \ldots, n_p, \frac{b}{2} + n_1\rangle$.

(9) *If* $p \geq 2$, *then* $\mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right) = \langle n_1, \ldots, n_p, \frac{b}{2} + n_1\rangle$.

Proof. Let $S = \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$ and $\overline{S} = \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-1}\right]\right)$. As $\gcd(a,b) = 2$ and $\gcd(a-1,b) = 1$, by Proposition 8 we have $S = \overline{S} \setminus \{b, \frac{b}{2}\}$ and by Theorems 7 and 9, we know that S is a pseudo-symmetric numerical semigroup with Frobenius number b.

(1) It is immediate from hypothesis that $\frac{b/2}{a/2} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p}$ is a proper Bézout sequence. If $a > 2$, as $\frac{b}{a-1} < \frac{\frac{b}{2}}{\frac{a}{2}-1}$, we get that $\frac{n_p}{c_p} < \frac{\frac{b}{2}}{\frac{a}{2}-1}$. By proceeding as in the proof of part (1) in Lemma 10, we obtain that $\frac{n_p}{c_p+1} < \frac{b}{a} = \frac{b/2}{a/2}$, and this proves (1).

(2) The proof is similar to that of assertion (2) in Lemma 10.

(3) It can be easily checked from hypothesis.

(4) It is immediate from Lemma 6 and (3).

(5) As n_1 and $\frac{b}{2}$ are different elements of the minimal generating system of $\overline{S}$, we have that $n_1 - \frac{b}{2} \notin \overline{S}$, and so $n_1 - \frac{b}{2} \notin S$. This implies either $b - (n_1 - \frac{b}{2}) \in S$ or $n_1 - \frac{b}{2} = \frac{b}{2}$. If $n_1 - \frac{b}{2} = \frac{b}{2}$, then $n_1 = b$ which is in contradiction with $b \notin S$. Hence $3\frac{b}{2} - n_1 \in \overline{S}$. By part (2) $\overline{\mathrm{S}}$ is minimally generated by $\left\{\frac{b}{2}, n_1, \ldots, n_p\right\}$, so there exist $\lambda, a_1, \ldots, a_p \in \mathbb{N}$ such that $3\frac{b}{2} - n_1 = \lambda\frac{b}{2} + a_1n_1 + \cdots + a_pn_p$, that is, $(3-\lambda)\frac{b}{2} = (a_1+1)n_1 + a_2n_2 + \cdots + a_pn_p$. Since $\left\{\frac{b}{2}, b\right\} \cap S = \emptyset$ we deduce that $\lambda = 0$, and therefore $3\frac{b}{2} = (a_1+1)n_1 + a_2n_2 + \cdots + a_pn_p$.

(6) We know from (2) that n_1 is a minimal generator of $\overline{S}$ and from Lemma 6 that $b \in \overline{S}$, so $n_1 \neq b$. This implies that $n_1 - b \notin \overline{S}$. From this we get either

$b-(n_1-b)\in S$ or $n_1-b=\frac{b}{2}$. Note that the case $n_1-b=\frac{b}{2}$ is not possible, because this would contradict the fact that n_1 is a minimal generator of $\overline{S}$. Hence $2b-n_1\in\overline{S}$ and this implies that $2b-n_1=\lambda\frac{b}{2}+a_1n_1+\cdots+a_pn_p$ for some $\lambda,a_1,\ldots,a_p\in\mathbb{N}$. Taking into account that $\left\{\frac{b}{2},b\right\}\cap S=\emptyset$, it follows that $\lambda\in\{0,1\}$ and so $2b\in\left\langle n_1,\ldots,n_p,\frac{b}{2}+n_1\right\rangle$.

(7) If $\frac{b/2}{a/2}<\frac{n_1}{c_1}<\frac{b}{a-1}$ is a Bézout sequence, then $\frac{a}{2}n_1-\frac{b}{2}c_1=1$ and $bc_1-n_1(a-1)=1$. It is straightforward to check from this that $n_1=3$.

