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Generalized skew derivations on nest algebras characterized by acting on zero products

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Abstract. Let $\mathcal{N}$ be a nest on a Banach space X with $N \in \mathcal{N}$ complemented in X whenever $N_- = N$, and let Alg$\mathcal{N}$ be the associated nest algebra. Assume that ϕ : Alg$\mathcal{N} \to$ Alg$\mathcal{N}$ is an automorphism and δ : Alg$\mathcal{N} \to$ Alg$\mathcal{N}$ is an additive map. It is shown that, if δ is ϕ-derivable at zero point (i.e., satisfies $\delta(A)B + \phi(A)\delta(B) = 0$ whenever $AB = 0$), then there exists an additive ϕ-derivation d : Alg$\mathcal{N} \to$ Alg$\mathcal{N}$ such that $\delta(A) = d(A) + \delta(I)A$ for all $A \in$ Alg$\mathcal{N}$. Moreover, by use of this result, the linear maps generalized ϕ-derivable at zero point are also characterized.

1. Introduction

Let $\mathcal{A}$ be an algebra with unit I and $\delta : \mathcal{A} \to \mathcal{A}$ an additive (linear) map. Recall that δ is called an additive (linear) derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$; if there is an additive (linear) derivation $\tau : \mathcal{A} \to \mathcal{A}$ such that $\delta(AB) = \delta(A)B + A\tau(B)$ for all $A,B \in \mathcal{A}$, then δ is called an additive (linear) generalized derivation and τ is the relating derivation. Derivations and generalized derivations are very important maps both in theory and applications, and have been studied intensively (see [5], [8], [11], [14], [15]).

Recently, more and more mathematicians are interested in characterizing the maps (generalized) derivable at some point. Recall that δ is derivable at

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some point $Z \in \mathcal{A}$ if $\delta(A)B + A\delta(B) = \delta(AB)$ for all $A, B \in \mathcal{A}$ with $AB = Z$; is generalized derivable at Z if there exists an additive map τ on $\mathcal{A}$ which is derivable at zero point such that $\delta(AB) = \delta(A)B + A\tau(B)$ for all $A, B \in \mathcal{A}$ with $AB = Z$. It is obvious that the condition of (generalized) derivable maps at some point is much weaker than the condition of being a (generalized) derivation. Jing *et al.* in [12] showed that, for the case of nest algebras on Hilbert spaces, the set of linear maps derivable at zero point with $\delta(I) = 0$ coincides with the set of inner derivations. Zhu and Xiong showed in [17] that every norm continuous linear map generalized derivable at zero point between finite nest algebras on Hilbert spaces is a generalized inner derivation (i.e., has the form $A \mapsto TA + AS$). Note that in [17], authors gave another definition for maps generalized derivable at zero point: δ is said to generalized derivable at zero point if $\delta(AB) = \delta(A)B + A\delta(B) - A\delta(I)B$ for all $A, B \in \mathcal{A}$ with $AB = 0$. It is easy to prove that their definition is a special case of ours. For other results, see [4], [10], [13], [18].

The concepts of (generalized) derivations have been generalized. Let ϕ be an automorphism of $\mathcal{A}$. An additive (linear) map $\delta : \mathcal{A} \to \mathcal{A}$ is called a ϕ-derivation if $\delta(AB) = \delta(A)B + \phi(A)\delta(B)$ for all $A, B \in \mathcal{A}$; if there is an additive (linear) ϕ-derivation $\tau : \mathcal{A} \to \mathcal{A}$ such that $\delta(AB) = \delta(A)B + \phi(A)\tau(B)$ for all $A, B \in \mathcal{A}$, then δ is called an additive (linear) generalized ϕ-derivation and τ is the relating ϕ-derivation. It is obvious that (generalized) ϕ-derivations are usual (generalized) derivations if ϕ is an identity map. The structure of ϕ-derivations has been studied (see, for example, [1], [2], [6]).

Motivated by the maps (generalized) derivable at some point, we give the concepts of the maps (generalized) ϕ-derivable at some point. We say that δ is ϕ-derivable at some point $Z \in \mathcal{A}$ if $\delta(A)B + \phi(A)\delta(B) = \delta(AB)$ for all $A, B \in \mathcal{A}$ with $AB = Z$; is generalized ϕ-derivable at Z if there exists an additive map τ on $\mathcal{A}$ which is ϕ-derivable at zero point such that $\delta(AB) = \delta(A)B + \phi(A)\tau(B)$ for all $A, B \in \mathcal{A}$ with $AB = Z$.

