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Disjointness preserving mappings on BSE Ditkin algebras

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Abstract. Let A and B be regular Banach function algebras. A linear map T defined from A into B is said to be *disjointness preserving or separating* if $f \cdot g \equiv 0$ implies $T(f) \cdot T(g) \equiv 0$ for all $f, g \in A$. We prove that if there exists a disjointness preserving bijection between two BSE Ditkin algebras with a BAI or if they are (supremum norm) isometric, then they are isomorphic as algebras.

1. Introduction

Since the 40's, when disjointness preserving mappings began to be used, many authors have studied them on several contexts. Among others, on Banach lattices (see e.g. [1], [2] or [6]), on spaces of continuous functions (see e.g. [14], [3], [7], [15] or [12]), on group algebras of locally compact Abelian groups ([8]), on Fourier algebras ([10] and [19]) and on some others (see e.g. [16] or [5]).

In [9], we extended the definition of disjointness preserving mappings to the class of regular Banach function algebras. Let us recall that a linear map T defined from a regular Banach function algebra A into such an algebra B is said to be *disjointness preserving or separating* if $f \cdot g \equiv 0$ implies $T(f) \cdot T(g) \equiv 0$ for all $f, g \in A$.

In [8] we proved that the existence of a disjointness preserving bijection between the group algebras of two locally compact Abelian groups implies that they are isomorphic as Banach algebras. A similar result was obtained in [10] for Fourier algebras and in [19] for generalized Fourier algebras.

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In this paper we extend the above results to a wider class of regular Banach function algebras which includes group algebras and Fourier algebras: the class of BSE Ditkin algebras with a BAI (bounded approximate identity). Let us recall that BSE algebras were introduced in [20] (see the definition in Section 3) motivated by the Bochner–Schoenberg–Eberlein characterization of the Fourier–Stieltjes transforms of measures on a locally compact abelian group. BSE Ditkin algebras with a BAI has recently attracted the attention of some authors. For example, one of the main results in [21] consists of an abstract analog of Cohen's Idempotent Theorem for such type of Banach algebras.

We prove here that if there exists a disjointness preserving bijection between two BSE Ditkin algebras with a BAI, then they are isomorphic as algebras. As a corollary we can deduce that two BSE Ditkin algebras with a BAI are isomorphic as Banach algebras if there exists a surjective supremum norm isometry between them.

2. Background

Let $(A, \|\cdot\|)$ be a commutative semisimple Banach algebra which may or may not have an identity element. Let Φ_A be the (locally compact) *structure space* of A. The *Gelfand transform* of $f \in A$ is denoted by $\hat{f}$. $\widehat{A}$ will stand for the point-separating subalgebra of $C_0(\Phi_A)$ consisting of all $\hat{f}$, $f \in A$. For the undefined concepts and notation used in this paper, the reader is referred to [18].

Next we gather the main results concerning disjointness preserving maps between regular Banach function algebras, which can be found in [9]:

In the sequel, let A and B be regular semisimple commutative Banach algebras, which is to say, regular Banach function algebras. Associated with a disjointness preserving map $T : A \longrightarrow B$, we can define a linear mapping $\hat{T} : \widehat{A} \longrightarrow \widehat{B}$ as $\hat{T}(\hat{f}) := \widehat{T(f)}$ for every $f \in A$. Since A and B are semisimple, it is easy to check that T is disjointness preserving if and only if $\hat{T}$ is disjointness preserving. In like manner, T is injective (resp. surjective) if and only if $\hat{T}$ is injective (resp. surjective).

If $\gamma \in \Phi_B$, let $\delta_\gamma \circ \hat{T} : \widehat{A} \to \mathbf{C}$ be the functional defined as $(\delta_\gamma \circ \hat{T})(\hat{f}) := \hat{T}(\hat{f})(\gamma)$ for all $f \in A$.

In general, a disjointness preserving map $T : A \longrightarrow B$ induces a continuous mapping h of Φ_B into $\Phi_A \cup \{\infty\}$, which may make no sense if A and B are not regular. We call h the *support map* of T. If T is continuous, then $\hat{T}$ is a *weighted composition map*; i.e., $(\delta_\gamma \circ \hat{T})(\hat{f}) = \hat{T}(\hat{f})(\gamma) = \kappa(\gamma)\hat{f}(h(\gamma))$ for all $\gamma \in \Phi_B$ and all $f \in A$, where the weight function $\kappa : \Phi_B \to \mathbf{C}$ is continuous, and the range of h is contained in Φ_A. If, in addition, T is surjective, then the point-separating property of $\widehat{B}$ easily implies that κ is nonvanishing on Φ_B.