(8) Since n_1 and n_p are different minimal generators of $\overline{S}$, we have that $n_1-n_p\notin\overline{S}$ and so $n_1-n_p\notin S$. This leads to either $b-(n_1-n_p)\in S$ or $n_1-n_p=\frac{b}{2}$. We note that the case $n_1-n_p=\frac{b}{2}$ is not possible because by (2) we know that $n_1,n_p,\frac{b}{2}$ are minimal generators of $\overline{S}$. So $b+n_p-n_1\in\overline{S}$ and there exist $\lambda,a_1,\ldots,a_p\in\mathbb{N}$ such that $b+n_p-n_1=\lambda\frac{b}{2}+a_1n_1+\cdots+a_pn_p$. We deduce from this that $\lambda\in\{0,1\}$ and in consequence that $b+n_p\in\langle n_1,\ldots,n_p,\frac{b}{2}+n_1\rangle$.

(9) We already know from (4) that $\langle n_1,\ldots,n_p,\frac{b}{2}+n_1\rangle\subseteq S$. Now we prove the other inclusion. Let $x\in S$. By Lemma 6 there exists a positive integer k such that $\frac{b/2}{a/2}<\frac{x}{k}<\frac{b}{a-1}$. We consider the following three cases:

   (a) If $\frac{n_1}{c_1}\le\frac{x}{k}\le\frac{n_p}{c_p}$, then by Lemma 6 we get that $x\in \mathrm{S}\left(\left[\frac{n_1}{c_1},\frac{n_p}{c_p}\right]\right)$. By Lemma 2 this implies that $x\in\langle n_1,\ldots,n_p\rangle$.

   (b) If $\frac{n_p}{c_p}<\frac{x}{k}<\frac{b}{a-1}$, then by Lemma 6 we obtain that $x\in \mathrm{S}\left(\left[\frac{n_p}{c_p},\frac{b}{a-1}\right]\right)$. As $\frac{n_p}{c_p}<\frac{b}{a-1}$ is a Bézout sequence, by Lemma 2 we get that $x\in\langle n_p,b\rangle$. Note that $x\neq b$ because $x\in S$, so $x=\lambda b+\mu n_p$ with $\lambda,\mu\in\mathbb{N}$ and $(\lambda,\mu)\neq(1,0)$. By using assertions (5), (6) and (8) we reach that $x\in\langle n_1,\ldots,n_p,\frac{b}{2}+n_1\rangle$.

   (c) If $\frac{b/2}{a/2}<\frac{x}{k}<\frac{n_1}{c_1}$, then by Lemma 6 we obtain that $x\in \mathrm{S}\left(\left[\frac{b/2}{a/2},\frac{n_1}{c_1}\right]\right)$. Since $\frac{b/2}{a/2}<\frac{n_1}{c_1}$ is a Bézout sequence, by Lemma 2 we get that $x\in\left\langle\frac{b}{2},n_1\right\rangle$. As we are assuming that $x\in S$, this implies that $x\notin\left\{\frac{b}{2},b\right\}$ and so $x=\lambda\frac{b}{2}+\mu n_1$ with $\lambda,\mu\in\mathbb{N}$ and $(\lambda,\mu)\notin\{(1,0),(2,0)\}$. By using parts (5) and (6) we obtain that $x\in\left\langle n_1,\ldots,n_p,\frac{b}{2}+n_1,b+n_1\right\rangle$.

Hence we have that $S\subseteq\left\langle n_1,\ldots,n_p,\frac{b}{2}+n_1,b+n_1\right\rangle$. Note that $b+n_1\in S$ and consequently $S=\left\langle n_1,\ldots,n_p,\frac{b}{2}+n_1,b+n_1\right\rangle$. To conclude the proof, we show that $b+n_1$ is not a minimal generator of S. By proceeding as in the proof of (8) we deduce that $n_p-n_1\notin S$. From this we get either $b-(n_p-n_1)\in S$ or $n_p-n_1=\frac{b}{2}$. The equality $n_p-n_1=\frac{b}{2}$ contradicts the fact that n_p is a minimal generator of $\overline{S}$. Thus $b+n_1-n_p\in S$ and we see from this that $b+n_1$ cannot be a minimal generator of S. □

Lemma 18 ([5, Theorem 7])**.** *The following conditions are equivalent:*

(1) *S is an irreducible numerical semigroup with* $\mathrm{m}(S) = \mathrm{e}(S) = 3$,

(2) *S is generated by* $\{3, x+3, 2x+3\}$ *with x not a multiple of* 3.