The purpose of this paper is to characterize the maps (generalized) ϕ-derivable at zero point on nest algebras on Banach spaces. Let $\mathcal{N}$ be a nest on a Banach space X with $N \in \mathcal{N}$ complemented in X whenever $N_- = N$, and let $\mathrm{Alg}\,\mathcal{N}$ be the associated nest algebra. Assume that $\phi : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is a ring automorphism and $\delta : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is an additive map ϕ-derivable at zero point. In Section 2, we show that δ has the form $\delta(A) = d(A) + \delta(I)A$ for all $A \in \mathrm{Alg}\,\mathcal{N}$, where $d : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is an additive ϕ-derivation (Theorem 2.1 and 2.2). In Section 3, by use of Theorem 2.1 and 2.2, we give a characterization of the maps generalized ϕ-derivable at zero point on $\mathrm{Alg}\,\mathcal{N}$ (Theorem 3.1).

Let X be a Banach space over the real or complex field $\mathbb{F}$. $\mathcal{B}(X)$ denotes the algebra of all bounded linear operators on X. A nest $\mathcal{N}$ on X is a chain of closed (under norm topology) subspaces of X which is closed under the formation of arbitrary closed linear span (denote by $\bigvee$) and intersection (denote by $\bigwedge$), and which includes $\{0\}$ and X. The nest algebra associated to the nest $\mathcal{N}$, denoted by $\mathrm{Alg}\,\mathcal{N}$, is the weak closed operator algebra consisting of all operators that leave $\mathcal{N}$ invariant, i.e.,

$$\mathrm{Alg}\,\mathcal{N} = \{T \in \mathcal{B}(X) : TN \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

When $\mathcal{N} \neq \{0, X\}$, we say that $\mathcal{N}$ is non-trivial. If $\mathcal{N}$ is trivial, then $\mathrm{Alg}\,\mathcal{N} = \mathcal{B}(X)$. For $N \in \mathcal{N}$, let $N_- = \bigvee\{M \in \mathcal{N} \mid M \subset N\}$. Denote $\mathcal{D}(\mathcal{N}) = \bigcup\{N \in \mathcal{N} \mid N_- \neq X\}$. It is clear that $\mathcal{D}(\mathcal{N})$ is dense in X. For more information on nest algebras, we refer to [7].

It is clear that every nest algebra on a finite dimensional space is isomorphic to an upper triangular block matrix algebra. Let $\mathcal{M}_n(\mathbb{F})$ denote the algebra of all $n \times n$ matrices over $\mathbb{F}$. Recall that an upper triangular block matrix algebra $\mathcal{T} = \mathcal{T}(n_1, n_2, \ldots, n_k)$ is a subalgebra of $\mathcal{M}_n(\mathbb{F})$ consisting of all $n \times n$ matrices of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1k} \\ 0 & A_{22} & \ldots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{kk} \end{pmatrix},$$

where $n_1, n_2, \ldots, n_k$ are finite sequence of positive integers satisfying $n_1 + n_2 + \cdots + n_k = n$ and $A_{ij} \in \mathcal{M}_{n_i \times n_j}(\mathbb{F})$, the space of all $n_i \times n_j$ matrices over $\mathbb{F}$.

2. Characterization of skew derivations

In this section, we consider the additive maps ϕ-derivable at zero point on nest algebras. The following are our main results.

Theorem 2.1. *Let $\mathcal{N}$ be a nest on an infinite dimensional Banach space X over the real or complex field $\mathbb{F}$ with $N \in \mathcal{N}$ complemented in X whenever $N_- = N$ and $\mathrm{Alg}\,\mathcal{N}$ the associated nest algebra. Assume that $\phi : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is a ring automorphism and $\delta : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is an additive map ϕ-derivable at zero point. Then there exist an invertible operator $T \in \mathrm{Alg}\,\mathcal{N}$, a scalar λ and an additive ϕ-derivation $d : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ such that $\delta(I) = \lambda T$ and $\delta(A) = d(A) + \delta(I)A = d(A) + \lambda TA$ for all $A \in \mathrm{Alg}\,\mathcal{N}$.*

Remark 2.1. If X is a Hilbert space, the assumption on nest $\mathcal{N}$ in the above theorem is superfluous.