The main result in [9] is the following:

Theorem 1. *Let $T : A \longrightarrow B$ be a disjointness preserving linear bijection. If A satisfies Ditkin's condition (i.e., if A is a Ditkin algebra), then*

1. *T is continuous*
2. *T^{-1} is disjointness preserving.*
3. *If also B satisfies Ditkin's condition, then the support map of T, h, is a homeomorphism of Φ_A onto Φ_B.*

As a consequence of this theorem and the above paragraphs, if there exists a disjointness preserving bijection T of A onto B, then $\hat{T}(\hat{f})(\gamma) = \kappa(\gamma)\hat{f}(h(\gamma))$ for all $f \in A$ and all $\gamma \in \Phi_B$. Since T^{-1} is also disjointness preserving and, consequently, continuous, we can write $\hat{T}^{-1}(\hat{g})(\zeta) = \Psi(\zeta)\hat{g}(h^{-1}(\zeta))$ for all $g \in B$ and all $\zeta \in \Phi_A$, where h^{-1} can be proved to be the inverse of the homeomorphism h. We will call $\kappa \in C(\Phi_B)$ and $\Psi \in C(\Phi_A)$ the *weight functions associated to T*.

3. The results

Let $\mathcal{A}$ be a semisimple commutative Banach algebra. A multiplier T on $\mathcal{A}$ is a bounded linear operator on $\mathcal{A}$ into itself which satisfies $T(f \cdot g) = f \cdot T(g) = T(f) \cdot g$ for all $f, g \in \mathcal{A}$. $M(\mathcal{A})$ denotes the commutative Banach algebra consisting of all multipliers on $\mathcal{A}$. By [17, Corollary 1.2.1], we may identify $M(\mathcal{A})$ with the normed algebra of all bounded continuous functions ϕ on $\Phi_{\mathcal{A}}$ such that $\phi\hat{\mathcal{A}} \subset \hat{\mathcal{A}}$. It is then apparent that multipliers are examples of disjointness preserving mappings.

Theorem 2. *Let A and B regular semisimple commutative Banach algebras. Then A and B are (algebra) isomorphic if and only if there exists a continuous disjointness preserving linear bijection between them whose (associated) weight functions are multipliers.*

Proof. Let us suppose that there exists a continuous disjointness preserving bijection T of A onto B whose (associated) weight functions are multipliers. First we claim that $(\hat{g} \circ h^{-1}) \in \widehat{A}$ for all $g \in B$. To prove this, let $\zeta \in \Phi_A$ and $f \in A$ such that $\hat{f}(\zeta) = 1$. Hence

$$\begin{aligned} 1 = \hat{f}(\zeta) = \hat{T}^{-1}(\hat{T}(\hat{f}))(\zeta) &= \Psi(\zeta) \cdot \hat{T}(\hat{f})(h^{-1}(\zeta)) \\ &= \Psi(\zeta) \cdot \kappa(h^{-1}(\zeta)) \cdot \hat{f}(h(h^{-1}(\zeta))) = \Psi(\zeta) \cdot \kappa(h^{-1}(\zeta)); \end{aligned}$$

that is, $\Psi(\zeta)\cdot\kappa(h^{-1}(\zeta)) = 1$ for all $\zeta \in \Phi_A$. On the other hand, from the fact that $\widehat{B}$ is an ideal in $M(B)$ (see [17]) and since, by hypothesis, $\kappa : \Phi_B \to \mathbf{C}$ belongs to $M(B)$, we infer that $\kappa\cdot\kappa\cdot(\hat{f}\circ h)$ belongs to $\widehat{B}$ for every $f \in A$. Consequently,

$$\begin{aligned}\hat{T}^{-1}(\kappa\cdot\kappa\cdot(\hat{f}\circ h))(\zeta) &= \Psi(\zeta)\cdot\kappa(h^{-1}(\zeta))\cdot\kappa(h^{-1}(\zeta))\cdot\hat{f}(h(h^{-1}(\zeta)))\\ &= \kappa(h^{-1}(\zeta))\cdot\hat{f}(\zeta)\end{aligned}$$

for all $\zeta \in \Phi_A$. This implies that the function $(\kappa\circ h^{-1})\cdot\hat{f}$ belongs to $\widehat{A}$ for all $f \in A$, which is to say that $(\kappa\circ h^{-1})$ belongs to $M(A)$. Hence, since $\widehat{A}$ is an ideal in $M(A)$ and the function $\Psi\cdot(\hat{g}\circ h^{-1})$ belongs to $\widehat{A}$, we have that $(\kappa\circ h^{-1})\cdot\Psi\cdot(\hat{g}\circ h^{-1}) = (\hat{g}\circ h^{-1})$ belongs to $\widehat{A}$ for all $g \in B$.