Remark 19. Under the notation of the previous Lemma, we point out that the integer $\mathrm{g}(S)$ is even and $x = \frac{\mathrm{g}(S)}{2}$.

Theorem 20. *Let* $\frac{b/2}{a/2} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p} < \frac{b}{a-1}$ *be a proper Bézout sequence.*

(1) *If* $p=1$, *then* $\left\{3, \frac{b}{2}+3, b+3\right\}$ *is the minimal generating system of* $\mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$.

(2) *If* $p \geq 2$, *then* $\{n_1, \ldots, n_p, \frac{b}{2} + n_1\}$ *is the minimal generating system of* $\mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$.

Proof.

(1) Let $S = \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$ and suppose that $p = 1$. From parts (7) and (4) in Lemma 17 it follows that $n_1 = 3$ and $\left\{3, \frac{b}{2}+3\right\} \subseteq S$. We know from Theorem 7 that $\mathrm{g}(S) = b$ and so $b + 3 \in S$. Hence $\left\langle 3, \frac{b}{2} + 3, b + 3\right\rangle \subseteq S$. As $\frac{b/2}{a/2} < \frac{n_1}{c_1}$ is a Bézout sequence, we have $\gcd(\frac{b}{2}, 3) = 1$. From this observation and the fact that $2\left(\frac{b}{2} + 3\right) > b + 3$, it is straightforward to check that the set $\left\{3, \frac{b}{2} + 3, b + 3\right\}$ is independent. This in particular implies that $\mathrm{e}(S) = 3$. Now the conclusion follows by applying Lemma 18 and Remark 19.

(2) We first prove that $\left\{n_1, \ldots, n_p, \frac{b}{2} + n_1\right\}$ is an independent set. Note that by Lemma 17 (2), $\frac{b}{2} \neq n_1$. We distinguish two cases:

   (a) Suppose that $n_1 > \frac{b}{2}$. Then by Lemma 4 we have $\frac{b}{2} < n_1 < \cdots < n_p < b$. Note that as a consequence of part (2) in Lemma 17 the set $\{n_1, \ldots, n_p\}$ is independent. Thus it is enough to show that $\frac{b}{2} + n_1 \notin \langle n_1, \ldots, n_p\rangle$. If $\frac{b}{2} + n_1 \in \langle n_1, \ldots, n_p\rangle$, then taking into account the inequalities $\frac{b}{2} < n_1 < \cdots < n_p < b$ we get that $\frac{b}{2} + n_1 = n_j$ for some $j \in \{2, \ldots, p\}$. But this is impossible because $\frac{b}{2} + n_1 > b$, so the set $\left\{n_1, \ldots, n_p, \frac{b}{2} + n_1\right\}$ is independent.

   (b) Assume now that $\frac{b}{2} > n_1$. We show that $\frac{\frac{b}{2}+n_1}{\frac{a}{2}+c_1} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p} < \frac{b}{a-1}$ is a proper Bézout sequence with adjacent ends. To see that this sequence is proper, it is enough to check that $\left(\frac{a}{2} + c_1\right)n_i - \left(\frac{b}{2} + n_1\right)c_i \geq 2$ for all $i \geq 2$ and $\left(\frac{a}{2} + c_1\right)b - \left(\frac{b}{2} + n_1\right)(a-1) \geq 2$. As by hypothesis we have $\frac{a}{2}n_i - \frac{b}{2}c_i \geq 2$ and $c_1 n_i - n_1 c_i \geq 1$ for all $i \geq 2$, we get that $\left(\frac{a}{2} + c_1\right)n_i - \left(\frac{b}{2} + n_1\right)c_i = \frac{a}{2}n_i - \frac{b}{2}c_i + c_1 n_i - n_1 c_i \geq 2$ for all $i \geq 2$. The inequality $\left(\frac{a}{2} + c_1\right)b - \left(\frac{b}{2} + n_1\right)(a-1) \geq 2$ follows easily by taking into account that $\frac{n_1}{c_1} < \frac{b}{a-1}$, and so that $bc_1 + n_1 - an_1 \geq 1$.