For the finite dimensional case, we have

Theorem 2.2. *Let $\mathbb{F}$ be the real or complex field, and n be positive integers greater than 1. Let $\mathcal{T} = \mathcal{T}(n_1, n_2, \ldots, n_k) \subseteq M_n(\mathbb{F})$ be an upper triangular block matrix algebra. Assume that $\phi : \mathcal{T} \to \mathcal{T}$ is a ring automorphism and $\delta : \mathcal{T} \to \mathcal{T}$ is an additive map ϕ-derivable at zero point. Then there exists an additive ϕ-derivation $d : \mathcal{T} \to \mathcal{T}$ such that $\delta(A) = d(A) + \delta(I)A$ for all $A \in \mathcal{T}$.*

As an immediate consequence of Theorem 2.1 and 2.2, we get the following corollary which generalizes the main theorem in [12].

Corollary 2.3. *Let $\mathcal{N}$ be a nest on a Banach space X over the real or complex field $\mathbb{F}$ with $N \in \mathcal{N}$ complemented in X whenever $N_- = N$ and $\mathrm{Alg}\,\mathcal{N}$ the associated nest algebra. Assume that $\delta : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is an additive map derivable at zero point. Then there exist a scalar λ and an additive derivation $d : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ such that $\delta(A) = d(A) + \lambda A$ for all $A \in \mathrm{Alg}\,\mathcal{N}$.*

Remark 2.2. By [8, Theorem 4.1], additive derivations on nest algebras on infinite dimensional Banach spaces are linear. Hence, if X is infinite dimensional in Corollary 2.3, then there exists an operator $S \in \mathrm{Alg}\,\mathcal{N}$ such that $\delta(A) = AS - SA + \lambda A$ for all $A \in \mathrm{Alg}\,\mathcal{N}$.

Before proving main theorems, we need the following lemma which appeared in [9].

Lemma 2.4 ([9, Lemma 3.2])**.** *Let $\mathcal{N}$ be a nest on a (real or complex) Banach space X. If $N \in \mathcal{N}$ is complemented in X whenever $N_- = N$, then the ideal $\mathrm{Alg}_{\mathcal{F}}\mathcal{N}$ of finite rank operators of $\mathrm{Alg}\,\mathcal{N}$ is contained in the linear span of the idempotents in $\mathrm{Alg}\,\mathcal{N}$. Moreover, for every rank one nilpotent operator F in $\mathrm{Alg}\,\mathcal{N}$, there exist idempotent operators P, Q in the nest algebra $\mathrm{Alg}\,\mathcal{N}$ such that $F = P - Q$.*

Proof of Theorem 2.1. We will prove the theorem by checking several claims.

Claim 1. There exist an invertible operator $T \in \mathrm{Alg}\,\mathcal{N}$ and some scalar λ such that $\delta(I) = \lambda T$.

For any idempotent $P \in \mathrm{Alg}\,\mathcal{N}$, it is obvious that $P(I-P) = (I-P)P = 0$. Since δ is ϕ-derivable at zero point on $\mathrm{Alg}\mathcal{N}$, we have

$$\begin{aligned} 0 = \delta(P(I-P)) &= \delta(P)(I-P) + \phi(P)\delta(I-P) \\ &= \delta(P) - \delta(P)P + \phi(P)\delta(I) - \phi(P)\delta(P) \end{aligned}$$

and

$$\begin{aligned}0 &= \delta((I-P)P) = \delta(I-P)P + \phi(I-P)\delta(P)\\ &= \delta(I)P - \delta(P)P + \phi(I)\delta(P) - \phi(P)\delta(P).\end{aligned}$$

Comparing the above two equations, we get

$$\delta(P) + \phi(P)\delta(I) = \phi(I)\delta(P) + \delta(I)P. \tag{2.1}$$

Since ϕ is an automorphism of $\operatorname{Alg}\mathcal{N}$, it is obvious that $\phi(I) = I$, and by [9], $\phi(A) = TAT^{-1}$ for all $A \in \operatorname{Alg}\mathcal{N}$, where $T \in \operatorname{Alg}\mathcal{N}$ is an invertible operator. So equation (2.1) yields $TPT^{-1}\delta(I) = \delta(I)P$, that is, $PT^{-1}\delta(I) = T^{-1}\delta(I)P$ for all idempotent $P \in \operatorname{Alg}\mathcal{N}$. By Lemma 2.4 and the fact that the set of finite rank operators is strongly dense in $\operatorname{Alg}\mathcal{N}$, it follows that there exists some $\lambda \in \mathbb{F}$ such that $T^{-1}\delta(I) = \lambda I$. Hence $\delta(I) = \lambda T$.

