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Two Schur-convex functions related to Hadamard-type integral inequalities

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Abstract. The Schur-convexity, the Schur-geometric convexity and the Schur-harmonic convexity of two mappings which related to Hadamard-type integral inequalities are researched. And three refinements of Hadamard-type integral inequality are obtained, as applications, some inequalities related to the arithmetic mean, the logarithmic mean and the power mean are established.

1. Introduction

Throughout the paper we assume that the set of n-dimensional row vector on real number field by $\mathbb{R}^n$, and $\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0,\ i = 1, \dots, n\}$. In particular, $\mathbb{R}^1$ and $\mathbb{R}^1_+$ denoted by $\mathbb{R}$ and $\mathbb{R}_+$ respectively.

Let f be a convex function defined on the interval $I \subseteq \mathbb{R} \to \mathbb{R}$ of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2} \tag{1}$$

is known as the Hadamard's inequality for convex function [1]. For some recent results which generalize, improve, and extend this classical inequality, see [2]–[8]and [15]–[17].

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When $f, -g$ both are convex functions satisfying $\int_a^b g(x)dx > 0$ and $f(\frac{a+b}{2}) \geq 0$, S.-J. Yang in [5] generalized (1) as

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \leq \frac{\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x}{\frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x}. \tag{2}$$

To go further in exploring (2), Lan He in [8] define two mappings L and F by $L : [a,b] \times [a,b] \to \mathbb{R}$,

$$L(x,y;f,g) = \left[\int_x^y f(t)\,\mathrm{d}t - (y-x)f\left(\frac{x+y}{2}\right)\right]\left[(y-x)g\left(\frac{x+y}{2}\right) - \int_x^y g(t)\,\mathrm{d}t\right]$$

and $F : [a,b] \times [a,b] \to \mathbb{R}$,

$$F(x,y;f,g) = g\left(\frac{x+y}{2}\right)\int_x^y f(t)\,\mathrm{d}t - f\left(\frac{x+y}{2}\right)\int_x^y g(t)\,\mathrm{d}t,$$

and established the following two theorems which are refinements of the inequality of (2).

Theorem A ([8]). *Let $f, -g$ both are convex functions on $[a,b]$. Then we have*

(i) *$L(a,y;f,g)$ is nonnegative increasing with y on $[a,b]$, $L(x,b;f,g)$ is nonnegative decreasing with x on $[a,b]$.*

(ii) *When $\int_b^a g(x)\,\mathrm{d}x > 0$ and $f\left(\frac{a+b}{2}\right) \geq 0$, for any $x, y \in (a,b)$ and $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha + \beta = 1$, we have the following refinement of (2)*

$$\begin{aligned}
\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} &\leq \frac{(b-a)f\left(\frac{a+b}{2}\right)}{2\int_a^b g(t)\,\mathrm{d}t} + \frac{\int_a^b f(t)\,\mathrm{d}t}{2(b-a)g\left(\frac{a+b}{2}\right)} \\
&\leq \frac{(b-a)f\left(\frac{a+b}{2}\right)}{2\int_a^b g(t)\,\mathrm{d}t} + \frac{\int_a^b f(t)\,\mathrm{d}t}{2(b-a)g\left(\frac{a+b}{2}\right)} + \frac{\alpha L(a,y;f,g) + \beta L(x,b;f,g)}{2(b-a)g\left(\frac{a+b}{2}\right)\int_a^b g(t)\,\mathrm{d}t} \\
&\leq \frac{\int_a^b f(t)\,\mathrm{d}t}{2\int_a^b g(t)\,\mathrm{d}t} + \frac{2f\left(\frac{a+b}{2}\right)}{2g\left(\frac{a+b}{2}\right)} \leq \frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t}.
\end{aligned} \tag{3}$$

Theorem B ([8]). *Let f, $-g$ both are nonnegative convex functions on $[a,b]$ satisfying $\int_a^b g(x)dx > 0$. Then we have the following two results:*

(i) *If f and $-g$ both are increasing, then $F(a,y;f,g)$ is nonnegative increasing with y on $[a,b]$, and we have the following refinement of (2)*

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \leq \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} + \frac{F(a,y;f,g)}{g\left(\frac{a+b}{2}\right)\int_a^b g(t)\,\mathrm{d}t} \leq \frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t}, \tag{4}$$

where $y \in (a,b)$.