Now we prove the condition on adjacent ends. We know from assumption that $\frac{b/2}{a/2} < \frac{n_1}{c_1}$. This implies that $\frac{b/2}{a/2} < \frac{\frac{b}{2}+n_1}{\frac{a}{2}+c_1} < \frac{n_1}{c_1}$ and so that $b\big(\frac{a}{2}+c_1\big) < a\big(\frac{b}{2}+n_1\big)$.

Now we check that $\frac{b}{a-1} < \frac{\frac{b}{2}+n_1}{\frac{a}{2}+c_1-1}$, that is, $b\big(\frac{a}{2}+c_1\big) - b < a\big(\frac{b}{2}+n_1\big) - \big(\frac{b}{2}+n_1\big)$. But this is clear from the hypothesis $\frac{b}{2} > n_1$ and from the inequality $b\big(\frac{a}{2}+c_1\big) < a\big(\frac{b}{2}+n_1\big)$.

Thus $\frac{\frac{b}{2}+n_1}{\frac{a}{2}+c_1} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p} < \frac{b}{a-1}$ is a proper Bézout sequence with adjacent ends. By applying Lemma 5 we deduce that the set $\big\{\frac{b}{2}+n_1, n_1,\ldots,n_p,b\big\}$ is independent, and so also is the set $\big\{\frac{b}{2}+n_1,n_1,\ldots,n_p\big\}$.

Finally, in view of assertion (9) in Lemma 17, the conclusion follows.

Remark 21. Given integers $2 \le a < b$ such that $\gcd(a,b) = 2$ and $\gcd(a-1,b) = 1$, and in view of Lemma 3 we can apply [3, Algorithm 5] to get the only proper Bézout sequence with ends $\frac{b/2}{a/2}$ and $\frac{b}{a-1}$. Hence by using Theorem 20 we obtain directly the minimal generating system of the pseudo-symmetric proportionally modular numerical semigroup $\mathrm{S}\big(\big]\frac{b}{a},\frac{b}{a-1}\big[\big)$.

Remark 22. As a consequence of Theorem 20, Lemma 2 and assertion (3) in Lemma 17, if $\frac{b/2}{a/2} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p} < \frac{b}{a-1}$ is a proper Bézout sequence, then we have $\mathrm{S}\big(\big]\frac{b}{a},\frac{b}{a-1}\big[\big) = \big\langle n_1,\ldots,n_p,\frac{b}{2}+n_1\big\rangle = \mathrm{S}\big(\big[\frac{\frac{b}{2}+n_1}{\frac{a}{2}+c_1},\frac{n_p}{c_p}\big]\big)$. By using Proposition 1 we get that

$$\mathrm{S}\left(\left]\frac{b}{a},\frac{b}{a-1}\right[\right)$$

$$=\left\{x \in \mathbb{N} : n_p\left(\frac{a}{2}+c_1\right)x \mod n_p\left(\frac{b}{2}+n_1\right) \le \left(n_p\left(\frac{a}{2}+c_1\right) - c_p\left(\frac{b}{2}+n_1\right)\right)x\right\}.$$

Example 23. Let $S = \mathrm{S}\big(\big]\frac{14}{4},\frac{14}{3}\big[\big)$. Since $\gcd(4,14) = 2$ and $\gcd(4-1,14) = 1$, by Theorems 7 and 9 we know that S is a pseudo-symmetric proportionally modular numerical semigroup with Frobenius number 14. Note that $\frac{7}{2} < \frac{4}{1} < \frac{9}{2} < \frac{14}{3}$ is a proper Bézout sequence. By Theorem 20 we obtain that $\{4, 9, \frac{14}{2}+4\}$ is the minimal generating system of S and so $S = \langle 4, 9, 11\rangle$. Finally by Remark 22 we get $S = \mathrm{S}([\frac{11}{3},\frac{9}{2}]) = \{x \in \mathbb{N} \mid 27x \mod 99 \le 5x\} = \mathrm{S}(27, 99, 5)$.