Claim 2. There exists an additive map $h : \mathbb{F} \to \mathbb{F}$ such that $\delta(\alpha I) = h(\alpha)T$ and $\delta(\alpha P) = \alpha\delta(P) + h(\alpha)TP - \alpha\lambda TP$ for every scalar $\alpha \in \mathbb{F}$ and every idempotent $P \in \operatorname{Alg}\mathcal{N}$.

For any idempotent P, since $\alpha P(I-P) = (I-P)\alpha P = 0$, we have

$$\begin{aligned}0 &= \delta(\alpha P(I-P)) = \delta(\alpha P)(I-P) + \phi(\alpha P)\delta(I-P)\\ &= \delta(\alpha P) - \delta(\alpha P)P + \phi(\alpha P)\delta(I) - \phi(\alpha P)\delta(P)\end{aligned}$$

and

$$\begin{aligned}0 &= \delta((I-P)\alpha P) = \delta(I-P)\alpha P + \phi(I-P)\delta(\alpha P)\\ &= \delta(I)\alpha P - \delta(P)\alpha P + \delta(\alpha P) - \phi(P)\delta(\alpha P).\end{aligned}$$

That is,

$$\begin{aligned}\delta(\alpha P) &= \delta(\alpha P)P + \phi(\alpha P)\delta(P) - \phi(\alpha P)\delta(I)\\ &= \delta(\alpha P)P + \phi(\alpha P)\delta(P) - \lambda\alpha TP \end{aligned}\tag{2.2}$$

and

$$\begin{aligned}\delta(\alpha P) &= \alpha\delta(P)P + \phi(P)\delta(\alpha P) - \alpha\delta(I)P\\ &= \alpha\delta(P)P + \phi(P)\delta(\alpha P) - \lambda\alpha TP.\end{aligned}\tag{2.3}$$

Comparing equation (2.2) with (2.3), we get

$$\alpha\delta(P)P + \phi(P)\delta(\alpha P) = \delta(\alpha P)P + \phi(\alpha P)\delta(P). \tag{2.4}$$

Similarly, by $P\alpha(I-P) = \alpha(I-P)P = 0$, one can get

$$\delta(\alpha I)P + \alpha\delta(P) = \delta(\alpha P)P + \alpha\phi(P)\delta(P) \tag{2.5}$$

and

$$\phi(P)\delta(\alpha I) + \alpha\delta(P) = \alpha\delta(P)P + \phi(P)\delta(\alpha P). \tag{2.6}$$

Combining equations (2.4)–(2.6), we have $\phi(P)\delta(\alpha I) = \delta(\alpha I)P$. By a similar argument with that of Claim 1, there exists a map $h : \mathbb{F} \to \mathbb{F}$ such that $\delta(\alpha I) = h(\alpha)T$. It is clear that h is an additive map. Now combining equation (2.3) with (2.6), we get $\delta(\alpha P) = \alpha\delta(P) + h(\alpha)TP - \alpha\lambda TP$.

Claim 3. $\delta(\alpha F) = h(\alpha)TF + \alpha\delta(F) - \alpha\lambda TF$ for all finite rank operator $F \in \operatorname{Alg}\mathcal{N}$ and $\lambda \in \mathbb{F}$.

Firstly, we'll prove that $\delta(\alpha x \otimes f) = h(\alpha)Tx \otimes f + \alpha\delta(x \otimes f) - \alpha\lambda Tx \otimes f$ for all rank one operator $x \otimes f \in \operatorname{Alg}\mathcal{N}$. We prove this by considering three cases.

Case 1. $\langle x, f\rangle = 1$. Then $x \otimes f$ is an idempotent. By Claim 2, the claim holds.

Case 2. $\langle x, f\rangle = 0$. By Lemma 2.4, Case 1 and the additivity of δ, the claim is true.

Case 3. $\langle x, f\rangle = \beta \neq 0, 1$. Then $P = \beta^{-1}x \otimes f$ is an idempotent. We first show that $h(\alpha\beta) = h(\alpha)\beta + \alpha h(\beta) - \lambda\alpha\beta$ for all $\alpha, \beta \in \mathbb{F}$. In fact, take any rank one square zero operator $x \otimes f \in \operatorname{Alg}\mathcal{N}$. By Case 2, for every $\alpha, \beta \in \mathbb{F}$, we have

$$\delta(\alpha\beta x \otimes f) = h(\alpha\beta)Tx \otimes f + \alpha\beta\delta(x \otimes f) - \lambda\alpha\beta Tx \otimes f.$$

