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A monotonicity property of Euler's gamma function

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Abstract. Let

$$\Delta(x) = \frac{\log\Gamma(x+1)}{x} \quad (-1 < x \neq 0), \quad \Delta(0) = -\gamma.$$

For all $n = 0, 1, 2, \dots$ and $x > -1$, we show that

$$(-1)^n \Delta^{(n+1)}(x) = (n+1)! \int_0^1 u^{n+1} \zeta(n+2, xu+1)\, du,$$

where ζ denotes the Hurwitz zeta function. This representation implies that Δ' is completely monotonic on $(-1, \infty)$. This extends a result published in 1996 by Grabner, Tichy, and Zimmermann, who proved that Δ is increasing and concave on $(-1, \infty)$.

1. Introduction and main results

In this note we are concerned with the function

$$\Delta(x) = \frac{\log\Gamma(x+1)}{x} \quad (-1 < x \neq 0), \quad \Delta(0) = -\gamma,$$

where Γ denotes Euler's gamma function and γ is Euler's constant. In the recent past, several authors studied interesting monotonicity properties of Δ as well as

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other functions defined in terms of Δ (see, for instance, [3], [4], [5], [7], [8], [9], [10]).

Grabner et al. [6] proved that Δ is increasing and concave on $(-1,\infty)$ and used their result to present an upper bound for the permanent of a $0-1$ matrix. Vogt and Voigt [11] showed that the function $x \mapsto \Delta(x) - \log(x+1) + 1$ is completely monotonic on $(-1,\infty)$.

We recall that a function $f : I \to \mathbf{R}$ is said to be completely monotonic, if f has derivatives of all orders and

$$0 \le (-1)^n f^{(n)}(x) \quad (n = 0, 1, 2, \dots;\ x \in I).$$

Completely monotonic functions have important applications in probability and potential theory, in numerical and asymptotic analysis, and in other fields. The main properties of these functions are given in [12, Chapter IV]. In [2] one can find a detailed list of references on this subject.

The result of Grabner et al. yields

$$0 \le (-1)^n \Delta^{(n+1)}(x) \quad (n = 0, 1;\ x > -1), \tag{1.1}$$

whereas the monotonicity theorem of Vogt and Voigt leads to the inequality

$$(-1)^n \Delta^{(n+1)}(x) \le \frac{n!}{(x+1)^{n+1}} \quad (n = 0, 1, 2, \dots;\ x > -1). \tag{1.2}$$

In view of (1.1), it is natural to ask whether Δ' is completely monotonic on $(-1,\infty)$. A positive answer to this question is given in the following theorem which provides, in addition, a closed form expression for $(-1)^n \Delta^{(n+1)}(x)$ in terms of the classical Hurwitz zeta function

$$\zeta(s,a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s} \quad (s > 1;\ a > 0).$$

Theorem 1.1. *Let n be a nonnegative integer and let x be a real number with $x > -1$. Then,*

$$(-1)^n \Delta^{(n+1)}(x) = (n+1)! \int_0^1 u^{n+1} \zeta(n+2, xu+1)\, du. \tag{1.3}$$

As a consequence, Δ' is completely monotonic on $(-1,\infty)$.

Remark 1.1. From (1.3), we have for $n \ge 1$:

$$\Delta^{(n)}(0) = (-1)^{n-1} \frac{n!}{n+1} \zeta(n+1),$$

where ζ denotes the Riemann zeta function.

Remark 1.2. If h' is completely monotonic on I, then $\exp(-h)$ is also completely monotonic on I. This result can be proved by applying the Leibniz rule and induction. Thus, setting $h = \Delta$ and $I = (-1, \infty)$, we conclude from Theorem 1.1 that the function

$$\Theta(x) = \Gamma(x+1)^{-1/x} \quad (-1 < x \neq 0), \quad \Theta(0) = \exp(\gamma),$$

is completely monotonic on $(-1, \infty)$.

A consequence of Theorem 1.1 is that the upper bound in (1.2) is asymptotically sharp, as stated in the following corollary.

Corollary 1.2. *For any nonnegative integer n, we have*

$$\lim_{x\to\infty} \frac{(x+1)^{n+1}}{n!} (-1)^n \Delta^{(n+1)}(x) = 1. \tag{1.4}$$

In order to prove Theorem 1.1 we need a lemma. It states that a certain function, defined in terms of the exponential function, is completely monotonic on $\mathbf{R}$.

