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On the regularity of group-valued additive arithmetical functions

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1. Introduction

a) Notations

$\mathbb{N}$ is the set of positive integers, $\mathbb{R}$ (resp. $\mathbb{T}$) is the set of real numbers (resp. complex numbers of modulus 1).

If G is a locally compact abelian group, G^* will denote the dual group of G, i.e. the locally compact abelian group of the continuous characters on G, and dm a Haar measure on G^*.

b) Position of the problem

Definition. Let G be an abelian group, and denote by $+$ the group operation. A function f is a G–valued additive arithmetical function if f is a $\mathbb{N} \to G$ function such that $f(m \cdot n) = f(m) + f(n)$ when $(m, n) = 1$.

Throughout this article, we shall assume that G is an abelian topological group. Then, it is a classical problem to give a characterization of the G–valued additive arithmetical functions satisfying the following condition (C);

$$\text{(C)} \qquad \lim_{n \to +\infty} (f(n+1) - f(n)) = 0\,.$$

This problem has been considered by P. Erdős in 1946 in the case $G = \mathbb{R}$, and he proved that a real valued additive arithmetical function f satisfies the condition (C) if and only if there exists a constant c such that $f(n) = c \cdot \log n$ for all n in $\mathbb{N}$ [2]. If $G = \mathbb{R}/\mathbb{Z}$, the solution has been provided by E. Wirsing in 1984: in this case, we have $f(n) = c \cdot \log n$ modulo 1 [7]. More recently, Z. Daróczy and I. Kátai solved this problem for metrical compactly generated locally compact abelian groups, and proved that a G–valued additive arithmetical function f satisfies the condition (C) if and only if there exists a continuous homomorphism $\varphi : \mathbb{R} \to G$ such that $f(n) = \varphi(\log n)$ [1]. This cannot be extended to all groups: I. Z. Ruzsa and R. Tijdeman proved that there exists a topology on the group of integers (whith no continuous characters), and an integer-valued function f satisfying the condition (C) [5], and I. Z. Ruzsa has an example in which f is real-valued function, and the group of the reals has a topology such that the continuous characters separate the elements of this group [4]. In this situation, it seems interesting to refine on the results obtained, and treat the case of a general abelian locally compact group, in which case the continuous characters separate the group.

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I thank also Prof. I. Z. Ruzsa who informed me of the existence of the result of himself and R. Tijdeman before it appears, and of his other example.

2. The result

The result presented in this article is the following:

Theorem. *Let G be an additive locally compact abelian group, G^* its dual, f an additive arithmetical function with values in G. The following statements are equivalent:*

i) *f satisfies the condition* $\lim(f(n+1)-f(n))=0,\ n\to+\infty$.
ii) *there exists a continuous homomorphism $\varphi:\mathbb{R}\to G$ such that for any n in $\mathbb{N}$, $f(n)=\varphi(\log n)$.*
iii) *for any X in G^*, there exists a real number $\tau(X)$ such that for any n in $\mathbb{N}$, $X(f(n))=\exp\{i\cdot\tau(X)\cdot\log n\}$*

Some remarks. 1. We denote by G^* the dual group of G, i.e. the group of the continuous characters on G. In the proof of the theorem, the Pontryagin duality principle will be essential, which says that in a locally compact abelian group G, the bidual group of G, i.e. the dual group of the dual group of G, is topologically and algebraically isomorphic to G. The abovementioned pathological example due to I. Z. Ruzsa shows that the separability of G by its continuous characters is not sufficient to obtain the same result.

2. Equivalence of the statements i) and ii) has been announced without proof in my note “Distribution des valeurs d’une fonction arithmétique additive à valeurs dans un groupe abélien localement compact métrisable”, (see *C. R. Acad. Sci.,* Paris, Sér. I T. 313, (1991), 345–348).