On the other hand,

$$\begin{aligned}\delta(\alpha\beta x \otimes f) &= h(\alpha)T\beta x \otimes f + \alpha\delta(\beta x \otimes f) - \lambda\alpha T\beta x \otimes f\\ &= h(\alpha)\beta Tx \otimes f + \alpha h(\beta)Tx \otimes f + \alpha\beta\delta(x \otimes f) - 2\lambda\alpha\beta Tx \otimes f.\end{aligned}$$

Comparing the above two equations, we get $(h(\alpha\beta) - h(\alpha)\beta - \alpha h(\beta) + \lambda\alpha\beta)Tx \otimes f = 0$, which implies that $h(\alpha\beta) = h(\alpha)\beta + \alpha h(\beta) - \lambda\alpha\beta$. Now by Claim 2, we have

$$\begin{aligned}\delta(\alpha x \otimes f) = \delta(\alpha\beta P) &= h(\alpha\beta)TP + \alpha\beta\delta(P) - \lambda\alpha\beta TP\\ &= h(\alpha)\beta TP + \alpha h(\beta)TP + \alpha\beta\delta(P) - 2\lambda\alpha\beta TP\\ &= h(\alpha)\beta TP + \alpha(h(\beta)TP + \beta\delta(P)) - 2\lambda\alpha\beta TP\\ &= h(\alpha)\beta TP + \alpha(\delta(\beta P) + \lambda\beta TP) - 2\lambda\alpha\beta TP\\ &= h(\alpha)\beta TP + \alpha\delta(\beta P) - \lambda\alpha\beta TP\\ &= h(\alpha)Tx \otimes f + \alpha\delta(x \otimes f) - \lambda\alpha Tx \otimes f.\end{aligned}$$

Since every finite rank operator in $\operatorname{Alg}\mathcal{N}$ is the sum of rank one operators in $\operatorname{Alg}\mathcal{N}$, the claim holds.

Claim 4. For any operator $A \in \operatorname{Alg}\mathcal{N}$ and any rank one operator $x \otimes f \in \operatorname{Alg}\mathcal{N}$, we have $\delta(Ax \otimes f) = \delta(A)x \otimes f + \phi(A)\delta(x \otimes f) - \lambda TAx \otimes f$.

We prove the claim by distinguishing three cases.

Case 1. $\langle x, f\rangle = 1$. Since $A(x\otimes f)(I - x\otimes f) = A(I - x\otimes f)(x\otimes f) = 0$, we have

$$\delta(Ax\otimes f) = \delta(Ax\otimes f)(x\otimes f) - \lambda TAx\otimes f + \phi(Ax\otimes f)\delta(x\otimes f)$$

and

$$\delta(Ax\otimes f)(x\otimes f) + \phi(Ax\otimes f)\delta(x\otimes f) = \delta(A)x\otimes f + \phi(A)\delta(x\otimes f).$$

Comparing the above two equations, it follows that the claim is true.

Case 2. $\langle x, f\rangle = 0$. By Case 1 and Lemma 2.4, the claim is true.

Case 3. $\langle x, f\rangle = \beta \neq 0, 1$. Then $x\otimes f = \beta P$ for rank one idempotent $P = \beta^{-1}x\otimes f \in \mathrm{Alg}\,\mathcal{N}$. Let $S = (I-P)A(I-P)$. It is clear that $S(\beta P) = (\beta P)S = 0$. So

$$\begin{aligned}\delta(S)(\beta P) + \phi(S)\delta(\beta P) &= \beta\delta(S)P + \beta\phi(S)\delta(P) + h(\beta)\phi(S)TP - \lambda\beta\phi(S)TP\\ &= \beta\delta(S)P + \beta\phi(S)\delta(P) = 0.\end{aligned} \tag{2.7}$$

Note that $A - S$ is a finite rank operator. By equation (2.7), Claims 2–3 and Case 1 of Claim 4, we have

$$\begin{aligned}\delta(Ax\otimes f) &= \delta(A(\beta P)) = \delta(\beta(A-S)P)\\ &= \delta(\beta(A-S))P + \phi(\beta(A-S))\delta(P) - \lambda\beta T(A-S)P\\ &= h(\beta)T(A-S)P + \beta\delta(A-S)P + \phi(\beta(A-S))\delta(P) - 2\lambda\beta T(A-S)P\\ &= h(\beta)TAP + \beta\delta(A)P + \beta\phi(A)\delta(P) - 2\lambda\beta TAP\\ &\quad - (\beta\delta(S)P + \beta\phi(S)\delta(P))\\ &= \delta(A)x\otimes f + \phi(A)(h(\beta)TP + \beta\delta(P)) - 2\lambda\beta TAP\\ &= \delta(A)x\otimes f + \phi(A)\delta(x\otimes f) - \lambda\beta TAP\\ &= \delta(A)x\otimes f + \phi(A)\delta(x\otimes f) - \lambda TAx\otimes f,\end{aligned}$$

completing the proof of the claim.