Lemma 1.3. *Let $N \geq 0$ be an integer and*

$$g_N(x) = \left[1 - e^{-x} \sum_{m=0}^{N} \frac{x^m}{m!}\right] x^{-N-1} \quad (x \neq 0), \quad g_N(0) = \frac{1}{(N+1)!}. \tag{1.5}$$

Then we have for $n \geq 0$ and $x \in \mathbf{R}$:

$$(-1)^n g_N^{(n)}(x) = \frac{1}{N!} \int_0^1 e^{-xu} u^{n+N}\, du. \tag{1.6}$$

In particular, g_N is completely monotonic on $\mathbf{R}$.

2. The proofs

Proof of Lemma 1.3. We get

$$g_N(x) = \frac{x^{-N-1}}{N!} \int_0^x e^{-t} t^N\, dt = \frac{1}{N!} \int_0^1 e^{-xu} u^N\, du.$$

Differentiation leads to (1.6). □

Proof of Theorem 1.1. Let $x > -1$, $t > 0$, and $n \geq 0$. We obtain

$$\Delta'(x) = \frac{\psi(x+1)}{x} - \frac{\log \Gamma(x+1)}{x^2} \quad (x \neq 0), \quad \Delta'(0) = \frac{\pi^2}{12},$$

where $\psi = \Gamma'/\Gamma$ denotes the digamma function. Using the integral formulas

$$\log \Gamma(z) = \int_0^\infty \left[(z-1)e^{-t} - \frac{e^{-t} - e^{-zt}}{1-e^{-t}} \right] \frac{dt}{t} \quad (z > 0)$$

and

$$\psi(z) = \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right] dt \quad (z > 0)$$

(see [1, pp. 258, 259]), we get

$$\Delta'(x) = \int_0^\infty \frac{te^{-t}}{1-e^{-t}} \, g_1(xt) \, dt, \tag{2.1}$$

where g_1 is given in (1.5). Applying (1.6) with $N = 1$ and

$$\frac{1}{1-e^{-t}} = \sum_{m=0}^\infty e^{-mt}$$

we obtain from (2.1) and Fubini's theorem

$$\begin{aligned}
(-1)^n \Delta^{(n+1)}(x) &= \int_0^\infty \frac{t^{n+1}e^{-t}}{1-e^{-t}} (-1)^n g_1^{(n)}(xt) \, dt \\
&= \int_0^\infty t^{n+1} e^{-t} \sum_{m=0}^\infty e^{-mt} \int_0^1 e^{-xtu} u^{n+1} \, du \, dt \\
&= \int_0^1 u^{n+1} \sum_{m=0}^\infty \int_0^\infty t^{n+1} e^{-t(m+xu+1)} \, dt \, du \\
&= \int_0^1 u^{n+1} \sum_{m=0}^\infty \frac{\Gamma(n+2)}{(m+xu+1)^{n+2}} \, du \\
&= (n+1)! \int_0^1 u^{n+1} \zeta(n+2, xu+1) \, du.
\end{aligned}$$

This completes the proof of Theorem 1.1. □

Proof of Corollary 1.2. Let $n \geq 0$ and $x > -1$. We make use of (1.3) and get

$$\begin{aligned}(-1)^n \Delta^{(n+1)}(x) &= (n+1)! \int_0^1 u^{n+1} \sum_{m=0}^{\infty} \frac{1}{(m+xu+1)^{n+2}}\, du \\ &\geq (n+1)! \int_0^1 u^{n+1} \int_0^{\infty} \frac{1}{(t+xu+1)^{n+2}}\, dt\, du \\ &= n! \int_0^1 \frac{u^{n+1}}{(xu+1)^{n+1}}\, du. \qquad (2.2)\end{aligned}$$

From (2.2) and (1.2) we obtain

$$\int_0^1 \left(\frac{xu+u}{xu+1}\right)^{n+1} du \leq \frac{(x+1)^{n+1}}{n!} (-1)^n \Delta^{(n+1)}(x) \leq 1.$$

Applying the dominated convergence theorem we conclude that (1.4) holds. □

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