Publ. Math. Debrecen
78/2 (2011), 469–484
DOI: 10.5486/PMD.2011.4851

Quasirecognition by prime graph of simple group $D_n(3)$

By B. KHOSRAVI (Tehran), Z. AKHLAGHI (Tehran) and M. KHATAMI (Tehran)

Abstract. Let G be a finite group. The prime graph $\Gamma(G)$ of G is defined as follows. The vertices of $\Gamma(G)$ are the primes dividing the order of G and two distinct vertices p and p' are joined by an edge if there is an element in G of order pp'. It is proved that $D_n(q)$, with disconnected prime graph, is quasirecognizable by their element orders.

In this paper as the main result, we show that $D_n(3)$, where $n \in \{p, p+1\}$ for an odd prime $p > 3$, is quasirecognizable by its prime graph.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. We construct the *prime graph* of G, which is denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct primes p and p' are joined by an edge if and only if G has an element of order pp'. Let $s(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1(G), \pi_2(G), \dots, \pi_{s(G)}(G)$ be the connected components of $\Gamma(G)$. Sometimes we use the notation π_i instead of $\pi_i(G)$. If $2 \in \pi(G)$ we always suppose $2 \in \pi_1(G)$. Let m and n be natural numbers. We write $m \sim n$ if and only if for every prime divisors $r \in \pi(m)$ and $s \in \pi(n), G$ has an element of order rs.

The *spectrum* of a finite group G, which is denoted by $\pi_e(G)$, is the set of its element orders. A subset X of the vertices of a graph is called an independent set if the induced graph on X has no edge. Let G be a finite group and $r \in \pi(G)$. We denote by $\rho(G)$, some independent set of vertices in $\Gamma(G)$ with the maximal

Mathematics Subject Classification: 20D05, 20D60, 20D08.
Key words and phrases: prime graph, simple group, recognition, quasirecognition.

number of elements. Also some independent set of vertices in $\Gamma(G)$ containing r with the maximal number of elements is denoted by $\rho(r,G)$. Also we put $t(G) = |\rho(G)|$ and $t(r,G) = |\rho(r,G)|$.

A finite group G is said to be *recognizable by spectrum* if the equality $\pi_e(H) = \pi_e(G)$ implies that $H \cong G$. A finite simple non-abelian group G is called *quasi-recognizable by spectrum* if each finite group H with $\pi_e(H) = \pi_e(G)$ has a unique non-abelian composition factor isomorphic to G (see [16]).

We denote by $k(\Gamma(G))$ the number of isomorphism classes of finite groups H satisfying $\Gamma(G) = \Gamma(H)$. Given a natural number n, a finite group G is called *n-recognizable by prime graph* if $k(\Gamma(G)) = n$. Usually a 1-recognizable group is called a recognizable group. A non-abelian simple group G is said to be *quasirecognizable by prime graph* if every finite group whose prime graph is $\Gamma(G)$ has a unique non-abelian composition factor isomorphic to G (see [16]).

Hagie in [9] determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. In [21] finite groups with the same prime graph as a CIT simple group are determined. It is proved that if $q = 3^{2n+1}$ $(n > 0)$, then the simple group ${}^2G_2(q)$ is uniquely determined by its prime graph [16, 35]. Also in [22] it is proved that $\mathrm{PSL}(2,p)$, where $p > 11$ is a prime number and $p \not\equiv 1 \pmod{12}$, is recognizable by prime graph. In [23] and [17], finite groups with the same prime graph as $\mathrm{PSL}(2,q)$, where q is not prime, are determined. In [1], [20], finite groups with the same prime graph as ${}^2F_4(q)$, where $q = 2^{2n+1} > 2$; $F_4(q)$, where $q = 2^n > 2$, are determined. Also in [15], it is proved that if p is a prime number which is not a Mersenne or Fermat prime and $p \neq 11$, 13, 19 and $\Gamma(G) = \Gamma(PGL(2,p))$, then G has a unique nonabelian composition factor which is isomorphic to $PSL(2,p)$ and if $p = 13$, then G has a unique nonabelian composition factor which is isomorphic to $PSL(2,13)$ or $PSL(2,27)$. Then it is proved that if p and $k > 1$ are odd and $q = p^k$ is a prime power, then $PGL(2,q)$ is uniquely determined by its prime graph [2] (see also [3], [6]). In [18], [19], [24], [25] finite groups with the same prime graph as $L_n(2)$ and $U_n(2)$ are obtained.