Claim 5. There exists an additive ϕ-derivation $d : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ such that $\delta(A) = d(A) + \lambda TA$ for all $A \in \mathrm{Alg}\,\mathcal{N}$, and therefore, the theorem is true.

For any $A, B \in \mathrm{Alg}\,\mathcal{N}$ and any rank one operator $x\otimes f \in \mathrm{Alg}\,\mathcal{N}$, by Claim 4, we have

$$\delta(ABx\otimes f) = \delta(AB)x\otimes f + \phi(AB)\delta(x\otimes f) - \lambda TABx\otimes f$$

and

$$\begin{aligned}\delta(ABx\otimes f) &= \delta(A)Bx\otimes f + \phi(A)\delta(Bx\otimes f) - \lambda TABx\otimes f\\ &= \delta(A)Bx\otimes f + \phi(A)\delta(B)x\otimes f + \phi(A)\phi(B)\delta(x\otimes f)\\ &\quad - \lambda\phi(A)TBx\otimes f - \lambda TABx\otimes f\\ &= \delta(A)Bx\otimes f + \phi(A)\delta(B)x\otimes f + \phi(AB)\delta(x\otimes f) - 2\lambda TABx\otimes f.\end{aligned}$$

Hence $(\delta(AB)-\delta(A)B-\phi(A)\delta(B)+\lambda TAB)x\otimes f=0$. Since $\mathcal{D}(\mathcal{N})$ is dense in X, we obtain that

$$\delta(AB)=\delta(A)B+\phi(A)\delta(B)-\lambda TAB. \tag{2.8}$$

Now let $d(A)=\delta(A)-\lambda TA$. By equation (2.8), it is easily checked that d is an additive ϕ-derivation on $\mathrm{Alg}\,\mathcal{N}$, and hence $\delta(A)=d(A)+\lambda TA$ for all $A\in\mathrm{Alg}\,\mathcal{N}$. Complete the proof. □

Now we give the proof of Theorem 2.2. The method is similar to that of Theorem 2.1.

Proof of Theorem 2.2. Firstly, note that, by [16], ϕ is in fact τ-linear, where τ is a field automorphism of $\mathbb{F}$. In the following, we'll use the fact repeatedly. We complete the proof by several claims.

Claim 1. $\delta(I)P=\phi(P)\delta(I)$ for all idempotent $P\in\mathcal{T}$.

By the same argument with that of equation (2.1) in Claim 1 of proof of Theorem 2.1, and noting that $\phi(I)=I$, it is obvious that $\phi(P)\delta(I)=\delta(I)P$.

Claim 2. $\delta(\alpha x\otimes f)=\phi(x\otimes f)\delta(\alpha I)+\alpha\delta(x\otimes f)-\alpha\delta(I)x\otimes f$ for all rank one operator $x\otimes f\in\mathcal{T}$.

We prove the claim by three cases.

Case 1. $\langle x,f\rangle=1$. Then $P=x\otimes f$ is an idempotent. Since $(I-P)\alpha P=0$, we have

$$\delta(\alpha P)=\alpha\delta(P)P+\phi(P)\delta(\alpha P)-\alpha\delta(I)P.$$

Similarly, by $P\alpha(I-P)=0$, we get

$$\phi(P)\delta(\alpha I)+\alpha\delta(P)=\alpha\delta(P)P+\phi(P)\delta(\alpha P).$$

Combining the above two equations, we get $\delta(\alpha P)=\alpha\delta(P)+\phi(P)\delta(\alpha I)-\alpha\delta(I)P$.

Case 2. $\langle x,f\rangle=0$. By Case 1 and Lemma 2.4, the claim is true.