(ii) *If f and $-g$ both are decreasing, then $F(a,y;f,g)$ is nonnegative decreasing with y on $[a,b]$, and we have the following refinement of* (2)

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \leq \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} + \frac{F(x,b;f,g)}{g\left(\frac{a+b}{2}\right)\int_a^b g(t)\,\mathrm{d}t} \leq \frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t}, \tag{5}$$

where $x \in (a,b)$.

The aim of this paper is to study the Schur-convexity of $L(x,y;f,g)$ and $F(x,y;f,g)$ with variables (x,y) in $[a,b]\times[a,b] \subseteq \mathbb{R}^2$, and study the Schur-geometric convexity and the Schur-harmonic convexity of $L(x,y;f,g)$ with variables (x,y) in $[a,b]\times[a,b] \subseteq \mathbb{R}^2_+$. We obtain the following results.

Theorem 1. *Let f and $-g$ both be convex function on $[a,b]$. Then*

(i) *$L(x,y;f,g)$ is Schur-convex on $[a,b]\times[a,b] \subseteq \mathbb{R}^2$, and $L(x,y;f,g)$ is Schur-geometrically convex and Schur-harmonic convex in $[a,b]\times[a,b] \subseteq \mathbb{R}^2_+$.*

(ii) *If $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$ or $0 \leq t_2 \leq t_1 \leq \frac{1}{2}$, then for $a<b$, we have*

$$\begin{aligned} 0 &\leq L(t_1a+(1-t_1)b, t_1b+(1-t_1)a;f,g) \\ &\leq L(t_2a+(1-t_2)b, t_2b+(1-t_2)a;f,g) \leq L(a,b;f,g), \end{aligned} \tag{6}$$

and for $0<a<b$, we have

$$0 \leq L\left(b^{t_2}a^{1-t_2}, a^{t_2}b^{1-t_2};f,g\right) \leq L\left(b^{t_1}a^{1-t_1}, a^{t_1}b^{1-t_1};f,g\right) \leq L(a,b;f,g) \tag{7}$$

and

$$\begin{aligned} 0 &\leq L\left(1/(t_2b+(1-t_2)a), 1/(t_2a+(1-t_2)b);f,g\right) \\ &\leq L\left(1/(t_1b+(1-t_1)a), 1/(t_1a+(1-t_1)b);f,g\right) \leq L(1/a,1/b;f,g). \end{aligned} \tag{8}$$

Theorem 2. *Let f and $-g$ both be nonnegative convex function on $[a,b]$. Then*

(i) *$F(x,y;f,g)$ is Schur-convex on $[a,b]\times[a,b] \subseteq \mathbb{R}^2$;*

(ii) *If $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$ or $0 \leq t_2 \leq t_1 \leq \frac{1}{2}$, then for $a<b$, we have*

$$\begin{aligned} 0 &\leq F(t_1a+(1-t_1)b, t_1b+(1-t_1)a;f,g) \\ &\leq F(t_2a+(1-t_2)b, t_2b+(1-t_2)a;f,g) \leq F(a,b;f,g). \end{aligned} \tag{9}$$

Theorem 3. *Let f and $-g$ both be convex function on $[a,b] \subseteq \mathbb{R}$. If $\int_b^a g(x)\,\mathrm{d}x > 0$ and $f\left(\frac{a+b}{2}\right) \geq 0$, then*

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \leq \frac{\int_a^b f(t)\,\mathrm{d}t - \int_{ta+(1-t)b}^{tb+(1-t)a} f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t - \int_{ta+(1-t)b}^{tb+(1-t)a} g(t)\,\mathrm{d}t} \leq \frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t}, \tag{10}$$

where $\frac{1}{2} \leq t < 1$ or $0 \leq t \leq \frac{1}{2}$.

Theorem 4. *Let f, $-g$ both are nonnegative convex functions on $[a,b]$ satisfying $\int_a^b g(x)dx > 0$, then for $a < b$, we have*

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \leq \frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t} - \frac{L(ta+(1-t)b, tb+(1-t)a; f, g)}{2(b-a)g\left(\frac{a+b}{2}\right)\int_a^b g(t)\,\mathrm{d}t} \leq \frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t}, \tag{11}$$

and for $0 < a < b$, we have

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \leq \frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t} - \frac{L\left(b^t a^{1-t}, a^t b^{1-t}; f, g\right)}{2(b-a)g\left(\frac{a+b}{2}\right)\int_a^b g(t)\,\mathrm{d}t} \leq \frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t}, \tag{12}$$

where $\frac{1}{2} \leq t \leq 1$ or $0 \leq t \leq \frac{1}{2}$.