In [10], it is proved that the simple group $D_n(q)$, with disconnected prime graph is quasirecognizable by spectrum. Also it is proved that $D_n(q)$ is recognizable by spectrum, for some $n \in \mathbb{N}$ and $q \in \{2,3,5\}$. In this paper we determine finite groups G such that $\Gamma(G) = \Gamma(D_n(3))$, where $D_n(3)$ has disconnected prime graph. We note that using these results we can give new proofs for some theorems in [10].

We note that a group which is recognizable by spectrum need not be even quasirecognizable by prime graph. Obviously the recognizability by prime graph implies the recognizability by spectrum.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [4], for example.

2. Preliminary results

Lemma 2.1 (see [34]). *A finite group G with disconnected prime graph $\Gamma(G)$ satisfies one of the following conditions:*

(a) *$s(G) = 2$ and G is a Frobenius group;*

(b) *$s(G) = 2$ and G is a 2-Frobenius group;*

(c) *there exists a nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$, where K is a nilpotent normal subgroup of G; furthermore K and $\overline{G}/S$ are trivial or $\pi_1(G)$-groups, $s(S) \geq s(G)$, and for every $2 \leq i \leq s(G)$, there exists $2 \leq j \leq s(S)$ such that $\pi_i(G) = \pi_j(S)$.*

Lemma 2.2 (see [31]). *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the following hold:*

(1) *Then there exists a finite non-abelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for a maximal normal soluble subgroup K of G.*

(2) *For every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K|.|\overline{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*

(3) *One of the following conditions holds:*

 (a) *$S \cong Alt_7$ or $A_1(q)$ for some odd q, and $t(S) = t(2, S) = 3$;*

 (b) *for every prime $r \in \pi(G)$ non-adjacent to 2 in $\Gamma(G)$, a Sylow r-subgroup of G is isomorphic to a Sylow r-subgroup of S; in particular $t(2, S) \geq t(2, G)$.*

Lemma 2.3 ((Zsigmondy Theorem) (see [36])). *Let p be a prime and let n be a positive integer. Then one of the following holds:*

(i) *there is a primitive prime p' for $p^n - 1$, that is , $p' \mid (p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$,*

(ii) *$p = 2$, $n = 1$ or 6,*

(iii) *p is a Mersenne prime and $n = 2$.*

Lemma 2.4 ((see [10], [32])).

(1) *If $G = A_{n-1}(q)$, then G contains a Frobenius subgroup with kernel of order q^{n-1} and cyclic complement of order $(q^{n-1} - 1)/(n, q - 1)$.*

(2) *If $G = C_n(q)$, then G contains a Frobenius subgroup with kernel of order q^n and cyclic complement of order $(q^n - 1)/(2, q - 1)$.*

(3) *If $G = {}^2D_n(q)$, and there exists a primitive prime divisor r of $q^{2n-2} - 1$, then G contains a Frobenius subgroup with kernel of order q^{2n-2} and cyclic complement of order r.*

(4) *If $G = B_n(q)$ or $D_n(q)$, and there exists a primitive prime divisor r_m of $q^m - 1$, where $m = n$ or $n - 1$ such that m is odd, then G contains a Frobenius subgroup with kernel of order $q^{m(m-1)/2}$ and cyclic complement of order r_m.*

Lemma 2.5 (see [28, Lemma 1]). *Let N be a normal subgroup of G. Assume that G/N is a Frobenius group with Frobenius kernel F and cyclic Frobenius complement C. If $(|N|, |F|) = 1$, and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$, where p is a prime factor of $|N|$.*

Lemma 2.6. *Let G be a finite group such that $s(G) \geq 2$ and K be a normal π_1-subgroup of G. Let S be a finite simple group such that $S \leq G/K$ and S is not a π_1-group. If $K \neq 1$, and S contains a Frobenius subgroup with kernel F and a cyclic complement C such that $(|F|, |K|) = 1$, then $r|C| \in \pi_e(G)$, for every prime divisor r of K.*

Proof. Since $KC_G(K)/K \trianglelefteq G/K$, so $S \cap KC_G(K)/K \trianglelefteq S$. Let F be contained in $KC_G(K)/K$. Hence $S \cap KC_G(K)/K \neq 1$ and so $S \cap KC_G(K)/K = S$, which implies that $S \leq KC_G(K)/K$. So there exists $m \in \pi(G) \setminus \pi_1(G)$ such that $m \in \pi(C_G(K))$. So for every $r \in \pi(K)$ we have $r \sim m$, which is a contradiction. Therefore F is not contained in $KC_G(K)/K$. Thus by Lemma 2.5, $r|C| \in \pi_e(G)$, for every prime divisor r of K. □

