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Absolutely indecomposable representations of a twisted group algebra of a finite p-group over a field of characteristic p

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Abstract. Let G be a non-cyclic finite p-group and K an infinite field of characteristic p. For every 2-cocycle $\lambda \in Z^2(G, K^*)$ such that the twisted group algebra $K^\lambda G$ is not uniserial, we find the integers $m \geq 1$ for which $K^\lambda G$ has infinitely many absolutely indecomposable representations of dimension m. The main results of the paper imply a solution of the second Brauer–Thrall conjecture for the twisted group algebras $K^\lambda G$, under some assumption on G and K.

Introduction

Let A be a finite-dimensional algebra with identity over an infinite field K. We say that A is of strongly unbounded representation type, if there exists an infinite sequence of positive integers d_i such that, for each i, the algebra A has an infinite number of indecomposable representations of dimension d_i.

Let $p \geq 2$ be a prime, G a finite group of order $|G|$, such that p divides $|G|$, G_p a Sylow p-subgroup of G and K a field of characteristic p. Higman [14] proved that if G_p is a cyclic group then the group algebra KG is of finite representation type, and if G_p is non-cyclic then KG is of unbounded representation type, that is KG has indecomposable representations of an arbitrary large dimension. Bondarenko and Drozd [8] have established that if G_p is non-cyclic then KG is tame if and only if $|G_p : G'_p| = 4$ (see also the paper [7] which uses the method of self-repeating matrix problems, first proposed in [20]). Let H be a non-cyclic

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abelian p-group and K be an infinite field of characteristic p. Bashev [6] showed that if $|H| = 4$ then KH has infinitely many absolutely indecomposable representations of dimension $2n$ for any integer $n \geq 1$, see also [1, Section V.5]. Gudivok [13] and Janusz [16], [17] proved that in the case $|H| \neq 4$ the group algebra KH has infinitely many absolutely indecomposable representations of each dimension $n \geq 2$. It follows, by Higman's theory of relative projectivity [14], that if K is an infinite field and KG is of infinite representation type then KG is of strongly unbounded representation type. This result is the confirmation of the second Brauer–Thrall conjecture for group algebras of finite groups (see [1, p. 138] and [22, pp. 341–2] for a formulation and a discussion of the conjecture).

Given a 2-cocycle $\lambda \in Z^2(G, K^*)$, we denote by $K^\lambda G$ a twisted group algebra of a finite group G over a field K of characteristic p corresponding to λ (see [18, p. 66]). We recall from [5] that $K^\lambda G$ is of finite representation type if and only if $K^\lambda G_p$ is a uniserial algebra.

In the present work we find all positive integers m for which a non-uniserial algebra $K^\lambda G_p$ has infinitely many absolutely indecomposable representations of dimension m.

Assume that K is an infinite field of characteristic p, G is a non-cyclic p-group, $K^\lambda G$ is a non-uniserial algebra and $d = \dim_K(K^\lambda G/ \operatorname{rad} K^\lambda G)$. Since $K^\lambda G$ is a local algebra (see [18, p. 74]), the dimension of every indecomposable representation of $K^\lambda G$ is a multiple of d. Let G be an abelian p-group and

$$l = \begin{cases} 1 & \text{if } 4d \neq |G|, \\ 2 & \text{if } 4d = |G|. \end{cases}$$

We prove that the algebra $K^\lambda G$ has infinitely many nonequivalent absolutely indecomposable representations of dimension nld for any integer $n \geq 2$ (Theorem 2.4). Suppose that G is a non-abelian p-group, G' is the commutant of G and

$$t = \begin{cases} 1 & \text{if } p \neq 2, \\ 2 & \text{if } p = 2. \end{cases}$$

If G' is non-cyclic or $pd \neq |G : G'|$ then $K^\lambda G$ has infinitely many nonequivalent absolutely indecomposable representations of dimension $nptd$ for any integer $n \geq 2$ (Propositions 3.4, 4.1 and Theorem 3.5). If G' is cyclic and $pd = |G : G'|$, under some assumptions on G or K, we prove in Propositions 4.4–4.9 that the algebra $K^\lambda G$ has infinitely many nonequivalent absolutely indecomposable representations of dimension $2nd$, for any integer $n \geq 1$. Hence the result is valid if one of the following conditions holds:

(i) $p \neq 2$ and $|G'| \geq p^2$,

(ii) $p = 2$ and $|G' \cap Z(D)| \geq 4$, where D is the subgroup of G such that $G' \subset D$ and $D/G' = \operatorname{soc}(G/G')$, or

(iii) $[K : K^p] \leq p^2$.

Assume now that G is an arbitrary finite group and K is an infinite field of characteristic p. By Higman's theory of relative projectivity [14], the algebras $K^\lambda G$ and $K^\lambda G_p$ are of the same representation type. Consequently, from the results obtained in this paper it follows that $K^\lambda G$ is of strongly unbounded representation type if $K^\lambda G_p$ is not a uniserial algebra and one of the following conditions holds:

(a) G_p is an abelian group,

(b) G'_p is a non-cyclic group,

(c) $p \neq 2$, G'_p is cyclic and $|G'_p| \geq p^2$,

(d) $p = 2$, G'_2 is cyclic and $|G'_2 \cap Z(D)| \geq 4$, where D is the subgroup of G_2 such that $G'_2 \subset D$ and $D/G'_2 = \operatorname{soc}(G_2/G'_2)$, or

(e) $[K : K^p] \leq p^2$.

Hence, the second Brauer–Thrall conjecture (see Remark 4.10) is valid for twisted group algebras of finite groups satisfying one of the conditions (a)–(d) and also for twisted group algebras of arbitrary finite groups over an infinite field K of characteristic p such that $[K : K^p] \leq p^2$.

Preliminaries

Throughout this paper, we use the following notations: K is an infinite field of characteristic p; K^* is the multiplicative group of K; $K^p = \{\alpha^p : \alpha \in K\}$; G is a finite p-group; G' is the commutant of G and G'' is the commutant of G'; $Z(G)$ is the center of G; e is the identity element of G; $|g|$ is the order of $g \in G$; $\operatorname{soc} B$ is the socle of an abelian p-group B. Moreover, we denote by $Z^2(G, K^*)$ the group of all K^*-valued normalized 2-cocycles of the group G, where we assume that G acts trivially on K^* (see [18, Chapter 1]).

Given a cocycle $\lambda : G \times G \to K^*$ in $Z^2(G, K^*)$, we denote by $K^\lambda G$ the twisted group algebra of the group G over the field K with the cocycle λ and by $\operatorname{rad} K^\lambda G$ the radical of $K^\lambda G$. We set $\overline{K^\lambda G} = K^\lambda G / \operatorname{rad} K^\lambda G$. We recall that in our case $\overline{K^\lambda G}$ is a finite purely inseparable field extension of K (see [18, p. 74]). A K-basis $\{u_g : g \in G\}$ of $K^\lambda G$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in G$ is called natural (corresponding to λ). All $K^\lambda G$-modules are assumed to be finite-dimensional left modules. If H is a subgroup of G, we often use the same symbol for an element $\lambda : G \times G \to K^*$ of $Z^2(G, K^*)$ and its restriction to $H \times H$. In this case, $K^\lambda H$ is a subalgebra of $K^\lambda G$.

If M is a $K^\lambda G$-module, we denote by $M_{K^\lambda H}$ the module M viewed as a $K^\lambda H$-module. If N is a $K^\lambda H$-module, $N^{K^\lambda G} = K^\lambda G \otimes_{K^\lambda H} N$ is the induced $K^\lambda G$-module. Let L be a field extension of K. If A is an L-algebra, we denote by A_K the algebra A viewed as a K-algebra.

Given $\lambda \in Z^2(G, K^*)$, the kernel $\mathrm{Ker}(\lambda)$ of λ is the union of all cyclic subgroups $\langle g \rangle$ of G such that the restriction of λ to $\langle g \rangle \times \langle g \rangle$ is a coboundary. We recall from Lemma 1 of [4] that $G' \subset \mathrm{Ker}(\lambda)$, $\mathrm{Ker}(\lambda)$ is a normal subgroup of G and the restriction of λ to $\mathrm{Ker}(\lambda) \times \mathrm{Ker}(\lambda)$ is a coboundary (see also [3, p. 197] for a simple proof). Up to cohomology in $Z^2(G, K^*)$, $\lambda_{g,a} = \lambda_{a,g} = 1$ for all $g \in G$ and $a \in \mathrm{Ker}(\lambda)$. In what follows, we assume that every cocycle $\lambda \in Z^2(G, K^*)$ under consideration satisfies this condition.

Let H be a normal subgroup of G, $H \subset \mathrm{Ker}(\lambda)$ and $T = G/H$. We set $\mu_{xH,yH} = \lambda_{x,y}$, for all $x, y \in G$. Then $\mu \in Z^2(T, K^*)$. Assume that $\{u_g : g \in G\}$ is a natural K-basis of $K^\lambda G$ corresponding to λ and $\{v_{gH} : g \in G\}$ is a natural K-basis of $K^\mu T$ corresponding to μ. The formula

$$f\left(\sum_{g \in G} \alpha_g u_g\right) = \sum_{g \in G} \alpha_g v_{gH}$$

defines a K-algebra epimorphism $f : K^\lambda G \to K^\mu T$ with the kernel $U = K^\lambda G \cdot \operatorname{rad} K^\lambda H$ (see [18, p. 88]). Hence $K^\lambda G/U \cong K^\mu T$. We recall that

$$\operatorname{rad} K^\lambda H = \bigoplus_{h \in H \setminus \{e\}} K(u_h - u_e)$$

is called the augmentation ideal of the group algebra $K^\lambda H$. If $G' \subset H$ then $K^\mu T$ is a commutative algebra. We often identify $u_g + U$ with v_{gH}.