2. Definitions and lemmas

We need the following definitions and lemmas.

Definition 1 ([9, 10]). Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(ii) Let $\Omega \subseteq \mathbb{R}^n$. The function $\varphi : \Omega \to \mathbb{R}$ be said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$.$\varphi$ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex.

Definition 2 ([11, 12]). Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$.

(i) $\Omega \subseteq \mathbb{R}_+^n$ is called a geometrical convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\boldsymbol{x}$ and $\boldsymbol{y} \in \Omega$, where α and $\beta \in [0,1]$ with $\alpha + \beta = 1$.

(ii) Let $\Omega \subseteq \mathbb{R}_+^n$. The function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur-geometrical convex function on Ω if $(\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. The function φ is said to be a Schur-geometrical concave on Ω if and only if $-\varphi$ is Schur-geometrical convex.

Definition 3 ([13]). Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$.

(i) $\Omega \subseteq \mathbb{R}_+^n$ is called a harmonic convex set if $(x_1 y_1/(\alpha x_1 + \beta y_1), \ldots, x_n y_n/(\alpha x_n + \beta y_n)) \in \Omega$ for all $\boldsymbol{x}$ and $\boldsymbol{y} \in \Omega$, where α and $\beta \in [0,1]$ with $\alpha + \beta = 1$.

(ii) Let $\Omega \subseteq \mathbb{R}^n_+$. The function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur-harmonic convex function on Ω if $(1/x_1, \dots, 1/x_n) \prec (1/y_1, \dots, 1/y_n)$ on Ω implies $\varphi(\boldsymbol{x} \le (\ge)\varphi(\boldsymbol{y})$. The function φ is said to be a Schur-harmonic concave on Ω if and only if $-\varphi$ is Schur-harmonic convex.

Lemma 1 ([9, 10]). *Let $\Omega \subseteq \mathbb{R}^n$ be a symmetric set and with a nonempty interior Ω^0, $\varphi : \Omega \to \mathbb{R}$ be a continuous on Ω and differentiable in Ω^0. Then φ is the Schur-convex (Schur-concave) function, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \ge 0(\le 0) \tag{13}$$

holds for any $\boldsymbol{x} = (x_1, \dots, x_n) \in \Omega^0$.

Lemma 2 ([11]). *Let $\Omega \subseteq \mathbb{R}^n_+$ be symmetric with a nonempty interior geometrically convex set. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial \varphi}{\partial x_1} - x_2\frac{\partial \varphi}{\partial x_2}\right) \ge 0(\le 0) \tag{14}$$

holds for any $\boldsymbol{x} = (x_1, \dots, x_n) \in \Omega^0$, then φ is a Schur-geometrical convex (Schur-geometrical concave) function.

Lemma 3 ([13]). *Let $\Omega \subseteq \mathbb{R}^n_+$ be symmetric with a nonempty interior harmonic convex set. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1^2\frac{\partial \varphi}{\partial x_1} - x_2^2\frac{\partial \varphi}{\partial x_2}\right) \ge 0(\le 0) \tag{15}$$

holds for any $\boldsymbol{x} = (x_1, \dots, x_n) \in \Omega^0$, then φ is a Schur-harmonic convex (Schur-harmonic concave) function.

Lemma 4 ([14]). *Let $a \le b, u(t) = ta + (1-t)b, v(t) = tb + (1-t)a$. If $1/2 \le t_2 \le t_1 \le 1$ or $0 \le t_1 \le t_2 \le 1/2$, then*

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a, b). \tag{16}$$

Lemma 5 ([18]). *Let I be an interval with nonempty interior on $\mathbb{R}$ and f be a continuous function on I. Then*

$$\Phi(a,b) = \begin{cases} \dfrac{1}{b-a}\int_a^b f(t)dt, & a, b \in I,\ a \ne b \\ f(a),\ a = b \end{cases}$$

Schur-convex (Schur-concave) on I^2 if and if f is convex(concave) on I.