Case 3. $\langle x,f\rangle=\beta\neq 0,1$. Then $P=\beta^{-1}x\otimes f$ is an idempotent. We first show that $\delta(\alpha\beta I)=\alpha\delta(\beta I)+\tau(\beta)\delta(\alpha I)-\alpha\beta\delta(I)$ for all $\alpha,\beta\in\mathbb{F}$. In fact, for any $g\in\mathbb{F}^n$, there exists $y\in\mathbb{F}^n$ such that $y\otimes g\in\mathcal{T}$ with $\langle y,g\rangle=0$. By Case 2, for every $\alpha,\beta\in\mathbb{F}$, we have

$$\delta(\alpha\beta y\otimes g)=\phi(y\otimes g)\delta(\alpha\beta I)+\alpha\beta\delta(y\otimes g)-\alpha\beta\delta(I)y\otimes g$$

and

$$\delta(\alpha\beta y\otimes g)=\tau(\beta)\phi(y\otimes g)\delta(\alpha I)+\alpha\phi(y\otimes g)\delta(\beta I)+\alpha\beta\delta(y\otimes g)-2\alpha\beta\delta(I)y\otimes g.$$

Comparing the above two equations, we get

$$\phi(y\otimes g)(\delta(\alpha\beta I)-\alpha\delta(\beta I)-\tau(\beta)\delta(\alpha I))+\alpha\beta\delta(I)y\otimes g=0. \qquad (2.9)$$

By Lemma 2.4 and Claim 1, it is clear that $\phi(A)\delta(I)=\delta(I)A$ for all $A\in\mathcal{T}$ with $A^2=0$. This and equation (2.9) yield $\phi(y\otimes g)(\delta(\alpha\beta I)-\alpha\delta(\beta I)-\tau(\beta)\delta(\alpha I)+\alpha\beta\delta(I))=0$. Since ϕ^{-1} is multiplicative, it follows that $(y\otimes g)\phi^{-1}(\delta(\alpha\beta I)-\alpha\delta(\beta I)-\tau(\beta)\delta(\alpha I)+\alpha\beta\delta(I))=0$. Since g is arbitrary, we get $\delta(\alpha\beta I)=\alpha\delta(\beta I)+\tau(\beta)\delta(\alpha I)-\alpha\beta\delta(I)$. Now by Case 1 and the τ-linearity of ϕ, it is easily checked that the claim holds.

Since every finite rank operator in $\mathcal{T}$ is the sum of rank one operators in $\mathcal{T}$, we have

$$\delta(\alpha F)=\phi(F)\delta(\alpha I)+\alpha\delta(F)-\alpha\delta(I)F \qquad (2.10)$$

for all finite rank operator $F\in\mathcal{T}$ and $\alpha\in\mathbb{F}$.

Claim 3. For any operator $A\in\mathcal{T}$ and any rank one operator $x\otimes f\in\mathcal{T}$, we have $\delta(Ax\otimes f)=\delta(A)x\otimes f+\phi(A)\delta(x\otimes f)-\delta(I)Ax\otimes f$.

The proof is similar to that of Claim 4 in the proof of Theorem 2.1, we omit it here.

Claim 4. There exists an additive ϕ-derivation $d:\mathcal{T}\to\mathcal{T}$ such that $\delta(A)=d(A)+\delta(I)A$ for all $A\in\mathcal{T}$, and therefore, Theorem 2.2 holds.

For any $A,\ B\in\mathcal{T}$ and any rank one operator $x\otimes f\in\mathcal{T}$, by calculating $\delta(ABx\otimes f)$ using two ways, similar to the proof of Claim 5 in the proof of Theorem 2.1, one can easily checked that

$$\delta(AB)=\delta(A)B+\phi(A)\delta(B)-\phi(A)\delta(I)B. \qquad (2.11)$$

Now let $d(A)=\delta(A)-\delta(I)A$. By equation (2.11), it is also easily checked that d is an additive ϕ-derivation on $\mathcal{T}$, and therefore $\delta(A)=d(A)+\delta(I)A$ for all $A\in\mathcal{T}$. □

Proof of Corollary 2.3. By Theorem 2.1 and 2.2, we only need to check that $\delta(I)=\lambda I$ for some scalar λ. In fact, by the proof of Claim 1 in Theorem 2.1, it is clear that $\delta(I)=\lambda I$ for some scalar λ and this proof is also applicable for finite dimensional case. Complete the proof. □

3. Characterization of generalized skew derivations

In this section, we consider the linear maps generalized ϕ-derivable at zero point on nest algebras. The following is our main result.