Let V be a finite-dimensional vector space over K and $\Gamma : G \to \mathrm{GL}(V)$ a projective representation of G with a 2-cocycle $\lambda \in Z^2(G, K^*)$. We refer to Γ as a λ-representation of G over the field K (see [18, p. 106]). The dimension of V is called the dimension of Γ. If we view V as a module over $K^\lambda G$ we say that V is the underlying module of the λ-representation Γ (see [10, p. 74]). Let $\mathrm{PGL}(V) = \mathrm{GL}(V)/K^* \cdot 1_V$ and $\pi : \mathrm{GL}(V) \to \mathrm{PGL}(V)$ be the canonical group homomorphism. The kernel of the homomorphism $\pi \circ \Gamma : G \to \mathrm{PGL}(V)$ is called the kernel of Γ and is denoted by $\mathrm{Ker}(\Gamma)$. If $\mathrm{Ker}(\Gamma) = \{e\}$, the representation Γ is called *faithful*. Recall that if G is a finite p-group, K is a field of characteristic

p and Γ is an irreducible λ-representation of G over K then $\mathrm{Ker}(\Gamma) = \mathrm{Ker}(\lambda)$ (see [3, p. 198]).

Let R be a finite-dimensional algebra with identity over a field K. All R-modules are assumed to be finite-dimensional left modules. We recall from [12, p. 437] that an R-module V is defined to be *absolutely indecomposable* if for every field extension L of K, $L \otimes_K V$ is an indecomposable module over $L \otimes_K R$. Applying Lemma 18.7 from [10, p. 72], we can see immediately that an R-module V is absolutely indecomposable if for every finite field extension L of K, $L \otimes_K V$ is an indecomposable module over $L \otimes_K R$. If an absolutely indecomposable R-module V is the underlying module of a representation Γ of R, we say that Γ is an absolutely indecomposable representation of R. Denote by $[M]$ the isomorphism class of R-modules that contains M. We denote by $\mathrm{AInd}(R, s)$ the set of all $[V]$, where V is an absolutely indecomposable R-module of K-dimension s. Moreover, we denote by $\mathrm{FAInd}(K^\lambda G, s)$ the set of all $[W]$, where W is the underlying $K^\lambda G$-module of a faithful absolutely indecomposable λ-representation of G over K of dimension s.

Let G be an abelian p-group, K a field of characteristic p and $\lambda \in Z^2(G, K^*)$. We recall from [5, p. 175–176] that the following statements hold:

(i) The algebra $K^\lambda G$ is uniserial if and only if $G = H \times B$, where H is a cyclic group and $K^\lambda B$ is a field.

(ii) If $G = H \times B$, where H is a cyclic group and $K^\lambda H$ is not a field, then the algebra $K^\lambda G$ is not uniserial if and only if $K^\lambda B$ is not a field.

(iii) Let $G = B \times \langle c_1 \rangle \times \cdots \times \langle c_s \rangle$ and $D_i = B \times \langle c_i \rangle$ for $i = 1, \ldots, s$. Assume that $K^\lambda B$ is a field and $K^\lambda D_i$ is not a field for every $i \in \{1, \ldots, s\}$. The algebra $K^\lambda G$ is not uniserial if and only if $s \geq 2$.

The reader is referred to [10], [18] and [19] for basic facts and notation from group representation theory and to [1] and [9] for terminology, notation and introduction to the representation theory of finite-dimensional algebras over a field.

1. On induced modules

Let R be an algebra with identity $1 \neq 0$ over a field K, $\mathrm{Aut}_K(R)$ the group of all K-automorphisms of R, $U(R)$ the unit group of R and H a finite group with the identity element e. By a crossed product of H over R we understand a K-algebra $R * H$ which is a free left R-module with a basis $\{u_h : h \in H\}$ such that

$$(ru_g)(su_h) = rs^{\sigma(g)}\lambda_{g,h}u_{gh}$$

for all $r, s \in R$, $g, h \in H$, where $\sigma : H \to \mathrm{Aut}_K(R)$ and $\sigma(e) = \mathrm{id}_R$, $\lambda : H \times H \to U(R)$ and $\lambda_{x,e} = \lambda_{e,x} = 1$ for any $x \in H$.

The element $1u_e = u_e$ is the identity element of $R * H$. Every u_h is a unit of $R * H$. The embedding of R into $R * H$ is given by $r \mapsto ru_e$.

Lemma 1.1. *Let R be a finite-dimensional algebra with identity over a field K of characteristic p, G a finite p-group and $A = R * G$.*

(i) *If V is an absolutely indecomposable R-module then the induced A-module $V^A = A \otimes_R V$ is absolutely indecomposable of K-dimension $|G| \cdot \dim_K V$.*

(ii) *If $\mathrm{AInd}(R, n)$ is an infinite set for some integer $n \geq 1$ then $\mathrm{AInd}(R * G, n|G|)$ is infinite as well.*

Proof. (i) See [19, p. 538].

(ii) Let $[V] \in \mathrm{AInd}(R, n)$. Then, by (i), $[V^A] \in \mathrm{AInd}(R * G, n|G|)$. Since $(V^A)_R \cong V \oplus W$, where W is an R-module, the set of all isomorphism classes $[V^A]$ is infinite, in view of the Krull–Schmidt Theorem. $\square$

Lemma 1.2. *Let L be a finite purely inseparable field extension of a field K of characteristic p, G a non-cyclic p-group and $f_n = nl[L : K]$, where*

$$l = \begin{cases} 1 & \text{if } |G : G'| \neq 4, \\ 2 & \text{if } |G : G'| = 4. \end{cases}$$

Then $\mathrm{AInd}\big((LG)_K, f_n\big)$ is infinite for any integer $n \geq 2$.

Proof. Let E_n be the identity matrix of order n, $J_n(\mu)$ the upper Jordan block of order n with the eigenvalue μ and

$$B_n(\mu) = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

the $n \times 1$-matrix, where $\mu \in K$ is a parameter. Denote by $H = \langle a \rangle \times \langle b \rangle$ an abelian group of order p^2.

If $p \neq 2$ then, by [13], the group H has the indecomposable linear matrix representations over K of the form:

$$\Gamma_\mu : a \mapsto \begin{pmatrix} E_n & J_n(\mu) \\ 0 & E_n \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} E_n & E_n \\ 0 & E_n \end{pmatrix};$$

$$\Delta_\mu : a \mapsto \begin{pmatrix} E_n & 0 & E_n \\ 0 & E_1 & 0 \\ 0 & 0 & E_n \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} E_n & E_n & 0 \\ 0 & E_n & B_n(\mu) \\ 0 & 0 & E_1 \end{pmatrix}. \tag{1}$$

Since K is an arbitrary field, these representations are absolutely indecomposable.

The group G has a normal subgroup N such that $G/N \cong H$. Hence in the case $p \neq 2$ the set $\operatorname{AInd}(KG, m)$ is infinite, for any integer $m \geq 2$.

Let $p = 2$. The group $H = \langle a \rangle \times \langle b \rangle$ has the absolutely indecomposable matrix representations Γ_μ of dimension $2n$ over K of the form (1). Let $D = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ and $T = \langle x \rangle \times \langle y \rangle$ be abelian groups of order 8 with $|x| = 4$. Gudivok in [13] established that these groups have the absolutely indecomposable matrix representations of dimension $2n+1$ over K of the form:

$$a \mapsto \begin{pmatrix} E_n & 0 & E_n \\ 0 & E_1 & 0 \\ E_1 & 0 & E_n \end{pmatrix}, \quad b \mapsto \begin{pmatrix} E_n & E_n & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_1 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} E_n & \mu E_n & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_1 \end{pmatrix};$$

$$x \mapsto \begin{pmatrix} E_n & E_n & 0 \\ 0 & E_n & B_n(\mu) \\ 0 & 0 & E_1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} E_n & 0 & E_n \\ 0 & E_1 & 0 \\ 0 & 0 & E_n \end{pmatrix},$$

where $\mu \in K$ is a parameter.

If G is a non-cyclic 2-group and $|G : G'| \neq 4$ then G has a normal subgroup N such that G/N is isomorphic to D or T. Hence in this case $\operatorname{AInd}(KG, m)$ is infinite for any integer $m \geq 2$. In the case $|G : G'| = 4$ the set $\operatorname{AInd}(KG, 2n)$ is infinite for any integer $n \geq 1$.

Suppose that $[L : K] = p^s$. Then $L = K(\theta_1, \ldots, \theta_s)$, where $\theta_1^p \in K$, $\theta_2^p \in K(\theta_1), \ldots, \theta_s^p \in K(\theta_1, \ldots, \theta_{s-1})$. Let $L_0 = K$ and $L_i = K(\theta_1, \ldots, \theta_i)$ for $i \in \{1, \ldots, s\}$. The group algebra $L_{i+1}G$ is a crossed product of the cyclic group of order p over the group algebra L_iG for every $i \in \{0, 1, \ldots, s-1\}$. Let $[V] \in \operatorname{AInd}(KG, nl)$. Applying Lemma 1.1 and transitivity of induction, we establish that V^{L_iG} is an absolutely indecomposable module over the K-algebra L_iG and $\dim_K V^{L_iG} = nlp^i$ for any $i \in \{1, \ldots, s\}$. Since

$$\left(V^{LG}\right)_{KG} \cong V \bigoplus \cdots \bigoplus V \quad (p^s \text{ times}),$$

we obtain

$$V^{LG} \cong W^{LG} \Leftrightarrow V \cong W.$$

Therefore $\mathrm{AInd}\big((LG)_K, f_n\big)$ is infinite for any $n \geq 2$, where $f_n = nl[L : K]$. □

We note that $|G : G'| = 4$ if and only if G is a dihedral, semidihedral or (generalized) quaternion group [15, p. 339].