5. The number of irreducible proportionally modular numerical semigroups with a given Frobenius number

In [11] it is presented an easy algorithm to compute the number of symmetric proportionally modular numerical semigroups with a given Frobenius number. In this section we obtain a similar method to get the number of pseudo-symmetric proportionally modular numerical semigroups with a given Frobenius number. Hence we have an algorithm to compute the number of irreducible proportionally modular numerical semigroups with a given Frobenius number. In view of Theorem 7, this procedure also counts the number of irreducible proportionally modular numerical semigroups with a given genus or singularity degree.

For two positive integers x, y such that $\gcd(x, y) = 1$, we denote by $x^{-1} \mod y$ the least positive integer u such that $xu \equiv 1 \mod y$.

Given a numerical semigroup S, if $\{n_1, n_2, \ldots, n_p\}$ is the minimal generating system of S, with $n_1 < n_2 < \cdots < n_p$ and $p \geq 2$, we will denote n_2 by $\mathrm{r}(S)$.

The next result is an easy consequence of the definitions and Lemmas 3 and 4 and is the key to prove Proposition 26.

Lemma 24. *Let $\{n_1, n_2, \ldots, n_p\}$ be the minimal generating system of a proportionally modular numerical semigroup S with $n_1 < n_2 < \cdots < n_p$. Then $\gcd(n_1, n_2) = 1$. Moreover, if $u \in \{1, \ldots, n_1 - 1\}$ and $v \in \{1, \ldots, n_2 - 1\}$ are such that $un_2 - vn_1 = 1$, then any proper Bézout sequence for $n_1, \ldots, n_p$, with fractions greater than one, is an extension of either $\frac{n_1}{u} < \frac{n_2}{v}$ or $\frac{n_2}{n_2 - v} < \frac{n_1}{n_1 - u}$.*

Under the same setting as in Lemma 24, we note that $u = n_2^{-1} \mod n_1$.

The next result is an immediate consequence of Lemma 24.

Lemma 25. *Let $n_1 < n_2 < \cdots < n_p$ be positive integers, let σ be a permutation of the set $\{1, \ldots, p\}$ and let $\frac{n_{\sigma(1)}}{c_1} < \cdots < \frac{n_{\sigma(p)}}{c_p}$ be a proper Bézout sequence with $1 < \frac{n_{\sigma(1)}}{c_1}$. Then*

$$\left\{ \frac{n_1}{n_2^{-1} \mod n_1}, \frac{n_1}{n_1 - (n_2^{-1} \mod n_1)} \right\} \cap \left\{ \frac{n_{\sigma(1)}}{c_1}, \ldots, \frac{n_{\sigma(p)}}{c_p} \right\} \neq \emptyset.$$

Proposition 26. *Let $b \geq 3$ be an integer and let $a, a' \in \{2, \ldots, b-1\}$. Let $S = \mathrm{S}\left(\left] \frac{b}{a}, \frac{b}{a-1} \right[\right)$ and $S' = \mathrm{S}\left(\left] \frac{b}{a'}, \frac{b}{a'-1} \right[\right)$. If $\mathrm{m}(S) = \mathrm{m}(S')$ and $\mathrm{r}(S) = \mathrm{r}(S')$, then either $a = a'$ or $a + a' = b + 1$.*

Proof. Suppose that $\{n_1, \ldots, n_p\}$ is the minimal generating system of S. By [9, Lemma 22 and Theorem 23], there exist positive integers $c_1, \ldots, c_p$ such that $\frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p}$ is a proper Bézout sequence with adjacent ends and $\frac{b}{a} < \frac{n_1}{c_1} < \cdots < \frac{n_p}{c_p} < \frac{b}{a-1}$. In view of Lemma 25 we consider the following cases:

(1) If $\frac{b}{a} < \frac{\mathrm{m}(S)}{\mathrm{r}(S)^{-1} \mod \mathrm{m}(S)} < \frac{b}{a-1}$, since these inequalities are equivalent to $\frac{b(\mathrm{r}(S)^{-1} \mod \mathrm{m}(S))}{\mathrm{m}(S)} < a < \frac{b(\mathrm{r}(S)^{-1} \mod \mathrm{m}(S))}{\mathrm{m}(S)} + 1$, it follows that there is at most one positive integer a such that $\frac{b}{a} < \frac{\mathrm{m}(S)}{\mathrm{r}(S)^{-1} \mod \mathrm{m}(S)} < \frac{b}{a-1}$.

(2) If $\frac{b}{a} < \frac{\mathrm{m}(S)}{\mathrm{m}(S)-(\mathrm{r}(S)^{-1} \mod \mathrm{m}(S))} < \frac{b}{a-1}$, as these inequalities are equivalent to $\frac{b(\mathrm{m}(S)-(\mathrm{r}(S)^{-1} \mod \mathrm{m}(S)))}{\mathrm{m}(S)} < a < \frac{b(\mathrm{m}(S)-(\mathrm{r}(S)^{-1} \mod \mathrm{m}(S)))}{\mathrm{m}(S)} + 1$, it follows that there is at most one positive integer a such that $\frac{b}{a} < \frac{\mathrm{m}(S)}{\mathrm{m}(S)-(\mathrm{r}(S)^{-1} \mod \mathrm{m}(S))} < \frac{b}{a-1}$.

Hence, if $\mathrm{m}(S) = \mathrm{m}(S')$, $\mathrm{r}(S) = \mathrm{r}(S')$ and $a \neq a'$, we can assume without any loss of generality that

$$\frac{b(\mathrm{r}(S)^{-1} \mod \mathrm{m}(S))}{\mathrm{m}(S)} < a < \frac{b(\mathrm{r}(S)^{-1} \mod \mathrm{m}(S))}{\mathrm{m}(S)} + 1 \quad \text{and}$$

$$\frac{b(\mathrm{m}(S) - (\mathrm{r}(S)^{-1} \mod \mathrm{m}(S)))}{\mathrm{m}(S)} < a' < \frac{b(\mathrm{m}(S) - (\mathrm{r}(S)^{-1} \mod \mathrm{m}(S)))}{\mathrm{m}(S)} + 1.$$

By adding the corresponding parts in the inequalities above we get that $b < a + a' < b + 2$, and so $a + a' = b + 1$. □

Theorem 27. *Let b and b' be integers both greater than or equal to 3. Let $a \in \{2, \ldots, b-1\}$ and $a' \in \{2, \ldots, b'-1\}$ and let $S = \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$ and $S' = \mathrm{S}\left(\left]\frac{b'}{a'}, \frac{b'}{a'-1}\right[\right)$. The following statements are equivalent:*

(1) $S = S'$,

(2) $\mathrm{g}(S) = \mathrm{g}(S')$, $\mathrm{m}(S) = \mathrm{m}(S')$ *and* $\mathrm{r}(S) = \mathrm{r}(S')$,

(3) $b = b'$ *and either* $a = a'$ *or* $a + a' = b + 1$.

Proof. The equivalence of assertions (1) and (2) is proved in [11, Corollary 24]. The equivalence of assertions (2) and (3) is a consequence of Theorem 7, Proposition 26 and Lemma 15. □

Given a set A, we denote by $\#A$ its cardinality. The following corollaries follow from Theorem 27.