Lemma 2.7 (see [29]). *Let G be a finite group and N a nontrivial normal p-subgroup, for some prime p, and set $K = G/N$. Suppose that K contains an element x of order m coprime to p such that $\langle \phi|_{\langle x \rangle}, 1|_{\langle x \rangle} \rangle > 0$ for every Brauer character ϕ of (an absolutely irreducible representation of) K in characteristic p. Then G contains elements of order pm.*

Lemma 2.8 (see [5, Remark 1]). *The equation $p^m - q^n = 1$, where p and q are primes and $m, n > 1$, has only one solution, namely $3^2 - 2^3 = 1$.*

Lemma 2.9 (see [5]). *With the exceptions of the relations $(239)^2 - 2(13)^4 = -1$ and $3^5 - 2(11)^2 = 1$ every solution of the equation*

$$p^m - 2q^n = \pm 1; \ p, \ q \text{ prime}; \ m, n > 1$$

has exponents $m = n = 2$; i.e. it comes from a unit $p - q.2^{1/2}$ of the quadratic field $Q(2^{1/2})$ for which the coefficients p and q are primes.

In the sequel we recall the concept of quadratic residue and the Legendre symbol from number theory.

Remark 2.10 (see [12]). Let $(k, n) = 1$. If there is an integer x such that $x^2 \equiv k \pmod{n}$, then k is called a *quadratic residue* $\pmod{n}$. Otherwise k is called a *quadratic nonresidue* $\pmod{n}$.

Let p be an odd prime. The symbol (a/p) will have the value 1 if a is a quadratic residue $\pmod{p}$, -1 if a is a quadratic nonresidue $\pmod{p}$, and zero if $p \mid a$. The symbol (a/p) is called the *Legendre symbol*. For computing the Legendre symbol we use the Law of quadratic reciprocity (for details see Chapter 5 of [12]).

Let p be a prime number and $(a, p) = 1$. Let $k \geq 1$ be the smallest positive integer such that $a^k \equiv 1 \pmod{p}$. Then k is called *the order of a with respect to p* and we denote it by $\mathrm{ord}_p(a)$. Obviously by the Fermat's little theorem it follows that $\mathrm{ord}_p(a) \mid (p-1)$. Also if $a^n \equiv 1 \pmod{p}$, then $\mathrm{ord}_p(a) \mid n$. Similarly if $q = p^\alpha$, then $\mathrm{ord}_q(a)$ is defined. If q is odd, some authors use the symbol $e(q, a)$ for $\mathrm{ord}_q(a)$.

Lemma 2.11 (see [12]). *Let p be an odd prime. Then $(-1/p) = (-1)^{(p-1)/2}$.*

Lemma 2.12 (see [33, Theorem 2.7]). *Let $G = D_n^\epsilon(q)$ be a finite simple group of Lie type over a field of characteristic p, where $\epsilon \in \{+, -\}$, and $D_n^+(q) = D_n(q)$, $D_n^-(q) = {}^2D_n(q)$. Define*

$$\eta(m) = m \text{ if } m \text{ is odd; otherwise } \eta(m) = m/2.$$

Suppose r and s are odd primes and $r, s \in \pi(G) \setminus \{p\}$, put $k = e(r, q)$, $l = e(s, q)$ and $1 \leq \eta(k) \leq \eta(l)$. Then r and s are nonadjacent if and only if $2\eta(k) + 2\eta(l) > 2n - (1 - \epsilon(-1)^{k+l})$ and k and l satisfy one of the following conditions:

(1) *$(-1)^{k+l} = 1$, $k \neq l$, and $\eta(l)/\eta(k)$ is not an odd integer;*

(2) *$(-1)^{k+l} = -1$, and if $\eta(k) = \eta(l)$ and $n/2$ is an odd integer, then $k \neq n/2$.*

3. Main results

Proposition 3.1. *Let G be a group with disconnected prime graph, such that $t(G) \geq 3$. Also let K be the maximal normal soluble subgroup of G. Then K is a nilpotent π_1-group.*