3. We shall need the following result:

Proposition. *If h is a function: $\mathbb{R} \to G$ and if for all χ in $G^*, \chi(h(a))$ tends to 1 when a tends to 0, then h is continuous at 0.*

Proof. Let V be a compact neighborhood of the origin of G, with characteristic function I_V. Since G is a topological abelian group, we can select W, a symmetric compact neighborhood of the origin of G with characteristic function I_W, such that $W+W \subset V$. Let $|W|$ be the measure of W, which is > 0, and L_W be defined by

$$L_W = \frac{1}{|W|} \times {I_W}^* I_W \,,$$

where * denotes the convolution. It is immediate that $0 \le L_W \le 1$, $L_W(0)=1$, and L_W is continuous, has its support in V and satisfies $L_W \le I_V$. Moreover, its Fourier transform $\mathcal{L}_W$ is continuous, non-negative, integrable by Plancherel theorem, hence invertible ([6], p.113 §30). Now, we remark that we have by inversion:

$$\begin{aligned} L_W(0) - L_W(u) &= \int_{G^*} \mathcal{L}_W(\chi) \cdot (\bar{\chi}(0) - \bar{\chi}(u)) \cdot dm(\chi) \\ &= \int_{G^*} \mathcal{L}_W(\chi) \cdot (1 - \bar{\chi}(u)) \cdot dm(\chi) \end{aligned}$$

for every u in G.

Using the inequality $1 - I_V(u) \le 1 - L_W(u)$, which we write as $I_V(0) - I_V(u) \le L_W(0) - L_W(u)$, we get that

$$I_V(0) - I_V(u) \le \int_{G^*} \mathcal{L}_W(\chi) \cdot (1 - \bar{\chi}(u)) \cdot dm(\chi) \,,$$

i.e.

$$1 - I_V(u) \le \int_{G^*} \mathcal{L}_W(\chi) \cdot (1 - \bar{\chi}(u)) \cdot dm(\chi) \,,$$

and a fortiori, we have, for every real a

$$1 - I_V(h(a)) \le \int_{G^*} \mathcal{L}_W(\chi) \cdot (1 - \bar{\chi}(h(a))) \cdot dm(\chi) \,.$$

Now, since $\mathcal{L}_W$ is integrable and for all χ in $G^*, \chi(h(a))$ tends to 1 when a tends to 0, the Lebesgue bounded convergence theorem gives that

$$\lim_{a \to 0} (1 - I_V(h(a))) = 0, \quad \text{and this implies that} \quad \lim_{a \to 0} h(a) = 0 \,.$$

3. Proof of the theorem

I. First, we remark that ii) implies i).

It is clear that if there exists a continuous homomorphism $\varphi : \mathbb{R} \to G$ such that for any n in $\mathbb{N}$, $f(n) = \varphi(\log n)$, by continuity, the additive function $f(n)$ will satisfy the condition (C).

II. We prove that i) implies iii).

1) By hypothesis, we have $\lim(f(n+1)-f(n)) = 0$, $n \to +\infty$, and if X is any element of G^*, by continuity, we get that $\lim X(f(n+1)) - f(n))=1$, $n \to +\infty$, which in fact means that $\lim\{X(f(n+1)) - X(f(n))\}=0$, $n \to +\infty$. Now, since the function defined on $\mathbb{N}$ by $X(f(n))$ is a $\mathbb{T}$–valued additive function, by Wirsing's result, there exists a real number $\tau(X)$ such that for any n in $\mathbb{N}$, $X(f(n)) = \exp\{i \cdot \tau(X) \cdot \log n\}$. This gives that i) implies iii).

III. We prove that iii) implies ii). We assume that iii) holds.

a) The correspondence $\tau : G^* \to \mathbb{R}$ is a well-defined function. Moreover, given X and X' in G^*, since for any n in $\mathbb{N}$ we have by definition $(X \cdot X')(f(n)) = X(f(n)) \cdot X'(f(n))$, we have

$$\begin{aligned}\exp\{i \cdot \tau(X \cdot X') \cdot \log n\} &= \exp\{i \cdot \tau(X) \cdot \log n\} \cdot \exp\{i \cdot \tau(X') \cdot \log n\} = \\ &= \exp\{i \cdot (\tau(X) + \tau(X')) \cdot \log n\}\,,\end{aligned}$$

and we get that $\tau(X \cdot X') = \tau(X) + \tau(X')$, which means that τ is a group homomorphism $G^* \to \mathbb{R}$. Now, for any fixed n in $\mathbb{N}$, the function $X \to \exp\{i \cdot \tau(X) \cdot \log n\}$ is continuous, since $X(f(n)) = \exp\{i \cdot \tau(X) \cdot \log n\}$.