Proposition 1.3. *Let G be a finite p-group, T a non-cyclic subgroup of G and*

$$l = \begin{cases} 1 & \text{if } |T : T'| \neq 4, \\ 2 & \text{if } |T : T'| = 4. \end{cases}$$

If $\lambda \in Z^2(G, K^*)$ and $T \subset \mathrm{Ker}(\lambda)$ then $\mathrm{AInd}(K^\lambda G, f_n)$ is infinite for every integer $n \geq 2$, where $f_n = nl|G : T|$.

Proof. Apply Lemmas 1.1, 1.2 and transitivity of induction. □

Lemma 1.4. *Let $R = K + Ku + Kv$ be the algebra over an infinite field K of characteristic p with the defining relations:*

$$u^2 = 0, \quad v^2 = 0, \quad uv = vu = 0. \tag{2}$$

The set $\mathrm{AInd}(R, 2n)$ is infinite, for any integer $n \geq 1$.

Proof. Denote by M_α an underlying R-module of the matrix representation

$$\Gamma_\alpha : 1 \mapsto \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix}, \quad u \mapsto \begin{pmatrix} 0 & J_n(\alpha) \\ 0 & 0 \end{pmatrix}, \quad v \mapsto \begin{pmatrix} 0 & E_n \\ 0 & 0 \end{pmatrix},$$

where E_n is the identity matrix of order $n \geq 1$ and $J_n(\alpha)$ is the upper Jordan block of order n with the eigenvalue α. By the proof of Lemma 1.2, M_α is an absolutely indecomposable R-module of K-dimension $2n$. If $\alpha \neq \beta$, the modules M_α and M_β are non-isomorphic. □

Proposition 1.5. *Let G be a finite p-group and R the algebra over an infinite field K of characteristic p with the defining relations* (2)*. Then $\mathrm{AInd}(R{*}G, 2n|G|)$ is infinite for any integer $n \geq 1$.*

Proof. Apply Lemmas 1.1 and 1.4. □

2. On absolutely indecomposable representations of a commutative twisted group algebra

The aim of this section is to prove Theorem 2.4 that is the first main result of the paper. We use this theorem to obtain other main results.

Lemma 2.1. *Let K be a non-perfect field of characteristic p and θ a root of the irreducible polynomial $X^{p^n} - \alpha \in K[X]$. If $\rho \in K(\theta)$, $\rho^{p^r} \in K$ and $\rho^{p^{r-1}} \notin K$, then $r \leq n$, $\rho \in K(\theta^{p^{n-r}})$ and $\rho \notin K(\theta^{p^{n-r+1}})$.*

Proof. There is the chain of fields

$$K \subset K(\theta^{p^{n-1}}) \subset K(\theta^{p^{n-2}}) \subset \cdots \subset K(\theta^p) \subset K(\theta),$$

where, for every $i \in \{0, 1, \ldots, n-1\}$, $K(\theta^{p^i})$ is a field extension of $K(\theta^{p^{i+1}})$ of degree p. Assume that $\rho \in K(\theta^{p^j})$ and $\rho \notin K(\theta^{p^{j+1}})$, for some $j \in \{0, 1, \ldots, n-1\}$. Then

$$\rho = \sum_{i=0}^{p-1} \sigma_i \theta^{ip^j},$$

where $\sigma_i \in K(\theta^{p^{j+1}})$, for every $i \in \{0, 1, \ldots, p-1\}$, and $\sigma_{i_0} \neq 0$ for some $i_0 \in \{1, \ldots, p-1\}$. It follows that $\rho^{p^{n-j}} \in K$. At the same time $\sigma_i^{p^{n-j-1}} \in K$, for every $i \in \{0, 1, \ldots, p-1\}$, and $\theta^{p^{n-1}}$ is a root of the irreducible polynomial $X^p - \alpha$ over K. Hence, $\rho^{p^{n-j-1}} \notin K$. Since p^r and p^{n-j} are degrees of minimal polynomials of the element ρ over K, we conclude that $r = n - j$. Consequently, $\rho \in K(\theta^{p^{n-r}})$ and $\rho \notin K(\theta^{p^{n-r+1}})$. $\square$

Lemma 2.2. *Let K be a non-perfect field of characteristic p and L a finite purely inseparable field extension of K. Suppose that $\alpha, \beta \in L^*$ and $\alpha, \beta \notin L^p$; $G = \langle a \rangle \times \langle b \rangle$ is an abelian group of type (p^{k+1}, p^{m+1}) where $k \geq 1$ and $m \geq 1$ are integers; $\mu \in Z^2(G, L^*)$, $\{u_g : g \in G\}$ is a natural L-basis of*

$$L^\mu G = \bigoplus_{i=0}^{p^{k+1}-1} \bigoplus_{j=0}^{p^{m+1}-1} L u_a^i u_b^j, \quad u_a^{p^{k+1}} = \alpha^p u_e, \quad u_b^{p^{m+1}} = \beta^p u_e;$$

$d = \dim_L \overline{L^\mu G}$ and

$$l = \begin{cases} 1 & \text{if } \ 4d \neq |G|, \\ 2 & \text{if } \ 4d = |G|. \end{cases}$$

The set $\mathrm{AInd}\big((L^\mu G)_K, f_n\big)$ is infinite for any integer $n \geq 2$, where $f_n = nld[L : K]$.

Proof. We can suppose that $k \le m$. Let $U = L^\mu G(u_a^{p^k} - \alpha u_e)$, $\widetilde{L^\mu G} = L^\mu G/U$ and $\tilde{w} = w + U$, for every $w \in L^\mu G$. Then

$$\widetilde{L^\mu G} = \bigoplus_{i=0}^{p^k-1} \bigoplus_{j=0}^{p^{m+1}-1} L\tilde{u}_a^i \tilde{u}_b^j,$$

where $\tilde{u}_a^{p^k} = \alpha\tilde{u}_e$ and $\tilde{u}_b^{p^{m+1}} = \beta^p \tilde{u}_e$.

Since $\alpha \notin L^p$, $F := L[\tilde{u}_a]$ is a field. We can view $\widetilde{L^\mu G}$ as a twisted group algebra of the group $\langle b\rangle$ of order p^{m+1} over the field F. Assume that $\beta\tilde{u}_e = \rho^{p^r}$, where $r \ge 1$, $\rho \in F$ and $\rho \notin F^p$. We have $r \le k$ and

$$\widetilde{L^\mu G}/\operatorname{rad}\widetilde{L^\mu G} \cong F(\theta), \quad \theta^{p^{m-r}} = \rho.$$

Hence $d = p^k \cdot p^{m-r} = p^{k+m-r}$ and, by Lemma 2.1,

$$\rho = \sum_{t=0}^{p^r-1} \alpha_t \tilde{u}_a^{tp^{k-r}}, \quad \alpha_t \in L.$$

Let

$$z = \sum_{t=0}^{p^r-1} \alpha_t u_a^{tp^{k-r}}.$$

Then

$$z^{p^{r+1}} = \sum_{t=0}^{p^r-1} \alpha_t^{p^{r+1}} u_a^{tp^{k+1}} = \sum_{t=0}^{p^r-1} \alpha_t^{p^{r+1}} \alpha^{pt} u_e = \left(\sum_{t=0}^{p^r-1} \alpha_t^{p^r} \alpha^t\right)^p u_e = \beta^p u_e.$$

Denote by H the subgroup of G generated by the elements $h_1 = a^{p^{k-r}}$ and $h_2 = b^{p^{m-r}}$. Then

$$L^\mu H = \bigoplus_{i,j=0}^{p^{r+1}-1} L u_{h_1}^i u_{h_2}^j,$$

where $u_{h_1} = u_a^{p^{k-r}}$, $u_{h_2} = u_b^{p^{m-r}}$, and we have

$$u_{h_1}^{p^{r+1}} = \alpha^p u_e, \quad u_{h_2}^{p^{r+1}} = \beta^p u_e.$$

Let $v_{h_2} = z^{-1} u_{h_2}$. Then

$$L^\mu H = L^\nu H = \bigoplus_{i,j=0}^{p^{r+1}-1} L u_{h_1}^i v_{h_2}^j, \quad v_{h_2}^{p^{r+1}} = u_e.$$

Denote by D the subgroup of H generated by the elements $c = h_1^{p^r}$ and h_2. If we set $v_c = u_{h_1}^{p^r}$, we get

$$L^\nu D = \bigoplus_{i=0}^{p-1} \bigoplus_{j=0}^{p^{r+1}-1} L v_c^i v_{h_2}^j, \quad v_c^p = \alpha^p u_e, \quad v_{h_2}^{p^{r+1}} = u_e.$$

Hence, in view of Lemmas 1.1, 1.2, the set $\mathrm{AInd}\big((L^\nu H)_K, n[L:K]\cdot|H:D|\big)$ is infinite, for every integer $n \geq 2$, because $L^\nu D$ is the group algebra of the non-cyclic group D over L and $|D| = p^{r+2} > 4$. Evidently, $|H:D| = p^r$.

Applying again Lemma 1.1, we conclude that the set

$$\mathrm{AInd}\big((L^\mu G)_K, n[L:K]p^r|G:H|\big)$$

is infinite. Since

$$p^r \cdot |G:H| = p^r \frac{|G|}{|H|} = p^r \frac{p^{k+1}\cdot p^{m+1}}{p^{r+1}\cdot p^{r+1}} = p^{k+m-r} = d,$$

the set $\mathrm{AInd}\big((L^\mu G)_K, n[L:K]d\big)$ is infinite, for any $n \geq 2$. We note that in this case $|G| = dp^{r+2}$, and therefore $|G| \neq 4d$.