Proof. By Lemma 2.2, there exists a finite non-abelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$. By [27, Lemma 8], we know that if G is a solvable group, then $t(G) \leq 2$. Therefore G is not solvable and so G is not a 2-Frobenius group. Now we prove that G is not a non-solvable Frobenius group. Let G be a non-solvable Frobenius group with Frobenius complement C and Frobenius kernel H. By [7], [11], $\{2,3,5\} \subseteq \pi(C)$ and $\Gamma(C)$ can be obtained from the complete graph with vertex set $\pi(C)$ by removing the edge $\{3,5\}$ and $\Gamma(H)$ is a complete graph. So $\pi_1(G) = \pi(C)$ and $\pi_2(G) = \pi(H)$. Since H is nilpotent and K is the maximal normal solvable subgroup, then $H \leq K$. Therefore $S \leq G/K$ and $|G/K| \mid |C|$. If $x \in \pi_2(G)$, then $x \in \pi(H)$ and $x \nsim 2$ in $\Gamma(G)$. Also $\pi(S) \subseteq \pi(C)$ and so $x \notin \pi(S)$. So by Lemma 2.2, $S \cong A_7$ or $S \cong A_1(q)$. Note that A_7 and $A_1(q)$ contain some Frobenius subgroups. So G/K contains a Frobenius subgroup with Frobenius complement of order m, where $\pi(m) \subseteq \pi_1(G)$. If $t \in \pi_2(G) \subseteq \pi(K)$, then by Lemma 2.5, we have $t \sim m$, which is a contradiction. Therefore by Lemma 2.1, G has a normal series $1 \trianglelefteq H \trianglelefteq N \trianglelefteq G$ such that H is a nilpotent π_1-group and in [34] it is proved that $H = S_{\pi_1}(G)$, where $S_{\pi_1}(G)$ is the maximal π_1-separable normal subgroup of G. Obviously K is π_1-separable. So $K \leq H$. Therefore K is a nilpotent π_1-group. □

Theorem 3.2. *Let G be a finite group such that $\Gamma(G) = \Gamma(D_{p+1}(3))$, where p is an odd prime. Then G is recognizable by prime graph, if $p > 3$. If $p = 3$, then $G \cong B_3(3)$, $C_3(3)$ $D_4(3)$ or $G/O_2(G) \cong \mathrm{Aut}(^2B_2(8))$.*

Proof. By Proposition 3.1, there exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for the maximal normal solvable subgroup K of G, and K is nilpotent. Moreover, $t(S) \geq t(G) - 1 = [3p/4]$. According to [33, Tables 1a–1c], we consider the following cases. In the sequel we denote by r_i, a primitive prime divisor of $3^i - 1$, and if S is a simple group of Lie type over $GF(q')$, where $q' = p_0^{\alpha}$, then u_i denotes a primitive prime divisor of ${q'}^i - 1$. By [10], we have $t(G) = [(3p+4)/4]$, $t(2,G) = 2$ and $t(3,G) = 3$ and by easy computation we can see that for every $m \in \pi_1(G) \setminus \pi(3^p+1)$, $\{m, r_p, r_{2p}\} \subseteq \rho(m,G)$. We know that $\pi_1(G) = \pi(3(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1))$ and $\pi_2(G) = \pi((3^p-1)/2)$.

Case 1. Let $S \cong A_n$, where either $n-2$, $n-1$ or n is a prime number. So $r_p \in \{n, n-1, n-2\}$ and $r_p^{\beta} = (3^p-1)/2$, for some $\beta > 0$. Therefore by Lemma 2.9, either $\beta = 1$ or $r_p = 11$. Let $\beta = 1$. Since $r_{2p} \mid (3^p+1)/4$, we have $r_{2p} \leq (3^p+1)/4 \leq (3^p-1)/2 - 6 \leq r_p - 6$. So $r_{2p} < n-3$. Hence $r_{2p} \sim 3$ in $\Gamma(S)$, by [33], which is a contradiction. Therefore $r_p = 11$ and $p = 5$ and so $S \cong A_n$, where $11 \leq n \leq 13$. On the other hand $r_{2p} = (3^p+1)/4 = 61$ and $r_{2(p-1)} = (3^{p-1}+1)/2 = 41$ do not belong to $\pi(S)$. So $r_{2p}, r_{2(p-1)} \in \pi(K) \cup \pi(\overline{G}/S)$.

But $\overline{G}/S \subseteq \mathrm{Out}(S)$ and so $r_{2p}, r_{2(p-1)} \in \pi(K)$, since $|\,\mathrm{Out}(S)| = 2$. Therefore $41 = r_{2p} \sim 61 = r_{2(p-1)}$ in $\Gamma(G)$, since K is nilpotent, which is a contradiction, since r_{2p} is only adjacent to divisors of $3^p + 1$.