Corollary 28. *Let $b \geq 3$ be an integer.*

(1) *If b is odd, then the number of symmetric proportionally modular numerical semigroups with Frobenius number b is equal to*

$$\#\left\{a \in \left\{2, \ldots, \frac{b+1}{2}\right\} : \gcd(a, b) = \gcd(a-1, b) = 1\right\}.$$

(2) *If b is even, then the number of pseudo-symmetric proportionally modular numerical semigroups with Frobenius number b is equal to*

$$\#\left\{a \in \{2, \ldots, b-1\} : \gcd(a, b) = 2, \gcd(a-1, b) = 1\right\}.$$

Assertion (1) in Corollary 28 is [11, Corollary 25]. The proof of (2) in Corollary 28 is an easy consequence of Theorems 7, 9 and 27, and Proposition 16. Note that the case $a + a' = b + 1$ is not possible when b is even.

Corollary 29. *Let $b \geq 3$ be an odd integer. Then b is prime if and only if there are exactly $\frac{b+1}{2} - 1$ symmetric proportionally modular numerical semigroups with Frobenius number b.*

For any positive integer b we define the set

$$\mathrm{X}(b) = \begin{cases} \{a \in \{1, \dots, b\} \mid \gcd(a, b) = \gcd(a-1, b) = 1\} & \text{if } b \text{ is odd,} \\ \{a \in \{1, \dots, b\} \mid \gcd(a, b) = 2,\ \gcd(a-1, b) = 1\} & \text{if } b \text{ is even,} \end{cases}$$

and the function $\chi(b) = \#\mathrm{X}(b)$.

By using this notation Corollary 28 can be reformulated as follows.

Corollary 30. *Let $b \geq 3$ be an integer.*

(1) *If b is odd, then the number of symmetric proportionally modular numerical semigroups with Frobenius number b is equal to $\frac{\chi(b)+1}{2}$.*

(2) *If b is even, then the number of pseudo-symmetric proportionally modular numerical semigroups with Frobenius number b is equal to $\chi(b)$.*

Assertion (1) in Corollary 30 is [11, Corollary 26].

We now study some properties of the function χ which determine it completely.

Lemma 31.

(1) $\chi(1) = \chi(2) = \chi(4) = 1$.

(2) $\chi(2^k) = 2^{k-2}$ *for any integer $k \geq 3$.*

(3) *If p is an odd prime number, then $\chi(p^k) = p^{k-1}(p-2)$ for any positive integer k.*

(4) *If b is an odd positive integer, then $\chi(b) = \chi(2b) = \chi(4b)$.*

(5) *If b is an odd positive integer, then $\chi(2^k b) = 2 \cdot \chi(2^{k-1} b)$ for any integer $k \geq 3$.*

Proof.

(1) These equalities are immediate from definitions.

(2) The number of integers a such that $1 \leq a \leq 2^k$ and $a = 2t$ with t odd is the same that the number of odd integers t such that $1 \leq t \leq 2^{k-1}$. The conclusion follows.

(3) See the proof of [11, Proposition 27].

(4) Let b be an odd positive integer. First we show that $\chi(b) = \chi(2b)$. To achieve this, we define a map f from $\mathrm{X}(b)$ to $\mathrm{X}(2b)$ as follows:

$$f(a) = \begin{cases} a & \text{if } a \text{ is even,} \\ a+b & \text{if } a \text{ is odd.} \end{cases}$$

One can easily check that f is well-defined. It is also immediate to show that the map g from $\mathrm{X}(2b)$ to $\mathrm{X}(b)$ given by

$$g(a') = \begin{cases} a' & \text{if } a' < b, \\ a' - b & \text{otherwise,} \end{cases}$$

is well-defined and is the inverse of f. This implies that f is bijective and in particular that $\chi(b) = \chi(2b)$.

Now we prove that $\chi(2b) = \chi(4b)$. To attain this, we define a map f from $\mathrm{X}(2b)$ to $\mathrm{X}(4b)$ as follows:

$$f(a) = \begin{cases} a & \text{if } a \text{ is not a multiple of 4,} \\ a+2b & \text{otherwise} \end{cases}$$

It is easy to see that f is well-defined and that the map g from $\mathrm{X}(4b)$ to $\mathrm{X}(2b)$ given by

$$g(a) = \begin{cases} a & \text{if } a < 2b, \\ a - 2b & \text{otherwise,} \end{cases}$$

is well-defined and is the inverse of f. Hence f is bijective and as a consequence we have $\chi(2b) = \chi(4b)$.