Now we assume that $\beta\tilde{u}_e \notin F^p$. Then

$$\widetilde{L^\mu G}/\operatorname{rad}\widetilde{L^\mu G} \cong F(\theta), \quad \theta^{p^m} = \beta\tilde{u}_e.$$

It follows that $d = p^{k+m}$. We denote by H the subgroup of G generated by the elements a^{p^k} and b^{p^m}. Because H is of type (p,p) and $L^\mu H$ is the group algebra of H over L, we conclude, by Lemma 1.2, that $\mathrm{AInd}\big((L^\mu H)_K, nl[L:K]\big)$ is infinite for any $n \geq 2$. The algebra $\big(L^\mu G\big)_K$ is a crossed product of G/H over $\big(L^\mu H\big)_K$. Now, Lemma 1.1 implies that the set $\mathrm{AInd}\big((L^\mu G)_K, nl[L:K]\cdot|G:H|\big)$ is infinite for any $n \geq 2$. Clearly, $|G:H| = d$. □

Remark 2.3. $d \cdot [L:K] = \dim_K \overline{L^\mu G}$, where $d = \dim_L \overline{L^\mu G}$.

We are now able to prove the first main result of this paper.

Theorem 2.4. *Let G be an abelian p-group, K an infinite field of characteristic p, $\lambda \in Z^2(G, K^*)$, $K^\lambda G$ a non-uniserial algebra, $d = \dim_K \overline{K^\lambda G}$ and*

$$l = \begin{cases} 1 & \text{if } 4d \neq |G|, \\ 2 & \text{if } 4d = |G|. \end{cases}$$

The set $\mathrm{AInd}(K^\lambda G, nld)$ is infinite, for arbitrary integer $n \geq 2$.

Proof. There exists a direct product decomposition $G = B\times\langle c_1\rangle\times\cdots\times\langle c_s\rangle$ such that $s\geq 2$, $F := K^\lambda B$ is a field and the twisted group algebra $K^\lambda D_i$ of the group $D_i := B\times\langle c_i\rangle$ is not a field for every $i\in\{1,\dots,s\}$. The field F is a finite purely inseparable field extension of K. First we suppose that $4d = |G|$. Then $p = 2$, $s = 2$ and

$$K^\lambda G = \bigoplus_{i_1=0}^{2^{n_1}-1}\bigoplus_{i_2=0}^{2^{n_2}-1} F u_{c_1}^{i_1}u_{c_2}^{i_2},\quad u_{c_1}^{2^{n_1}} = \alpha_1^2 u_e,\quad u_{c_2}^{2^{n_2}} = \alpha_2^2 u_e,$$

where $|c_i| = 2^{n_i}$, $\alpha_i\in F^*$ for $i = 1,2$ and $d = [F:K]\cdot 2^{n_1+n_2-2}$. Hence $K^\lambda G = F^\mu H$, where $H = \langle c_1\rangle\times\langle c_2\rangle$. Let

$$x_i = c_i^{2^{n_i-1}},\quad v_{x_i} = \alpha_i^{-1}u_{c_i}^{2^{n_i-1}},\quad i = 1,2,$$

and $T = \langle x_1\rangle\times\langle x_2\rangle$. Then $v_{x_i}^2 = u_e$, therefore $F^\mu T$ is the group algebra of the group T of order 4 over F. Now, Lemmas 1.1 and 1.2 imply that $\mathrm{AInd}(K^\lambda G, 2n[F:K]\cdot|H:T|)$ is infinite for every $n\geq 2$. Since $[F:K]\cdot|H:T| = d$, the set $\mathrm{AInd}(K^\lambda G, 2nd)$ is infinite, for any $n\geq 2$.

Now assume that $4d\neq |G|$ and let

$$D = \begin{cases} B, & \text{if } s = 2,\\ B\times\langle c_3\rangle\times\cdots\times\langle c_s\rangle, & \text{if } s\neq 2.\end{cases}$$

Clearly, $G = D\times H$ with $H = \langle c_1\rangle\times\langle c_2\rangle$. Then $V := K^\lambda G\cdot\operatorname{rad}K^\lambda D$ is an ideal of $K^\lambda G$ and $K^\lambda G/V\cong L^\mu H$, where L is a finite purely inseparable field extension of K and $L\cong K^\lambda D/\operatorname{rad}K^\lambda D$. We have

$$L^\mu H = \bigoplus_{i_1=0}^{p^{n_1}-1}\bigoplus_{i_2=0}^{p^{n_2}-1} L u_{c_1}^{i_1}u_{c_2}^{i_2},\quad u_{c_1}^{p^{n_1}} = \alpha_1^{p^{m_1}}u_e,\quad u_{c_2}^{p^{n_2}} = \alpha_2^{p^{m_2}}u_e,$$

where $p^{n_j} = |c_j|$, $\alpha_j\in L^*$ and $m_j > 0$ for every $j\in\{1,2\}$. If $m_j\geq n_j$, we can assume that $\alpha_j = 1$. If $m_j < n_j$, we assume that $\alpha_j\notin L^p$. Since there exists a K-algebra isomorphism $\overline{K^\lambda G}\cong\overline{L^\mu H} = L^\mu H/\operatorname{rad}L^\mu H$, we get $\dim_K\overline{L^\mu H} = d$.

We start with the case $|H|\neq 4\cdot\dim_L\overline{L^\mu H}$. If $\alpha_1 = \alpha_2 = 1$, then $d = [L:K]$ and, by Lemma 1.2, $\mathrm{AInd}\big((L^\mu H)_K, n[L:K]\big)$ is infinite, for any $n\geq 2$. Hence $\mathrm{AInd}(K^\lambda G, nd)$ is infinite, for any $n\geq 2$. If $\alpha_1 = 1$, $\alpha_2\notin L^p$ then $\overline{K^\lambda G}\cong L(\theta)$, where θ is a root of the irreducible polynomial $X^{p^{n_2-m_2}} - \alpha_2$ over L. It follows that $d = [L:K]\cdot p^{n_2-m_2}$. Since the order of the group $\langle c_1\rangle\times\langle c_2^{p^{n_2-m_2}}\rangle$ is not equal to 4, then, by Lemmas 1.1 and 1.2, $\mathrm{AInd}\big((L^\mu H)_K, n[L:K]p^{n_2-m_2}\big)$ is infinite for every $n \geq 2$. Hence $\mathrm{AInd}(K^\lambda G, nd)$ is infinite, for any integer $n \geq 2$.

Let $\alpha_1 \notin L^p$ and $\alpha_2 \notin L^p$. Denote by θ_i a root of the irreducible polynomial

$$X^{p^{n_i-m_i}} - \alpha_i$$

over L for $i = 1, 2$. If $[L(\theta_1, \theta_2) : L] = p^{n_1-m_1+n_2-m_2}$, then

$$d = [L : K] \cdot p^{n_1-m_1+n_2-m_2}.$$

Since the order of the group $\langle c_1^{p^{n_1-m_1}} \rangle \times \langle c_2^{p^{n_2-m_2}} \rangle$ is not equal to 4, then, by Lemmas 1.1 and 1.2, the set $\mathrm{AInd}\big((L^\mu H)_K, nd\big)$ is infinite for every $n \geq 2$. Consequently $\mathrm{AInd}(K^\lambda G, nd)$ is infinite for arbitrary $n \geq 2$.

Now, let $[L(\theta_1, \theta_2) : L] < p^{n_1-m_1+n_2-m_2}$ and $X = \langle x_1 \rangle \times \langle x_2 \rangle$ be a group of type $\big(p^{n_1-m_1+1}, p^{n_2-m_2+1}\big)$. There exists an L-algebra epimorphism $L^\mu H \to L^\nu X$, where

$$L^\nu X = \bigoplus_{j_1=0}^{p^{n_1-m_1+1}-1} \bigoplus_{j_2=0}^{p^{n_2-m_2+1}-1} L v_{x_1}^{j_1} v_{x_2}^{j_2}, \quad v_{x_1}^{p^{n_1-m_1+1}} = \alpha_1^p v_e, \quad v_{x_2}^{p^{n_2-m_2+1}} = \alpha_2^p v_e.$$

Let $d' = \dim_L \overline{L^\nu X}$. By assumption, $d' < p^{n_1-m_1+n_2-m_2}$. According to Lemma 2.2, $\mathrm{AInd}\left((L^\nu X)_K, nd'[L:K]\right)$ is infinite for every $n \geq 2$, because $|X| \neq 4d'$. But $d'[L:K] = d$. Hence $\mathrm{AInd}(K^\lambda G, nd)$ is infinite for any $n \geq 2$.

We may suppose that in previous reasonings $H = \langle c_{i_1} \rangle \times \langle c_{i_2} \rangle$ and $|H| \neq 4 \dim_L \overline{L^\mu H}$, where $i_1 < i_2$ and $i_1, i_2 \in \{1, \dots, s\}$. Let us consider the case where $p = 2$, $H = \langle c_{i_1} \rangle \times \langle c_{i_2} \rangle$ and $|H| = 4 \dim_L \overline{L^\mu H}$ for all distinct $i_1, i_2 \in \{1, \dots, s\}$. Since $4d \neq |G|$, there exists a K-algebra homomorphism of $K^\lambda G$ onto

$$M^\nu C = \bigoplus_{i_1=0}^{2^{n_1+1}-1} \bigoplus_{i_2=0}^{2^{n_2+1}-1} \bigoplus_{i_3=0}^{2^{n_3+1}-1} M v_{c_1}^{i_1} v_{c_2}^{i_2} v_{c_3}^{i_3}, \quad v_{c_j}^{2^{n_j+1}} = \alpha_j^2 v_e \quad \text{for} \quad j = 1, 2, 3,$$

where M is a finite purely inseparable field extension of K, $C = \langle c_1 \rangle \times \langle c_2 \rangle \times \langle c_3 \rangle$, $n_j \geq 0$, $\alpha_j \in M^*$ and $\dim_M \overline{M^\nu C} = 2^{n_1+n_2+n_3}$. The algebra $M^\nu C$ contains the group algebra MT, where

$$T = \langle c_1^{2^{n_1}} \rangle \times \langle c_2^{2^{n_2}} \rangle \times \langle c_3^{2^{n_3}} \rangle$$

is of type $(2, 2, 2)$. By Lemmas 1.1 and 1.2, $\mathrm{AInd}\big((M^\nu C)_K, n[M:K] \cdot |C:T|\big)$ is infinite for any $n \geq 2$. Since $[M:K] \cdot |C:T| = d$, the set $\mathrm{AInd}(K^\lambda G, nd)$ is infinite for any $n \geq 2$. $\square$

Let us remark that Sobolewska in [23] has found some infinite subsets of the set of all integers $m \geq 1$ for which a non-uniserial twisted group algebra of a finite abelian p-group over an infinite field of characteristic p has infinitely many indecomposable representations of dimension m.