Case 2. Let $S \cong A_{n-1}(q)$, where $q = p_0^\alpha$, $n \in \{p', p'+1\}$ and p' is an odd prime.

Let $p = 3$. Then $p_0 \in \pi_1(S) \subseteq \pi_1(D_4(3)) = \{2,3,5,7\}$. Since 13 is the only primitive prime of $p_0^{p'\alpha} - 1$, easily we can see that $q = p_0 = 3$ and $p' = 3$. So $S \cong A_2(3)$ or $A_3(3)$. If $S \cong A_2(3)$, then $5, 7 \in \pi(K)$ and so $5 \sim 7$, since K is nilpotent, which is a contradiction. If $S \cong A_3(3)$, then $7 \in \pi(K)$. Also S contains a Frobenius subgroup with kernel of order 3^3 and cyclic complement of order 13, which implies that $7 \sim 13$, that is a contradiction.

Let $p = 5$. So $\pi_1(S) \subseteq \{2,3,5,7,13,41,61\}$ and 11 is the only primitive prime of $q^{p'} - 1$. Now similarly and by easy calculation we get a contradiction. Therefore $p > 5$.

We claim that $\pi(q+1) \subseteq \{2,3,5\}$. Let $2 \neq x \in \pi(q+1)$. By [33, Propositions 2.1, 3.1], we have $x \sim t$, for every $t \in \pi_1(S)$. Let $A = \{r_{2p}, r_{2(p-1)}, r_{2(p-2)}\}$. By Lemma 2.12 we can show that A is an independent set of $\Gamma(G)$. Using Lemma 2.2, $|A \setminus \pi(S)| \leq 1$ and so let $r, s \in A \cap \pi_1(S)$, for $r \neq s$. Then $x \sim r$ and $x \sim s$. Let $r = r_{2(p-1)}$ and $s = r_{2(p-2)}$. Then using the orders of maximal tori of $D_{p+1}(3)$, we see that there are maximal tori T and T' of $D_{p+1}(3)$ such that $r_{2(p-1)} \mid |T|$, $r_{2(p-2)} \mid |T'|$ and $x \mid (|T|, |T'|)$. If $d = (|T|, |T'|)$, then easily we can see that $\pi(d) \subseteq \{2,3,5\}$. Similarly for every $r, s \in A$, we get the result. So $\pi(q+1) \subseteq \{2,3,5\}$.

Since $p_0 \in \pi(S)$, it follows that $p_0 \in \pi(G)$. Let p_0 be a primitive prime of $3^t - 1$, where $t \leq 2p$. We claim that $t \notin \{2,3,4,6,8\}$. Otherwise, for example if $t = 6$, then $p_0 = 7$, $q = 7^\alpha$ and $\pi(q+1) \subseteq \{2,3,5\}$, which implies that $q = 7$ or $q = 49$. Let $S \cong A_{n-1}(49)$. If $n \neq 3$, then $1201 \in \pi(49^4 - 1) \subseteq \pi_1(S)$ and by [33], $t(1201, S) = 4$. On the other hand, by the orders of maximal tori in $\Gamma(G)$, $t(1201, G) \geq 7$, since 1201 is a primitive prime of $3^{300} - 1$ and so $p \geq 100$. Therefore by Lemma 2.2, we get a contradiction. Therefore $S \cong A_2(49)$ and $\pi((3^p - 1)/2) = \pi((49^3 - 1)/48)$, which is a contradiction. If $S \cong A_{n-1}(7)$, then we get a contradiction similarly.

By [33], we have $\rho(p_0, S) = \{p_0, u_{n-1}, u_n\}$. Let $t \geq 5$ be odd. Then $\{p_0, r_p, r_{2p}, r_{2(p-1)},\ r_{2(p-2)}\} \subseteq \rho(p_0, G)$, by Lemma 2.12, and we get a contradiction by Lemma 2.2.

Let $t \geq 10$ be even and $t/2 \equiv \epsilon \pmod 2$, where $\epsilon \in \{0,1\}$. Then $\{p_0, r_p, r_{2(p-2+\epsilon)}, r_{p-2}, r_{p-4}\} \subseteq \rho(p_0, G)$, and we get a contradiction by Lemma 2.2.