(5) Let b be an odd positive integer and let $k \geq 3$ be an integer. First of all we show that $\mathrm{X}(2^{k-1}b) \subseteq \mathrm{X}(2^k b)$. If $a \in \mathrm{X}(2^{k-1}b)$, then $1 \leq a \leq 2^{k-1}b$, $\gcd(a, 2^{k-1}b) = 2$ and $\gcd(a-1, 2^{k-1}b) = 1$. As we are assuming that b is odd and $k \geq 3$, it is clear that $1 \leq a \leq 2^k b$, $\gcd(a, 2^k b) = 2$ and $\gcd(a-1, 2^k b) = 1$. This means that $a \in \mathrm{X}(2^k b)$.

Next we show that $\{a + 2^{k-1}b \mid a \in \mathrm{X}(2^{k-1}b)\} \subseteq \mathrm{X}(2^k b)$. To see this, consider an element a such that $1 \leq a \leq 2^{k-1}b$, $\gcd(a, 2^{k-1}b) = 2$ and $\gcd(a-1, 2^{k-1}b) = 1$. Note that $\gcd(a + 2^{k-1}b, 2^k b) = \gcd(a + 2^{k-1}b, 2^{k-1}b)$ because $a + 2^{k-1}b$ is an integer of the form $2t$ with t odd. Moreover we have $\gcd(a + 2^{k-1}b, 2^{k-1}b) = \gcd(a, 2^{k-1}b) = 2$. Thus we get that $\gcd(a +$

$2^{k-1}b, 2^k b) = 2$. Similarly as $a+2^{k-1}b-1$ is odd, we have that $\gcd(a+2^{k-1}b-1, 2^k b) = \gcd(a+2^{k-1}b-1, 2^{k-1}b)$ and this is equal to $\gcd(a-1, 2^{k-1}b) = 1$. Thus $\gcd(a+2^{k-1}b-1, 2^k b) = 1$.

Hence we have that

$$\mathrm{X}(2^{k-1}b) \cup \{a + 2^{k-1}b \mid a \in \mathrm{X}(2^{k-1}b)\} \subseteq \mathrm{X}(2^k b).$$

Now we show the opposite inclusion. Suppose that $a' \in \mathrm{X}(2^k b)$. If $a' < 2^{k-1}b$, since we are assuming that $\gcd(a', 2^k b) = 2$ and $\gcd(a'-1, 2^k b) = 1$, we get that $\gcd(a', 2^{k-1}b) = 2$ and $\gcd(a'-1, 2^{k-1}b) = 1$, and so that $a' \in \mathrm{X}(2^{k-1}b)$. If $a' > 2^{k-1}b$, then we call $a = a' - 2^{k-1}b$. Analogously it is immediate to see that $a \in \mathrm{X}(2^{k-1}b)$. Hence we have

$$\mathrm{X}(2^{k-1}b) \cup \{a + 2^{k-1}b \mid a \in \mathrm{X}(2^{k-1}b)\} = \mathrm{X}(2^k b).$$

To conclude the proof, we note that $\mathrm{X}(2^{k-1}b) \cap \{a+2^{k-1}b \mid a \in \mathrm{X}(2^{k-1}b)\} = \emptyset$. $\square$

A *multiplicative number theoretic function* is a function whose domain is the set of positive integers and such that $f(m \cdot n) = f(m) \cdot f(n)$ for all pairs of relatively prime positive integers m and n.

The following result extends [11, Proposition 28] and is a consequence of the Chinese Remainder Theorem and Lemma 31.

Corollary 32. *χ is a multiplicative number theoretic function.*

Taking into account Corollary 30, Lemma 31 and Corollary 32 we can easily compute the number of irreducible proportionally modular numerical semigroups with a given Frobenius number.

Example 33. The number of irreducible proportionally modular numerical semigroups with Frobenius number 1000 is equal to $\chi(1000) = \chi(2^3 5^3) = \chi(2^3)\chi(5^3) = 2^1 5^2 (5-2) = 150$.

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