Corollary 2.5. *Let G be an abelian p-group, K a non-perfect field of characteristic p, $\lambda \in Z^2(G, K^*)$, $d = \dim_K \overline{K^\lambda G}$ and*

$$l = \begin{cases} 1 & \text{if } 4d \neq |G|, \\ 2 & \text{if } 4d = |G|. \end{cases}$$

If $\operatorname{Ker}(\lambda) = \{e\}$ and $K^\lambda G$ is not uniserial, then the set $\operatorname{FAInd}(K^\lambda G, nld)$ is infinite for any integer $n \geq 2$.

Note that Corollary 2.5 generalizes Corollary 2.4 in [2].

Corollary 2.6. *Let G be a finite p-group, K an infinite field of characteristic p, $\lambda \in Z^2(G, K^*)$, $d = \dim_K \overline{K^\lambda G}$ and*

$$l = \begin{cases} 1 & \text{if } 4d \neq |G : G'|, \\ 2 & \text{if } 4d = |G : G'|. \end{cases}$$

If $K^\lambda G / K^\lambda G \cdot \operatorname{rad} K^\lambda G'$ is not a uniserial algebra then $\operatorname{AInd}(K^\lambda G, nld)$ is infinite for every integer $n \geq 2$.

Remark 2.7. If we assume in Corollary 2.6 that the cocycle λ is trivial, then $K^\lambda G$ is the group algebra, $d = 1$, and we obtain the result of Gudivok [13].

3. On absolutely indecomposable representations of a non-commutative twisted group algebra $K^\lambda G$ with $p \dim_K \overline{K^\lambda G} \neq |G : G'|$

First we prove two useful lemmas that can be used to reduce a general case to the case of a p-group G such that G' is an elementary abelian group of type (p, p) or a group of order p.

Lemma 3.1. *Let G be a non-abelian p-group with non-cyclic commutant G'. Then G contains a normal subgroup H such that $H \subset G'$ and $(G/H)' = G'/H$ is an elementary abelian group of type (p, p).*

Proof. Let $\Phi(G')$ be the Frattini subgroup of G'. Then $\Phi(G') \lhd G$ and $G'/\Phi(G')$ is a non-cyclic elementary abelian p-group. Since $\big(G/\Phi(G')\big)'=G'/\Phi(G')$, in what follows, we assume that G' is a non-cyclic elementary abelian p-group.

Let $|G'| = p^n$ and $n > 2$. Let us choose an element $a \neq e$ in G' such that $a \in Z(G)$. Then the commutant of the factor group $G/\langle a\rangle$ is a non-cyclic elementary abelian group of order p^{n-1}. If $n-1 > 2$, we inductively continue the above construction. $\square$

Lemma 3.2. *Let G be a non-abelian p-group, $G' = \langle c\rangle$, $H = \langle c^p\rangle$, $\lambda \in Z^2(G, K^*)$ and $V = K^\lambda G(u_c^p - u_e)$. Then V is an ideal of $K^\lambda G$ and $K^\lambda G/V \cong K^\mu T$, where $T = G/H$ and $\mu_{xH,yH} = \lambda_{x,y}$ for all $x, y \in G$. If $K^\lambda G$ is not uniserial then $K^\mu T$ is not uniserial either.*

Proof. Since $G' \subset \operatorname{Ker}(\lambda)$ and H is a normal subgroup of G, V is an ideal of $K^\lambda G$ and $K^\lambda G/V \cong K^\mu T$. Let $\widehat{K^\lambda G} = K^\lambda G/V$ and $\hat{w} = w + V$ for every $w \in K^\lambda G$. Because V is a nilpotent ideal of the algebra $K^\lambda G$, we have

$$\operatorname{rad} \widehat{K^\lambda G} = \big(\operatorname{rad} K^\lambda G + V\big)/V = \big(\operatorname{rad} K^\lambda G\big)/V.$$

Assume that $\widehat{K^\lambda G}$ is a uniserial algebra. Then $\operatorname{rad}\widehat{K^\lambda G} = \widehat{K^\lambda G}\cdot\hat{\theta}$, for some $\theta \in \operatorname{rad} K^\lambda G$ (see [9, p. 170]). Consequently, for every $w \in \operatorname{rad} K^\lambda G$ there is $z \in K^\lambda G$ such that $w + V = (z + V)(\theta + V)$. It follows that $w - z\theta \in (\operatorname{rad} K^\lambda G)^2$, because

$$u_c^p - u_e = (u_c - u_e)^p \in (\operatorname{rad} K^\lambda G)^2.$$

This yields

$$w + (\operatorname{rad} K^\lambda G)^2 = \big[z + (\operatorname{rad} K^\lambda G)^2\big]\cdot\big[\theta + (\operatorname{rad} K^\lambda G)^2\big].$$

Hence the radical of the algebra $K^\lambda G/(\operatorname{rad} K^\lambda G)^2$ is a principal left ideal; thus $K^\lambda G/(\operatorname{rad} K^\lambda G)^2$ is uniserial. But $K^\lambda G/(\operatorname{rad} K^\lambda G)^2$ is uniserial if and only if $K^\lambda G$ is uniserial (see [9, p. 172]). Therefore $K^\lambda G$ is a uniserial algebra. $\square$

Lemma 3.3. *Let G be a non-abelian p-group with $G' = \langle c\rangle$ of order p, H an abelian subgroup of G such that $G' \subset H$ and $G' \neq H$. Assume that $\lambda \in Z^2(G, K^*)$, $K^\lambda G$ is not uniserial and $K^\lambda G/K^\lambda G(u_c - u_e)$ is uniserial. If $K^\lambda H/K^\lambda H(u_c - u_e)$ is not a field then $K^\lambda H$ is not a uniserial algebra.*

Proof. If $H = A \times \langle c\rangle$ then $K^\lambda A \cong K^\lambda H/K^\lambda H(u_c - u_e)$ is not a field. Hence $K^\lambda H$ is not uniserial.

Now let $\langle c\rangle$ not be a direct factor of H. Then (see [11, p. 119]) there exists a decomposition of H into a direct product $H = \langle h_1\rangle \times \cdots \times \langle h_n\rangle$, where $c \in \langle h_n\rangle$ and $|c| < |h_n|$. If $K^\lambda H$ is a uniserial algebra, $|h_n| = p^{t+1}$ and $c = h_n^{p^t}$, then

$$K^\lambda H = \bigoplus_{i_1,\dots,i_n} K u_{h_1}^{i_1} \dots u_{h_n}^{i_n},$$

where

$$u_{h_j}^{|h_j|} = \delta_j u_e, \quad \delta_j \in K^* \quad \text{for} \quad j = 1, \dots, n-1,$$
$$u_{h_n}^{p^t} = \alpha u_c, \quad \alpha \in K^*$$

and

$$L := \bigoplus_{i_1,\dots,i_{n-1}} K u_{h_1}^{i_1} \dots u_{h_{n-1}}^{i_{n-1}}$$

is a field. Since $K^\lambda H / K^\lambda H(u_c - u_e)$ is not a field, we have $\alpha u_e = \theta^p$ for some $\theta \in L$. From the equality

$$\left(\theta^{-1} u_{h_n}^{p^{t-1}} - u_e\right)^p = u_c - u_e$$

it follows that $u_c - u_e \in (\operatorname{rad} K^\lambda H)^2$. Because $K^\lambda G$ is a local algebra, we conclude that $\operatorname{rad} K^\lambda H \subset \operatorname{rad} K^\lambda G$, consequently $K^\lambda G(u_c - u_e) \subset (\operatorname{rad} K^\lambda G)^2$. By hypothesis, the algebra $K^\lambda G / K^\lambda G(u_c - u_e)$ is uniserial. Arguing as in the proof of Lemma 3.2, we show that $K^\lambda G / (\operatorname{rad} K^\lambda G)^2$ is a uniserial algebra. Then $K^\lambda G$ is also uniserial and we get a contradiction. Therefore, $K^\lambda H$ is not uniserial. $\square$

Proposition 3.4. *Let G be a non-abelian p-group, K an infinite field of characteristic p, $\lambda \in Z^2(G, K^*)$, $d = \dim_K \overline{K^\lambda G}$ and*

$$l = \begin{cases} 1 & \text{if } |G' : G''| \neq 4, \\ 2 & \text{if } |G' : G''| = 4. \end{cases}$$

If $K^\lambda G$ is not a uniserial algebra and $d = |G : G'|$ then the set $\operatorname{AInd}(K^\lambda G, nld)$ is infinite, for any integer $n \geq 2$.