Lemma 6. *Let f and $-g$ both be convex function on $[a,b]\subseteq\mathbb{R}$. If $\int_b^a g(x)\,\mathrm{d}x\geq 0$ and $f\left(\frac{a+b}{2}\right)\geq 0$, then*

$$L(a,b;f,g)\leq 2(b-a)\left[g\left(\frac{a+b}{2}\right)\int_a^b f(t)\,\mathrm{d}t-f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,\mathrm{d}t\right]. \quad (17)$$

Proof.

$$\begin{aligned}L(a,b;f,g)=&\left[\int_a^b f(t)\,\mathrm{d}t-(b-a)f\left(\frac{a+b}{2}\right)\right]\left[(b-a)g\left(\frac{a+b}{2}\right)-\int_a^b g(t)\,\mathrm{d}t\right]\\ =&\,(b-a)g\left(\frac{a+b}{2}\right)\int_a^b f(t)\,\mathrm{d}t-\int_a^b f(t)\,\mathrm{d}t\int_a^b g(t)\,\mathrm{d}t\\ &-(b-a)^2 f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)+(b-a)f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,\mathrm{d}t. \quad (18)\end{aligned}$$

Combining (18) with (3.6) and (3.7) in [10], it following that (17) is hold. □

3. Proofs of the main results

Proof of Theorem 1. (i) It is clear that $L(x,y;f,g)$ is symmetric with x, y. Without loss of generality, we may assume $y\geq x$. Directly calculating yields

$$\begin{aligned}\frac{\partial L}{\partial y}=&\left[f(y)-f\left(\frac{x+y}{2}\right)-\frac{y-x}{2}f'\left(\frac{x+y}{2}\right)\right]\left[(y-x)g\left(\frac{x+y}{2}\right)-\int_x^y g(t)\,\mathrm{d}t\right]\\ &+\left[\int_x^y f(t)\,\mathrm{d}t-(y-x)f\left(\frac{x+y}{2}\right)\right]\left[g\left(\frac{x+y}{2}\right)+\frac{y-x}{2}g'\left(\frac{x+y}{2}\right)-g(y)\right],\\ \frac{\partial L}{\partial x}=&\left[-f(x)+f\left(\frac{x+y}{2}\right)-\frac{y-x}{2}f'\left(\frac{x+y}{2}\right)\right]\left[(y-x)g\left(\frac{x+y}{2}\right)-\int_x^y g(t)\,\mathrm{d}t\right]\\ &+\left[\int_x^y f(t)\,\mathrm{d}t-(y-x)f\left(\frac{x+y}{2}\right)\right]\left[-g\left(\frac{x+y}{2}\right)+\frac{y-x}{2}g'\left(\frac{x+y}{2}\right)+g(x)\right].\end{aligned}$$

By Lagrange mean value theorem, there is $\xi\in((x+y)/2,y)$ such that

$$f(y)-f\left(\frac{x+y}{2}\right)=\left(y-\frac{x+y}{2}\right)f'(\xi)=\frac{y-x}{2}f'(\xi).$$

Since f is convex, f' is increasing, we have $f'(\xi)\geq f'\left(\frac{x+y}{2}\right)$, so

$$f(y)-f\left(\frac{x+y}{2}\right)-\frac{y-x}{2}f'\left(\frac{x+y}{2}\right)\geq 0.$$

By the same arguments, we have

$$-f(x)+f\left(\frac{x+y}{2}\right)-\frac{y-x}{2}f'\left(\frac{x+y}{2}\right)\le 0.$$

Similarly, since $-g$ is convex, we have

$$g\left(\frac{x+y}{2}\right)+\frac{y-x}{2}g'\left(\frac{x+y}{2}\right)-g(y)\ge 0$$

and

$$-g\left(\frac{x+y}{2}\right)+\frac{y-x}{2}g'\left(\frac{x+y}{2}\right)+g(x)\le 0.$$