In like manner, we can prove that $\hat{f}\circ h$ belongs to $\widehat{B}$ for all $f \in A$. Since $h : \Phi_B \longrightarrow \Phi_A$ is a homeomorphism, the mapping $\hat{T}_h : \widehat{A} \longrightarrow \widehat{B}$, defined as $\hat{T}_h(\hat{f}) := \hat{f}\circ h$, is a surjective algebra isomorphism, which, by semisimplicity, provides the desired algebra isomorphism of A onto B.

The converse is clear. $\square$

Theorem 3. *Let A and B be Ditkin algebras. Then A and B are (algebra) isomorphic if and only if there exists a disjointness preserving bijection between them whose weight functions are multipliers.*

PROOF. Combine Theorems 1 and 2. $\square$

Next we show that Ditkin algebras with a BAI have local units thanks to the Cohen Factorization Theorem ([13]).

Proposition 1. *Let A be a Ditkin algebra which has an approximate identity of bound b. Then for each compact $K \subset \Phi_A$ and each $\epsilon > 0$ there exists $k \in A$ such that $\hat{k}$ has compact support, $\hat{k} \equiv 1$ on K and $\|k\| < b + \epsilon$.*

PROOF. Since A is regular, we can find $f \in A$ such that $\hat{f} \equiv 1$ on K. By Cohen Factorization Theorem, given $\delta > 0$, we can write $f = f_1 f_2$, where $f_1, f_2 \in A$, $\|f_1\| \leq b$ and $\|f - f_2\| < \delta$. Hence, if we define $g_1 := f_1 - f_1(f - f_2)$, then $\hat{g}_1 \equiv 1$ on K and $\|g_1\| < b(1+\delta)$. By [18, p. 205], we know that there exists $g_2 \in A$ such that $\hat{g_2}$ has compact support and $||g_1 - g_2|| < \delta$. Hence we can now define the following function in A:

$$k = g_2 \sum_{n=0}^{\infty} (g_1 - g_2)^n.$$

It is apparent that $\hat{k}$ has compact support and that, if $x \in K$, then

$$\hat{k}(x) = \hat{g}_2(x)\frac{1}{1 - \hat{g}_1(x) + \hat{g}_2(x)} = 1.$$

Furthermore, by choosing an appropiate δ,

$$\|k\| \leq \frac{b(1+2\delta)}{1-\delta} < b + \epsilon$$

as was to be proved. $\square$

Let A be a commutative Banach algebra. A complex-valued function κ on $\Phi_{\mathcal{A}}$ is said to satisfy the BSE-condition if there exists $C > 0$ such that, for every finite collection $c_1, \dots, c_n$ of complex numbers and $\alpha_1, \dots, \alpha_n$ in $\Phi_{\mathcal{A}}$,

$$\left| \sum_{j=1}^{n} c_j \kappa(\alpha_j) \right| \leq C \left\| \sum_{j=1}^{n} c_j \alpha_j \right\|_{A^*}$$

where A^* denotes the dual space of A. This condition is motivated by the Bochner–Schoenberg–Eberlein theorem, which characterizes the Fourier–Stieltjes transforms of measures on a locally compact abelian group. A commutative Banach algebra A without order is called a BSE-algebra ([20]) if the continuous functions on $\Phi_{\mathcal{A}}$ satisfying the BSE-condition are precisely the functions of the form $\hat{w}$ where $w \in M(A)$.

Lemma 1. *Let A be a Ditkin algebra with BAI and B a BSE Ditkin algebra. Let $T : A \longrightarrow B$ be a disjointness preserving bijection. Then the weight function κ belongs to $M(B)$.*

Proof. Let $\{\alpha_1, \dots, \alpha_n\}$ be a subset of $\Phi_{\mathcal{B}}$ and $\epsilon > 0$. By Proposition 1, there exists $f \in A$ such that $\|f\| < b + \epsilon$ and $\hat{f}(h(\alpha_i)) = 1$ for $i = 1, \dots, n$.