Proof. We have $\overline{K^\lambda G} = K^\lambda G / K^\lambda G \cdot \operatorname{rad} KG'$. Hence $\operatorname{rad} K^\lambda G = K^\lambda G \cdot \operatorname{rad} KG'$. Since $K^\lambda G$ is not uniserial, G' is non-cyclic. By Proposition 1.3, $\operatorname{AInd}(K^\lambda G, nld)$ is infinite for any integer $n \geq 2$. $\square$

Our second main result of this paper is the following theorem.

Theorem 3.5. *Let G be a non-abelian p-group, K an infinite field of characteristic p, $\lambda \in Z^2(G, K^*)$, $d = \dim_K \overline{K^\lambda G}$ and*

$$l = \begin{cases} 1 & \text{if } p \neq 2, \\ 2 & \text{if } p = 2. \end{cases}$$

Assume that $K^\lambda G$ is not a uniserial algebra and $pd < |G : G'|$. Then the set $\operatorname{AInd}(K^\lambda G, nlpd)$ is infinite, for any integer $n \geq 2$.

Proof. Let $\{u_g : g \in G\}$ be a natural K-basis of $K^\lambda G$, $U = K^\lambda G \cdot \operatorname{rad} KG'$, $\widetilde{K^\lambda G} = K^\lambda G/U$ and $\tilde{w} = w + U$ for every $w \in K^\lambda G$. Assume that $G/G' = \langle a_1 G' \rangle \times \cdots \times \langle a_m G' \rangle$, where $|a_j G'| = p^{s_j}$ for $j = 1, \ldots, m$. Then

$$\widetilde{K^\lambda G} = \bigoplus_{i_1, \ldots, i_m} K \tilde{u}_{a_1}^{i_1} \ldots \tilde{u}_{a_m}^{i_m},$$

where $\tilde{u}_{a_j}^{p^{s_j}} = \gamma_j \tilde{u}_e, \quad \gamma_j \in K^*$. The algebra $\widetilde{K^\lambda G}$ is a twisted group algebra of the non-cyclic abelian p-group G/G' over K.

If $\widetilde{K^\lambda G}$ is not uniserial, then, by Corollary 2.6, $\operatorname{AInd}(K^\lambda G, nld)$ is infinite for every $n \geq 2$.

Assume now that $\widetilde{K^\lambda G}$ is uniserial and the K-algebra

$$F := \bigoplus_{i_1, \ldots, i_{m-1}} K \tilde{u}_{a_1}^{i_1} \ldots \tilde{u}_{a_{m-1}}^{i_{m-1}}$$

is a field. We have $F = (K^\lambda D + U)/U$, where D is the subgroup of G generated by G' and the elements $a_1, \ldots, a_{m-1}$. Evidently

$$\widetilde{K^\lambda G} = \bigoplus_{i_m = 0}^{p^{s_m} - 1} F \tilde{u}_{a_m}^{i_m}.$$

Since $\dim_K \widetilde{K^\lambda G} = |G : G'|$, $\dim_K(\widetilde{K^\lambda G} / \operatorname{rad} \widetilde{K^\lambda G}) = d$ and $pd < |G : G'|$, we conclude that $\widetilde{K^\lambda G}$ is not a field. There exists an element

$$\rho = \sum_{i_1 = 0}^{p^{s_1} - 1} \cdots \sum_{i_{m-1} = 0}^{p^{s_{m-1}} - 1} \alpha_{i_1, \ldots, i_{m-1}} u_{a_1}^{i_1} \ldots u_{a_{m-1}}^{i_{m-1}}, \tag{3}$$

where $\alpha_{i_1, \ldots, i_{m-1}} \in K$, such that $\tilde{\rho}^{p^r} = \gamma_m^{-1} \tilde{u}_e$ with $2 \leq r \leq s_m$ and $\tilde{\rho} \notin F^p$, if $r < s_m$. We have $d = |D : G'| \cdot p^{s_m - r}$, and hence $dp^r = |G : G'|$.

In view of Lemmas 3.1 and 3.2, we can suppose that G' is an elementary abelian group of type (p,p) or a group of order p. Denote by H the subgroup of G generated by G' and the elements

$$a_1^p, \dots, a_{m-1}^p, a_m^{p^{s_m-r+1}}.$$

Now we show that H is abelian. Assume that $G' = \langle c_1\rangle \times \langle c_2\rangle$, where $|c_1| = |c_2| = p$ and $c_1 \in Z(G)$. Since the center of the factor group $G/\langle c_1\rangle$ contains $c_2\langle c_1\rangle$, we have $g^{-1}c_2g = c_2c_1^i$ for any $g \in G$. This implies $c_2g^p = g^pc_2$ for every $g \in G$. If $h \in G$ then $g^{-1}hg = hc_1^rc_2^s$ for some $r, s \in \{0, 1, \dots, p-1\}$. It follows that $g^{-p}hg^p = hc_1^{t_1}c_2^{t_2}$, where $t_1 = pr + is\frac{p(p-1)}{2}$, $t_2 = ps$. Hence, if $p \neq 2$ then $hg^p = g^ph$. If $p = 2$ then $g^{-2}hg^2 = hc_1^{is}$, $g^{-2}h^2g^2 = h^2$. In the case $G' = \langle c_1\rangle$, $|c_1| = p$ we obtain $g^{-1}c_1g = c_1$, $g^{-1}hg = hc_1^r$, $g^{-p}hg^p = h$ for arbitrary $g, h \in G$.

Let S be the subgroup of H generated by G' and the elements $a_1^p, \dots, a_{m-1}^p$; $T = S/G'$ and

$$w = \sum_{i_1=0}^{p^{s_1}-1} \cdots \sum_{i_{m-1}=0}^{p^{s_{m-1}}-1} \alpha^p_{i_1,\dots,i_{m-1}} u_{a_1}^{pi_1} \dots u_{a_{m-1}}^{pi_{m-1}} \quad \text{(see (3))}.$$

Then $w \in K^\lambda S$ and

$$\left(wu_{a_m}^{p^{s_m-r+1}}\right)^{p^{r-1}} \equiv u_e \pmod{K^\lambda H \cdot \operatorname{rad} KG'}.$$

It follows that $K^\lambda H/K^\lambda H \cdot \operatorname{rad} KG'$ is the group algebra of the cyclic group of order p^{r-1} over the field

$$L := (K^\lambda S + K^\lambda H \cdot \operatorname{rad} KG')/K^\lambda H \cdot \operatorname{rad} KG'.$$

Clearly,

$$L \cong K^\lambda S/K^\lambda S \cap K^\lambda H \cdot \operatorname{rad} KG' \cong K^\lambda S/K^\lambda S \cdot \operatorname{rad} KG' \cong K^\mu T,$$

where $\mu_{xG',yG'} = \lambda_{x,y}$ for all $x, y \in S$. Thus, $\dim_K \overline{K^\lambda H} = |T|$.

If G' is the group of type (p,p), then, by Proposition 1.3, $K^\lambda H$ is of infinite representation type. Hence $K^\lambda H$ is not uniserial. If G' is the group of order p, then $K^\lambda H$ is not uniserial either, by Lemma 3.3. By Theorem 2.4, $\operatorname{AInd}(K^\lambda H, nl|T|)$ is infinite for every $n \geq 2$. Since H is a normal subgroup of G, the algebra $K^\lambda G$ is a crossed product of G/H over $K^\lambda H$. Applying Lemma 1.1, we conclude that $\operatorname{AInd}(K^\lambda G, nl|T| \cdot |G : H|)$ is infinite for every integer $n \geq 2$. This completes the proof, because

$$|T| \cdot |G : H| = \frac{|G : G'|}{|H : S|} = \frac{|G : G'|}{p^{r-1}} = pd. \qquad \square$$

4. On absolutely indecomposable representations of a non-commutative twisted group algebra $K^\lambda G$ with $p \cdot \dim_K \overline{K^\lambda G} = |G : G'|$

Assume that G is a non-abelian p-group, K is a field of characteristic p, $\lambda \in Z^2(G, K^*)$ and $d = \dim_K \overline{K^\lambda G}$. If K is perfect then $K^\lambda G$ is the group algebra of G over K (see [18, p. 43]). In this case $\overline{K^\lambda G} \cong K$. Since G/G' is non-cyclic, $|G : G'| \neq p$. Therefore, if $pd = |G : G'|$ then K is a non-perfect field.

The subalgebra $K^\lambda G'$ of $K^\lambda G$ is the group algebra of G' over K. Let $\{u_g : g \in G\}$ be a natural K-basis of $K^\lambda G$, $U = K^\lambda G \cdot \operatorname{rad} KG'$, $\widetilde{K^\lambda G} = K^\lambda G/U$ and $\tilde{w} = w + U$ for every $w \in K^\lambda G$. We have $\widetilde{K^\lambda G} = K^\mu T$, where $T = G/G'$ and $\mu_{xG', yG'} = \lambda_{x,y}$ for all $x, y \in G$. Suppose that $pd = |G : G'|$. Since $\widetilde{K^\lambda G}/\operatorname{rad}\widetilde{K^\lambda G} \cong \overline{K^\lambda G}$, there exists a direct product decomposition $T = A \times \langle bG'\rangle$ such that $F := K^\mu A$ is a field and

$$K^\mu T = \bigoplus_{i=0}^{p^n-1} F\tilde{u}_b^i, \quad \tilde{u}_b^{p^n} = \rho^p,$$

where $p^n = |bG'|$, $\rho \in F^*$ and $\rho \notin F^p$ for $n > 1$. Moreover, $d = |A| \cdot p^{n-1}$. The algebra $K^\mu T$ is uniserial.

Proposition 4.1. *Let G be a non-abelian p-group, K a non-perfect field of characteristic p, $\lambda \in Z^2(G, K^*)$ and $d = \dim_K \overline{K^\lambda G}$. Assume that G' is non-cyclic, $pd = |G : G'|$ and*

$$l = \begin{cases} 1 & \text{if } |G' : G''| \neq 4, \\ 2 & \text{if } |G' : G''| = 4. \end{cases}$$

Then $\operatorname{AInd}(K^\lambda G, nlpd)$ is infinite for any $n \geq 2$.