Let K be a fixed compact neighborhood of the origin in G^*. We denote by m a Haar measure on G^*, and by m^* its restriction to K. Since K is a compact neighborhood of the origin in G^*, m^* is a regular positive bounded measure on K. Now, let a be a real number, a_n and b_n two sequences in $\mathbb{N}$ such that

$$\lim_{n\to+\infty} \log\left(\frac{a_n}{b_n}\right) = a\,.$$

We define a function $\psi_a : K \to \mathbb{T}$ by $\psi_a(X) = \exp\{i \cdot a \cdot \tau(X)\}$. We have

$$\begin{aligned}\psi_a(X) = \exp\{i \cdot a \cdot \tau(X)\} &= \lim_{n\to+\infty} \exp\{i \cdot \log\left(\frac{a_n}{b_n}\right) \cdot \tau(X)\} = \\ = \lim_{n\to+\infty} \exp\{i \cdot \log a_n \cdot \tau(X)\} &\cdot \overline{\exp\{i \cdot \log b_n \cdot \tau(X)\}} = \\ \lim_{n\to+\infty} X(f(a_n)) &\cdot \overline{X(f(b_n))}\,.\end{aligned}$$

This limit $\psi_a(X)$ is independent of the approximating sequence, and for a fixed a, $\psi_a(X)$ is a limit of a sequence of continuous functions, and as a consequence, $\psi_a(X)$ is m^*–measurable on K.

It is a classical fact the m^*–measurability of the homomorphism $\psi_a(X)$ implies its continuity on G^* (see [6] p.66 §18)

b) We now use the Pontryagin duality theorem. First of all, we recall the duality principle:

Let A be a locally compact abelian group, A^* its dual. The Pontryagin duality theorem says that for every character π of A^*, there exists an element g of A such that $\pi(X) = X(g)$ for all X of A^* (see, for instance, [3. Ch. 11 §1.2 p.172]).

We now remark that for any given real a, the function $\psi_a(X)$ is in the dual of G^*, since it is a continuous group homomorphism $G^* \to \mathbb{T}$. Hence we get that there exists some element $\varphi(a)$ in G such that $\psi_a(X) = X(\varphi(a))$. Moreover, φ is a group homomorphism $\mathbb{R} \to G$, since we have $X(\varphi(a+b)) = X(\varphi(a) + \varphi(b))$, as it can be seen from the equalities

$$\begin{aligned} X(\varphi(a+b)) &= \psi_{a+b}(X) = \exp\{i \cdot (a+b) \cdot \tau(X)\} = \\ &= \exp\{i \cdot a \cdot \tau(X)\} \cdot \exp\{i \cdot b \cdot \tau(X)\} = \psi_a(X) \cdot \psi_b(X) = \\ &= X(\varphi(a)) \cdot X(\varphi(b)) = X(\varphi(a) + \varphi(b))\,. \end{aligned}$$

But we have $\psi_a(X) = \exp\{i \cdot a \cdot \tau(X)\}$ by definition, which implies that, for any X in G^*, $X(\varphi(a))$ tends to 1 when a tends to 0, and the Proposition gives that φ is continuous in 0. Since φ is an homomorphism, φ is continuous on $\mathbb{R}$.

Now, since for any X in G^* and any real a we have $X(\varphi(a)) = \psi_a(X)$, a fortiori, if $a = \log n$ with n in $\mathbb{N}$, we shall have $X(f(n)) = \exp\{i \cdot \log n \cdot \tau(X)\} = \psi_{\log n}(X) = X(\varphi(\log n))$, which gives $X(f(n)) = X(\varphi(\log n))$, and so, we obtain that for any n in $\mathbb{N}$, we have $f(n) = \varphi(\log n)$ and this ends the proof of the theorem.

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