Theorem 3.1. *Let $\mathcal{N}$ be a nest on a Banach space X over the real or complex field $\mathbb{F}$ with $N \in \mathcal{N}$ complemented in X whenever $N_- = N$ and $\mathrm{Alg}\,\mathcal{N}$ the associated nest algebra. Assume that $\phi : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is an automorphism and $\delta : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is a linear map generalized ϕ-derivable at zero point associated with a linear map τ which is ϕ-derivable at zero point (i.e. $\delta(A)B+\phi(A)\tau(B) = 0$ whenever $AB = 0$). Then there exists a linear ϕ-derivation $d : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ such that $\delta(A) = d(A) + \delta(I)A$ for all $A \in \mathrm{Alg}\,\mathcal{N}$.*

To prove Theorem 3.1, we need a lemma, which is an independent interest on its own. Let $\mathcal{A}$ be a ring (an algebra). Recall that an additive (a linear) map $\delta : \mathcal{A} \to \mathcal{A}$ is called a left (right) multiplier if $\delta(AB) = \delta(A)B$ ($\delta(AB) = A\delta(B)$) for all $A, B \in \mathcal{A}$; is called a local multiplier if for every $A \in \mathcal{A}$ there is a multiplier $\delta_A : \mathcal{A} \to \mathcal{A}$ such that $\delta(A) = \delta_A(A)$. In [3], Brešar proved that, if $\mathcal{A}$ is prime ring containing a nonzero idempotent and δ satisfies that $A\delta(B) = 0$ whenever $AB = 0$, then δ is a right multiplier. Here we consider similar problems on nest algebras. Note that nest algebras are not prime.

Lemma 3.2. *Let $\mathcal{N}$ be a nest on a Banach space X over the real or complex field $\mathbb{F}$ with $N \in \mathcal{N}$ complemented in X whenever $N_- = N$ and $\mathrm{Alg}\,\mathcal{N}$ the associated nest algebra. Assume that $L : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is a linear map. If L satisfies $L(A)B = 0$ whenever $AB = 0$ (in particular, if L is a local multiplier), then $L(A) = L(I)A$ for all $A \in \mathrm{Alg}\,\mathcal{N}$, that is, L is a left multiplier.*

Proof. For any idempotent $P \in \mathrm{Alg}\,\mathcal{N}$ and any scalar α, since $\alpha(I-P)P = \alpha P(I-P) = 0$, we have

$$L(\alpha I)P = L(\alpha P)P \quad \text{and} \quad L(\alpha P) = L(\alpha P)P,$$

and so

$$L(\alpha P) = L(\alpha I) = \alpha L(I)P = L(I)\alpha P.$$

By Lemma 2.4, we get

$$L(F) = L(I)F \text{ for all finite rank operator } F \in \mathrm{Alg}\,\mathcal{N}. \tag{3.1}$$

For any operator $A \in \mathrm{Alg}\,\mathcal{N}$ and any idempotent $P \in \mathrm{Alg}\,\mathcal{N}$, since $AP(I-P) = A(I-P)P = 0$, we have

$$L(AP) = L(AP)P \quad \text{and} \quad L(AP)P = L(A)P.$$

It follows that

$$L(AP) = L(A)P. \tag{3.2}$$

Now taking any rank one idempotent $P \in \mathrm{Alg}\,\mathcal{N}$, noting that AP is finite rank, by equation (3.1) and (3.2), we have $L(A)P = L(I)AP$ for all A. Hence $L(A)x \otimes f = L(I)Ax \otimes f$ for all $A \in \mathrm{Alg}\,\mathcal{N}$ and all rank one operator $x \otimes f \in \mathrm{Alg}\,\mathcal{N}$. Since

$\mathcal{D}(\mathcal{N})$ is dense in X, we obtain that $L(A) = L(I)A$. Complete the proof. $\square$

Proof of Theorem 3.1. Since τ is ϕ-derivable at zero point, we have $AB = 0 \Rightarrow \tau(A)B + \phi(A)\tau(B) = 0$. By assumption, we get

$$AB = 0 \Rightarrow \delta(A)B - \tau(A)B = 0. \tag{3.3}$$

Let $L(A) = \delta(A) - \tau(A)$ for all A. It is clear that L is additive on $\mathrm{Alg}\,\mathcal{N}$. Moreover, by equation (3.3), we get $AB = 0 \Rightarrow L(A)B = 0$. By using of Lemma 3.2, Theorem 2.1 and 2.2, we obtain that $L(A) = L(I)A$, that is, $\delta(A) = \tau(A) + (\delta(I) - \tau(I))A = d(A) + \delta(I)A$ for all $A \in \mathrm{Alg}\,\mathcal{N}$, where $d : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is a ϕ-derivation. $\square$

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