Proof. Apply Proposition 1.3. □

Lemma 4.2. *Let $p \neq 2$ and G be a non-abelian p-group with cyclic commutant G'. Then $[a, b]^p = e$ for all $a, b \in G$ such that $a^p, b^p \in G'$.*

Proof. Let $G' = \langle c\rangle$, $|c| = p^m$ and $m \geq 2$. If $g \in G$ and $g^p \in G'$ then $g^{-1}cg = c^r$, where $r \equiv 1 \pmod{p^{m-1}}$. It follows that $g^{-1}c^pg = c^p$. Let $a, b \in G$ and $a^p, b^p \in G'$. Suppose that $a^{-1}ca = c^r$ and $b^{-1}ab = ac^i$. Then $b^{-1}a^pb = a^pc^{it}$, where $t = 1 + r + \cdots r^{p-1}$. It is easy to see that $t \equiv p \pmod{p^m}$. Hence, $b^{-1}a^pb = a^pc^{ip}$.

Let $H = \langle c^p \rangle$. If $a^p \in H$, then $b^{-1}a^p b = a^p$ and we conclude that $ip \equiv 0 \pmod{p^m}$. If $a^p \notin H$, then we may assume that $a^p = c$. This implies that $b^{-1}ab = a^{1+pi}$. We have $b^p = a^{pj}$ for some j, since $b^p \in G'$. Therefore, $b^{-p}ab^p = a$. At the same time

$$b^{-p}ab^p = a^{(1+pi)^p}.$$

From this we deduce that $(1+pi)^p \equiv 1 \pmod{p^{m+1}}$. Thus $pi \equiv 0 \pmod{p^m}$ and $[a,b]^p = e$. □

Lemma 4.3. *Let G be a non-abelian 2-group with cyclic commutant G' and D the subgroup of G such that $G' \subset D$ and $D/G' = \operatorname{soc}(G/G')$.*

(i) *If $|G' \cap Z(D)| \geq 4$, then $[a,b]^4 = e$, for all $a, b \in D$.*

(ii) *If $G' \subset Z(D)$, then $[a,b]^2 = e$, for all $a, b \in D$.*

Proof. (i) Let $G' = \langle c \rangle$ and $|c| = 2^m$. If $m = 2$ then $G' \subset Z(D)$. At first let us investigate the case $m > 2$. If $g \in D$, then $g^{-1}cg = c^r$, where $r \equiv 1 \pmod{2^{m-1}}$. It follows that $g^{-1}c^2 g = c^2$. Assume that $a, b \in D$, $a^{-1}ca = c^r$ and $b^{-1}ab = ac^i$. Then $b^{-1}a^2 b = a^2 c^{i(1+r)}$. Let $H = \langle c^2 \rangle$. If $a^2 \in H$ then $b^{-1}a^2 b = a^2$, hence $i(1+r) \equiv 0 \pmod{2^m}$. From this we obtain $2i \equiv 0 \pmod{2^m}$. If $a^2 \notin H$, then we may suppose that $a^2 = c$ and $r = 1$. We get $b^{-1}c^2 b = c^2$ and $b^{-1}c^2 b = c^2 c^{4i}$, thus $c^{4i} = e$.

(ii) Now assume that $G' \subset Z(D)$. Then $b^{-1}a^2 b = a^2 c^{2i}$ and $b^{-1}a^2 b = a^2$. Hence $c^{2i} = e$. □

We are now able to prove the third main result of this paper.

Proposition 4.4. *Let $p \neq 2$, G be a non-abelian p-group with cyclic commutant G', K a non-perfect field of characteristic p, $\lambda \in Z^2(G, K^*)$ and $d = \dim_K \overline{K^\lambda G}$. Asssume that $K^\lambda G$ is not uniserial, $pd = |G : G'|$ and $|G'| > p$. Then the set $\operatorname{AInd}(K^\lambda G, nd)$ is infinite, for any integer $n \geq 2$.*

Proof. Denote by D the subgroup of G such that $G' \subset D$ and $D/G' = \operatorname{soc}(G/G')$. Let $G' = \langle c \rangle$ and $T = \langle c^p \rangle$. By Lemma 4.2, D/T is abelian. In view of Lemma 3.2, we can assume that $|G'| = p$ and D is an abelian group. By Lemma 3.3, $K^\lambda D$ is not uniserial, since $K^\lambda D / K^\lambda D \cdot \operatorname{rad} KG'$ is not a field. Let $d_1 = \dim_K \overline{K^\lambda D}$. Then $pd_1 = |D : G'|$. By Theorem 2.4, $\operatorname{AInd}(K^\lambda D, nd_1)$ is infinite for every $n \geq 2$. Consequently, by Lemma 1.1, $\operatorname{AInd}(K^\lambda G, nd_1|G : D|)$ is infinite. Hence $\operatorname{AInd}(K^\lambda G, nd)$ is infinite, for any $n \geq 2$, because $d_1|G : D| = p^{-1}|D : G'| \cdot |G : D| = p^{-1}|G : G'| = d$. □

Proposition 4.5. *Let G be a non-abelian 2-group with cyclic commutant G', K a non-perfect field of characteristic 2, $\lambda \in Z^2(G, K^*)$, $d = \dim_K \overline{K^\lambda G}$*

and $2d = |G : G'|$. *Denote by* D *the subgroup of* G *such that* $G' \subset D$ *and* $D/G' = \mathrm{soc}(G/G')$. *If* $|G' \cap Z(D)| \geq 4$ *and* $K^\lambda G$ *is not a uniserial algebra, the set* $\mathrm{AInd}(K^\lambda G, 2nd)$ *is infinite, for any integer* $n \geq 1$.

Proof. Apply Lemma 4.3 and modify the arguments used in the proof of Proposition 4.4. □

Proposition 4.6. *Let* G *be a non-abelian* p*-group,* K *a non-perfect field of characteristic* p, $\lambda \in Z^2(G, K^*)$, $H = \mathrm{Ker}(\lambda)$ *and* $d = \dim_K \overline{K^\lambda G}$. *Suppose also that* $K^\lambda G$ *is not uniserial,* $pd = |G : G'|$, $H \neq G'$ *and let*

$$l = \begin{cases} 1 & \text{if } |H : H'| \neq 4, \\ 2 & \text{if } |H : H'| = 4. \end{cases}.$$

Then the set $\mathrm{AInd}(K^\lambda G, nld)$ *is infinite, for arbitrary integer* $n \geq 2$.

Proof. Let $U = K^\lambda G \cdot \mathrm{rad}\, KH$ and $\widetilde{K^\lambda G} = K^\lambda G/U$. Since $G' \subset H$, $G' \neq H$, $\dim_K \widetilde{K^\lambda G} = |G : H|$ and $\widetilde{K^\lambda G}/\mathrm{rad}\,\widetilde{K^\lambda G} \cong \overline{K^\lambda G}$, we get $|G : H| = d$ and $U = \mathrm{rad}\, K^\lambda G$. By the hypothesis, $K^\lambda G$ is not uniserial, hence U is not a principal left ideal of $K^\lambda G$. It follows that H is a non-cyclic p-group, and, by Proposition 1.3, $\mathrm{AInd}(K^\lambda G, nld)$ is infinite, for arbitrary $n \geq 2$. □

Proposition 4.7. *Let* G *be a non-abelian* p*-group with cyclic commutant* G', K *a non-perfect field of characteristic* p, $\lambda \in Z^2(G, K^*)$ *and* $d = \dim_K \overline{K^\lambda G}$. *Assume also that* $K^\lambda G$ *is not uniserial,* $pd = |G : G'|$ *and* $\mathrm{soc}(G/G') = \langle a_1 G'\rangle \times \cdots \times \langle a_m G'\rangle$, *where* $[a_i, a_j] = e$, *for all* $i, j \in \{1, \ldots, m-1\}$. *Then the set* $\mathrm{AInd}(K^\lambda G, 2nd)$ *is infinite, for any integer* $n \geq 1$.

Proof. In view of Lemma 3.2, we may suppose that $G' = \langle c\rangle$ is of order p. Denote by H the subgroup of G generated by the elements $c, a_1, \ldots, a_m$. First, assume that H is abelian. By Lemma 3.3, $K^\lambda H$ is not a uniserial algebra, since $K^\lambda H/K^\lambda H(u_c - u_e)$ is not a field. We also have $\dim_K \overline{K^\lambda H} = p^{m-1}$ and $|G : H| = dp^{1-m}$. By Theorem 2.4 and Lemma 1.1, $\mathrm{AInd}(K^\lambda G, 2nd)$ is infinite for any integer $n \geq 1$.

In what follows, we may assume that $[a_j, a_m] = e$ for all $j = 1, \ldots, m-2$ and $[a_{m-1}, a_m] = c$. Let D be the subgroup of H generated by the elements $a_1, \ldots, a_{m-1}, c$. If $K^\lambda D/K^\lambda D(u_c - u_e)$ is not a field, then, by Lemma 3.3, $K^\lambda D$ is not a uniserial algebra. By Theorem 2.4 and Lemma 1.1, $\mathrm{AInd}(K^\lambda G, 2nd)$ is infinite for any integer $n \geq 1$, since in this case $\dim_K \overline{K^\lambda D} = p^{m-2}$ and $|G : D| = dp^{2-m}$.