Let $\{c_1, \dots, c_n\} \subset \mathbf{C}$. Then, since $\hat{T}$ is continuous (Theorem 1 (1)), we have

$$\begin{aligned} \left| \sum_{i=1}^{n} c_i \cdot \kappa(\alpha_i) \right| &= \left| \sum_{i=1}^{n} c_i \cdot \hat{T}(\hat{f})(\alpha_i) \right| \|\hat{T}(\hat{f})\| \left\| \sum_{i=1}^{n} c_i \delta_{\alpha_i} \right\|_{A^*} \\ &\leq \|\hat{T}\| (b+\epsilon) \left\| \sum_{i=1}^{n} c_i \delta_{\alpha_i} \right\|_{A^*} \end{aligned}$$

Consequently, κ satisfies the BSE-condition and, as B is a BSE algebra, $\kappa \in M(B)$. $\square$

Theorem 4. *Let A and B be BSE Ditkin algebras with BAI. Then A and B are isomorphic as Banach algebras if and only if there exists a disjointness preserving bijection between them.*

Proof. It is a straightforward consequence of Lemma 1 and Theorem 3. $\square$

Corollary 1. *Let A and B be BSE Ditkin algebras with BAI. Then A and B are isomorphic as Banach algebras if and only if $\hat{A}$ and $\hat{B}$ are $\|\cdot\|_\infty$-isometric; i.e., there exists a linear bijection T of A onto B such that $\|\hat{f}\|_\infty = \|\hat{T}(\hat{f})\|_\infty$ for all $f \in A$.*

Proof. Suppose that T is a linear bijection of A onto B with $\|\hat{f}\|_\infty = \|\hat{T}(\hat{f})\|_\infty$ for all $f \in A$. By [4, Theorem 4.1 and Lemma 2.1]) we know that

$$\partial B = \bigcup_{\zeta \in \partial A} \{\gamma \in \Phi_B : |\hat{f}(\zeta)| = |\hat{T}(\hat{f})(\gamma)| \text{ for all } f \in A\},$$

where ∂A and ∂B stand for the Shilov boundaries of $\hat{A}$ and $\hat{B}$ respectively. But, since $\hat{A}$ is a regular subalgebra of $C_0(\Phi_A)$, it is well known that the Shilov boundary of $\hat{A}$ coincides with Φ_A. Hence, we indeed have

$$\Phi_B = \bigcup_{\zeta \in \Phi_A} \{\gamma \in \Phi_B : |\hat{f}(\zeta)| = |\hat{T}(\hat{f})(\gamma)| \text{ for all } f \in A\}.$$

The remainder of this part of the proof consists of checking that T is disjointness preserving and applying Theorem 4. Assume, contrary to what we claim, that there are $\hat{f}, \hat{g} \in A$ with disjoint cozero sets such that $\hat{T}(\hat{f}) \cdot \hat{T}(\hat{g}) \not\equiv 0$. Let us choose $\gamma_0 \in \Phi_B$ such that $|\hat{T}(\hat{f})(\gamma_0)| > 0$ and $|\hat{T}(\hat{g})(\gamma_0)| > 0$. In virtue of the paragraph above, there exists $\zeta_0 \in \Phi_A$ such that $|\hat{u}(\zeta_0)| = |\hat{T}(\hat{u})(\gamma_0)|$ for all $u \in A$. Since the cozero sets of $\hat{f}$ and $\hat{g}$ are disjoint, we have that either $\hat{f}(\zeta_0) = 0$ or $\hat{g}(\zeta_0) = 0$, which yields that either $\hat{T}(\hat{f})(\gamma_0) = 0$ or $\hat{T}(\hat{g})(\gamma_0) = 0$. This contradiction proves that T is disjointness preserving.

On the other hand, suppose that S is an algebra isomorphism from A onto B. Then there exists a homeomorphism $h : \Phi_B \longrightarrow \Phi_A$ such that $\hat{S}(\hat{f}) = \hat{f} \circ h$. It follows that $\hat{S}$ is a $\|\cdot\|_\infty$-isometry from $\hat{A}$ onto $\hat{B}$. □

Remark 1. The above corollary is not true for general Banach function algebras. Indeed, H^∞, the Banach algebra of bounded analytic functions on the open unit disk, and H_0^∞, the subalgebra of all elements in H^∞ which vanish at the origin, are isometric but are not algebraically isomorphic.

A similar situation can be found in [11], where the authors provide two isometric semisimple commutative Banach algebras which are not isomorphic as Banach algebras.

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