And by Hadamard's inequality (1), it follows that $(y-x)g\big(\frac{x+y}{2}\big)-\int_x^y g(t)\,\mathrm{d}t\ge 0$ and $\int_x^y f(t)\,\mathrm{d}t-(y-x)f\big(\frac{x+y}{2}\big)\ge 0$. So $\frac{\partial L}{\partial y}\ge 0$ and $\frac{\partial L}{\partial x}\le 0$, further $(y-x)\big(\frac{\partial L}{\partial y}-\frac{\partial L}{\partial x}\big)\ge 0$ and $(x-y)\big(x^2\frac{\partial L}{\partial x}-y^2\frac{\partial L}{\partial y}\big)\ge 0$. Notice that from $y\ge x$, we have $\ln x-\ln y\le 0$, and then $(\ln x-\ln y)\big(x\frac{\partial L}{\partial x}-y\frac{\partial L}{\partial y}\big)\ge 0$. According to Lemma 1, Lemma 2 and Lemma 3, it follows that $L(x,y;f,g)$ is Schur-convex in $[a,b]\times[a,b]\subseteq\mathbb{R}^2$, and $L(x,y;f,g)$ is Schur-geometrical convex and Schur-harmonic convex in $[a,b]\times[a,b]\subseteq\mathbb{R}^2_+$.

(ii) From Lemma 4, we have

$$\begin{aligned}\big(\ln\sqrt{ab},\ln\sqrt{ab}\big)&\prec\big(\ln(b^{t_2}a^{1-t_2}),\ln(a^{t_2}b^{1-t_2})\big)\\&\prec\big(\ln(b^{t_1}a^{1-t_1}),\ln(a^{t_1}b^{1-t_1})\big)\prec(\ln a,\ln b).\end{aligned}\tag{19}$$

By (i) in Theorem 1, from (16) and (19) it follows that (6), (8) and (7) are hold.

The proof of Theorem 1 is completed. □

Proof of Theorem 2. (i) It is clear that $F(x,y;f,g)$ is symmetric. Without loss of generality, we may assume $y\ge x$. Directly calculating yields

$$\begin{aligned}\frac{\partial F}{\partial y}&=\frac{1}{2}g'\left(\frac{x+y}{2}\right)\int_x^y f(t)\,\mathrm{d}t+g\left(\frac{x+y}{2}\right)f(y)\\&\quad-\frac{1}{2}f'\left(\frac{x+y}{2}\right)\int_x^y g(t)\,\mathrm{d}t-f\left(\frac{x+y}{2}\right)g(y),\\\frac{\partial F}{\partial x}&=\frac{1}{2}g'\left(\frac{x+y}{2}\right)\int_x^y f(t)\,\mathrm{d}t-g\left(\frac{x+y}{2}\right)f(x)\\&\quad-\frac{1}{2}f'\left(\frac{x+y}{2}\right)\int_x^y g(t)\,\mathrm{d}t+f\left(\frac{x+y}{2}\right)g(x),\end{aligned}$$

and then

$$
\begin{aligned}
&(y-x)\left(\frac{\partial F}{\partial y}-\frac{\partial F}{\partial x}\right)\\
&\qquad=(y-x)\left[g\left(\frac{x+y}{2}\right)(f(x)+f(y))-f\left(\frac{x+y}{2}\right)(g(x)+g(y))\right].
\end{aligned}
$$

Since f and $-g$ both be convex function on $[a,b]$, $f(x)+f(y)\geq 2f\left(\frac{x+y}{2}\right)$ and $g\left(\frac{x+y}{2}\right)\geq\frac{g(x)+g(y)}{2}$, and then $g\left(\frac{x+y}{2}\right)(f(x)+f(y))-f\left(\frac{x+y}{2}\right)(g(x)+g(y))\geq 0$, so $(y-x)\left(\frac{\partial F}{\partial y}-\frac{\partial F}{\partial x}\right)\geq 0$. From Lemma 1, it follows that $F(x,y;f,g)$ is Schur-convex on $[a,b]\times[a,b]$.

(ii) By (i) in Theorem 2, from (16) it follows that the (9) is hold.

The proof of Theorem 2 is completed. □

Proof of Theorem 3. By the Theorem 2, for $\frac{1}{2}\leq t<1$ or $0\leq t\leq\frac{1}{2}$, we have

$$
\begin{aligned}
&F(ta+(1-t)b,tb+(1-t)a;f,g)\\
&\qquad=g\left(\frac{a+b}{2}\right)\int_{ta+(1-t)b}^{tb+(1-t)a}f(t)\,\mathrm{d}t-f\left(\frac{a+b}{2}\right)\int_{ta+(1-t)b}^{tb+(1-t)a}g(t)\,\mathrm{d}t\\
&\qquad\leq g\left(\frac{a+b}{2}\right)\int_a^b f(t)\,\mathrm{d}t-f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,\mathrm{d}t=F(a,b;f,g).
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right)\left[\int_a^b g(t)\,\mathrm{d}t-\int_{ta+(1-t)b}^{tb+(1-t)a}g(t)\,\mathrm{d}t\right]\\
&\qquad\qquad\leq g\left(\frac{a+b}{2}\right)\left[\int_a^b f(t)\,\mathrm{d}t-\int_{ta+(1-t)b}^{tb+(1-t)a}f(t)\,\mathrm{d}t\right],
\end{aligned}
$$

which is equivalent to left inequality in (10).