Now we investigate the case when $K^\lambda D/K^\lambda D(u_c - u_e)$ is a field. Let $u_{a_m}^p = \gamma u_x$, where $\gamma \in K^*$ and $x \in G'$. Since $K^\lambda H/K^\lambda H(u_c - u_e)$ is not a field, there exists an element $v \in K^\lambda D$ such that

$$v^p \equiv \gamma^{-1} u_e \big(\mathrm{mod} K^\lambda D(u_c - u_e)\big).$$

Put $w = vu_{a_m} - u_e$. Then $w^p \equiv 0 \pmod{K^\lambda H(u_c - u_e)}$. It is easy to verify that

$$\operatorname{rad} K^\lambda H = K^\lambda H(u_c - u_e) + K^\lambda H w. \tag{4}$$

Let $V = K^\lambda G \operatorname{rad} K^\lambda H$. Because H is a normal subgroup of G, V is a nilpotent ideal of the algebra $K^\lambda G$. The K-dimension of the quotient algebra $K^\lambda G/V$ is equal to $p^{m-1} \cdot |G : H| = p^{-1}|G : G'| = d$. It follows that V is the radical of $K^\lambda G$. Since $K^\lambda G$ is not uniserial, $\operatorname{rad} K^\lambda G$ is not a principal left ideal of $K^\lambda G$. This implies that $\operatorname{rad} K^\lambda H$ is not a principal left ideal of $K^\lambda H$, and $K^\lambda H$ is not a uniserial algebra.

Let $\widehat{K^\lambda H} = K^\lambda H/(\operatorname{rad} K^\lambda H)^2$ and $\hat{y} = y + (\operatorname{rad} K^\lambda H)^2$ for every $y \in K^\lambda H$. Since $K^\lambda H$ is not uniserial, $\widehat{K^\lambda H}$ is not uniserial either (see [9, p. 172]). The radical of $\widehat{K^\lambda H}$ is equal to $\operatorname{rad} K^\lambda H/(\operatorname{rad} K^\lambda H)^2$. Hence, in view of equation (4), we get

$$\operatorname{rad} \widehat{K^\lambda H} = \widehat{K^\lambda H}(\hat{u}_c - \hat{u}_e) + \widehat{K^\lambda H}\hat{w},$$

and $(\hat{u}_c - \hat{u}_e)^2 = \hat{0}$, $\hat{w}^2 = \hat{0}$, $(\hat{u}_c - \hat{u}_e)\hat{w} = \hat{w}(\hat{u}_c - \hat{u}_e) = \hat{0}$. Since $K^\lambda H = K^\lambda D + \operatorname{rad} K^\lambda H$, the algebra $\widehat{K^\lambda H}$ is the K-linear span of the elements

$$\hat{u}_{a_1}^{i_1} \cdots \hat{u}_{a_{m-1}}^{i_{m-1}}, \quad \hat{u}_{a_1}^{i_1} \cdots \hat{u}_{a_{m-1}}^{i_{m-1}}(\hat{u}_c - \hat{u}_e), \quad \hat{u}_{a_1}^{i_1} \cdots \hat{u}_{a_{m-1}}^{i_{m-1}}\hat{w},$$

where $i_r = 0, 1, \dots, p-1$, for all $r = 1, \dots, m-1$. We prove that these elements are K-linearly independent.

Let

$$S = \left\{ \sum_{i_1=0}^{p-1} \cdots \sum_{i_{m-1}=0}^{p-1} \alpha_{i_1,\dots,i_{m-1}} u_{a_1}^{i_1} \cdots u_{a_{m-1}}^{i_{m-1}} : \alpha_{i_1,\dots,i_{m-1}} \in K \right\}.$$

Since $K^\lambda D/K^\lambda D(u_c - u_e)$ is a field, $\operatorname{rad} K^\lambda D = K^\lambda D(u_c - u_e)$ and $S \cap \operatorname{rad} K^\lambda D = \{0\}$. If

$$\hat{x} + \hat{y}(\hat{u}_c - \hat{u}_e) + \hat{z}\hat{w} = \hat{0} \tag{5}$$

for some $x, y, z \in S$, then $x \in \operatorname{rad} K^\lambda H$. Thus x is a nilpotent element and $x \in \operatorname{rad} K^\lambda D$. Hence $x = 0$. If $y \neq 0$, then there exists $y_1 \in S$ such that $y_1 y \equiv u_e$

(mod $K^\lambda D(u_c - u_e)$). It follows that $\hat{y}_1\hat{y} = \hat{u}_e + \hat{y}_2(\hat{u}_c - \hat{u}_e)$ for some $y_2 \in S$. Hence, in view of (5), we get $\hat{u}_c - \hat{u}_e + \hat{z}_1\hat{w} = \hat{0}$ for some $z_1 \in S$. It follows that $\operatorname{rad}\widehat{K^\lambda H}$ is a principal left ideal of the algebra $\widehat{K^\lambda H}$. This contradiction proves that $y = 0$. If $z \neq 0$, then from $\hat{z}\hat{w} = \hat{0}$ we obtain $\hat{w} = \hat{0}$. Therefore, $x = y = z = 0$.

From $u_{a_{m-1}}u_{a_m} = u_c u_{a_m} u_{a_{m-1}}$ we deduce $\hat{u}_{a_{m-1}}\hat{w} = \big(\hat{w} + \hat{u}_c - \hat{u}_e\big)\hat{u}_{a_{m-1}}$. It is clear that $\hat{u}_{a_j}\hat{w} = \hat{w}\hat{u}_{a_j}$ for $j = 1, \dots, m-2$. Hence $\widehat{K^\lambda H}$ is a crossed product of D/G' over the K-algebra $R = K\hat{u}_e + K(\hat{u}_c - \hat{u}_e) + K\hat{w}$. By Proposition 1.5, $\operatorname{AInd}(\widehat{K^\lambda H}, 2n|D/G'|)$ is infinite, for any integer $n \geq 1$, and, hence, the set $\operatorname{AInd}(K^\lambda H, 2n|D/G'|)$ is infinite. It follows, by applying Lemma 1.1, that $\operatorname{AInd}(K^\lambda G, 2nd)$ is infinite, since $|D/G'| \cdot |G : H| = d$. □

Proposition 4.8. *Let G be a non-abelian p-group with cyclic commutant G', K a non-perfect field of characteristic p, $\lambda \in Z^2(G, K^*)$ and $d = \dim_K \overline{K^\lambda G}$. Assume also that G/G' has at most three invariants that equal p, $K^\lambda G$ is not a uniserial algebra and $pd = |G : G'|$. Then $\operatorname{AInd}(K^\lambda G, 2nd)$ is infinite, for any integer $n \geq 1$.*

Proof. In view of Lemma 3.2, we may suppose that $G' = \langle c \rangle$ is of order p. Let $G/G' = \langle a_1G' \rangle \times \dots \times \langle a_mG' \rangle$, where $m \geq 3$ and $|a_jG'| = p^{k_j}$ for $j = 1, \dots, m$. Assume also that $k_{m-2} = k_{m-1} = k_m = 1$, $[a_{m-1}, a_m] = c$ and $[a_{m-2}, a_{m-1}] = c^t$. If we set $b_j = a_j^{p^{k_j - 1}}$ for $j = 1, \dots, m-3$, and $b_{m-2} = a_{m-2}a_m^t$, $b_{m-1} = a_{m-1}$, $b_m = a_m$, then

$$\operatorname{soc}(G/G') = \langle b_1G' \rangle \times \cdots \times \langle b_mG' \rangle$$

and $[b_r, b_s] = e$ for $r, s \in \{1, \dots, m-1\}$. By Proposition 4.7, the set $\operatorname{AInd}(K^\lambda G, 2nd)$ is infinite, for any integer $n \geq 1$. □

Our final main result of this paper is the following proposition.

Proposition 4.9. *Let G be a non-abelian p-group with cyclic commutant G', K a non-perfect field of characteristic p, $\lambda \in Z^2(G, K^*)$ and $d = \dim_K \overline{K^\lambda G}$. Assume also that $[K : K^p] \leq p^2$, $K^\lambda G$ is not a uniserial algebra and $pd = |G : G'|$. Then the set $\operatorname{AInd}(K^\lambda G, 2nd)$ is infinite, for any integer $n \geq 1$.*

Proof. Let $G/G' = \langle a_1G' \rangle \times \cdots \times \langle a_mG' \rangle$ and $G' = \langle c \rangle$. There is an isomorphism $K^\lambda G / K^\lambda G(u_c - u_e) \cong K^\mu T$, where $T = G/G'$ and $\mu_{xG', yG'} = \lambda_{x,y}$, for all $x, y \in G$. Renumbering the set $\{a_1, \dots, a_m\}$, if necessary, the algebra $K^\mu T$ is a twisted group algebra of the cyclic group $\langle a_mG' \rangle$ over the field $K^\mu D$, where $D = \langle a_1G' \rangle \times \cdots \times \langle a_{m-1}G' \rangle$. We recall that if K is a non-perfect field of characteristic p and $[K : K^p] = p^t$, then t is the largest element of the set that consists of all integers $r \geq 1$ such that a K-algebra of the form

$$K[X]/(X^p - \alpha_1) \otimes_K \cdots \otimes_K K[X]/(X^p - \alpha_r)$$

is a field, for some $\alpha_1, \ldots, \alpha_r \in K$ (see [3, p. 200]). Thus, if $[K : K^p] = p$ then $m = 2$, and if $[K : K^p] = p^2$ then $m \in \{2, 3\}$. By Proposition 4.8, the set $\mathrm{AInd}(K^\lambda G, 2nd)$ is infinite, for any integer $n \geq 1$. $\square$

Remark 4.10. The second Brauer–Thrall conjecture is discussed in the paper [21]. The reader is referred to [22, Section XIX.3] for a discussion of the second Brauer–Thrall conjecture and the tame-wild problem for algebras, that is related to the problems discussed in the paper (see [22, pp. 341–4, 355–6]).

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