Since f is convex on $[a,b]$, by Lemma 5, it follows that $\frac{1}{y-x}\int_x^y f(t)\,\mathrm{d}t$ is Schur convex on $[a,b]\times[a,b]$, and since $-g$ is convex on $[a,b]$, i.e. g is concave on $[a,b]$, by Lemma 5, it follows that $\frac{1}{y-x}\int_x^y g(t)\,\mathrm{d}t$ is Schur concave on $[a,b]\times[a,b]$, and then

$$
\frac{\frac{1}{y-x}\int_x^y f(t)\,\mathrm{d}t}{\frac{1}{y-x}\int_x^y g(t)\,\mathrm{d}t}=\frac{\int_x^y f(t)\,\mathrm{d}t}{\int_x^y g(t)\,\mathrm{d}t}
$$

is Schur convex on $[a,b]\times[a,b]$. Therefore, from (16) we have

$$\frac{\int_{ta+(1-t)b}^{tb+(1-t)a} f(t)\,\mathrm{d}t}{\int_{ta+(1-t)b}^{tb+(1-t)a} g(t)\,\mathrm{d}t} \le \frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t}.$$

The above inequality equivalent to the right inequality in (10).

The proof of Theorem 3 is completed. □

Proof of Theorem 4. By the Theorem 1, for $a<b$, we have

$$L(ta+(1-t)b, tb+(1-t)a; f,g) \le L(a,b;f,g), \tag{20}$$

and for $0<a<b$, we have

$$L\left(b^t a^{1-t}, a^t b^{1-t}; f,g\right) \le L(a,b;f,g). \tag{21}$$

Combining (17) with (20) and (21) respectively, it is deduced that (11) and (12) are hold.

The proof of Theorem 4 is completed. □

4. Applications

Corollary 1. *Let $a,b\in\mathbb{R}_+$ with $a<b$, and let $u=tb+(1-t)a$, $v=ta+(1-t)b$, $\frac{1}{2}\le t<1$ or $0\le t\le\frac{1}{2}$. Then for $1\le r\le 2$, we have*

$$\left(\frac{2}{a+b}\right)^r \le \frac{r[(\ln b-\ln a)-(\ln u-\ln v)]}{2(b-a)(1-t)} \le \frac{r(\ln b-\ln a)}{b-a}. \tag{22}$$

Proof. For $1\le r\le 2$, taking $f(x)=x^{-1}$ and $g(x)=x^{r-1}$, then f and $-g$ both be convex function on $[a,b]$. From Theorem 3, it follows that (22) is hold.

The proof of Corollary 1 is completed. □

Remark 1. Taking $r=1$, from (22), we have

$$\frac{2}{a+b} \le \frac{(\ln b-\ln a)-(\ln u-\ln v)}{2(b-a)(1-t)} \le \frac{\ln b-\ln a}{b-a} \tag{23}$$

(23) is a refinement of the following Ostle–Terwilliger inequality [19]:

$$\frac{\ln b-\ln a}{b-a} \ge \frac{2}{a+b}. \tag{24}$$

Corollary 2. *Let $a, b \in \mathbb{R}_+$ with $a < b$, and let $u = tb + (1-t)a, v = ta + (1-t)b$, $\frac{1}{2} \le t < 1$ or $0 \le t \le \frac{1}{2}$. Then for $1 \le r \le 2$, we have*

$$\frac{a+b}{2} \le \left[\frac{(b^{2r} - a^{2r}) - (u^{2r} - v^{2r})}{2(b^r - a^r) - 2(u^r - v^r)}\right]^{\frac{1}{r}} \le \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}}. \tag{25}$$

Proof. For $1 \le r \le 2$, taking $f(x) = x^{2r-1}$ and $g(x) = x^{r-1}$, then f and $-g$ both be convex function on $[a, b]$, from Theorem 3, it is easy to prove that (25) is hold.

The proof of Corollary 2 is completed. □

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