Therefore $p_0 = 3$ and so the set of primitive primes of $3^{\alpha p'} - 1$ is equal to the set of primitive primes of $(3^p - 1)/2$, which implies that $\alpha p' = p$, by Lemma 2.3. Then $\alpha = 1$ and $p' = p$. Therefore $\{r_{2p}, r_{2(p-1)}\} \subseteq \pi_1(G) \setminus \pi_1(S)$ and we get a contradiction, since $\pi(\overline{G}/S) \subseteq \{2\}$, K is nilpotent and $r_{2(p-1)} \nsim r_{2p}$ in $\Gamma(G)$.

If $S \cong {}^2A_{n-1}(q)$, then we get a contradiction, similarly.

Case 3. Let $S \cong A_1(q)$, where $4 \mid (q+1)$ and $q = p_0^\alpha$. Therefore $\pi_1(S) = \pi(q+1)$, $\pi_2(S) = \pi(q)$ and $\pi_3(S) = \pi((q-1)/2)$. Also $t(S) = 3$. Therefore $[3p/4] \leq 3$. So $p \leq 5$.

Let $p = 3$ and $\pi_2(S) = \pi(q) = \{13\}$. So $p_0 = 13$ and $\pi(q^2-1) \subseteq \{2,3,5,7\}$. If $5 \in \pi(q^2-1)$, then $\mathrm{ord}_5(13) = 4$, and so $2 \mid \alpha$. Hence we get a contradiction, since $17 \in \pi(13^4-1) \subseteq \pi(q^2-1)$. Therefore $\pi(q^2-1) \subseteq \{2,3,7\}$ and so $q = 13$. Since $5 \notin \pi(S)$, by [13, Theorem 15.13], the Brauer character table in characteristic 5 and the ordinary character table of S are the same. By [4] we have $\langle \phi|_{\langle x \rangle}, 1|_{\langle x \rangle} \rangle > 0$, for every irreducible character ϕ in S, where $x \in S$ is an element of order 7. Since $5 \in \pi(K)$, by Lemma 2.7, $5 \sim 7$ in $\Gamma(G)$, which is a contradiction

Let $p = 3$ and $\pi_3(S) = \pi((q-1)/2) = \{13\}$. So $p_0^\alpha - 1 = 2.13^\beta$, for some $\beta > 0$, and $p_0 \in \pi_1(G) = \{2,3,5,7\}$, which implies that $q = 27$, by Lemma 2.9. So we have $5 \in \pi(G) \setminus \pi(S)$. Therefore $5 \in \pi(K)$. On the other hand, S has a Frobenius subgroup with Frobenius kernel of order 27 and Frobenius complement of order 13, which implies that $13 \sim 5$, a contradiction.

Let $p = 5$ and $\pi_2(S) = \pi(q) = \{11\}$. So $p_0 = 11$ and $\pi(q^2-1) \subseteq \{2,3,5,7,13,41,61\}$. If $7, 13 \in \pi(S)$, then $3 \mid \alpha$. Hence we get a contradiction, since $19 \in \pi(11^3-1) \subseteq \pi(q^2-1)$. If $41 \in \pi(S)$, then we get a contradiction, similarly. So $7, 13, 41 \in \pi(K) \cup \pi(\overline{G}/S)$ and so we get a contradiction by Lemma 2.2, since $\{7, 13, 41\}$ is an independent set.

Let $p = 5$ and $\pi_3(S) = \pi((q-1)/2) = \{11\}$. So $p_0^\alpha - 1 = 2.11^\beta$, for some $\beta > 0$, and $p_0 \in \pi_1(G) = \{2,3,5,7,13,41,61\}$. By Lemma 2.9, the only solution for the equation is $(p_0, \alpha, \beta) = (3, 5, 2)$. So $41 \notin \pi(S)$ and $41 \notin \pi(\overline{G}/S) \subseteq \{2, \alpha\}$. Therefore $41 \in \pi(K)$. On the other hand, by Lemma 2.4, S contains a Frobenius subgroup with Frobenius kernel of order 3^5 and Frobenius complement of order $(3^5-1)/2$. Therefore by Lemma 2.6, $41 \sim 11$, which is a contradiction.

If $S \cong A_1(q)$, where $4 \mid (q-1)$ or $2 \mid q$, then we get a contradiction similarly.

Case 4. Let $S \cong G_2(q)$, where $q = p_0^\alpha$. Therefore $\pi_2(S) = \pi(q^2 - \epsilon q + 1)$, where $\epsilon = \pm 1$. Since $t(S) = 3$, we have $[3p/4] \leq 3$. Then $p \leq 5$.

Let $p = 3$. Then $r_{2p} = 7$, $\pi(q^2 - \epsilon q + 1) = \{13\}$ and $\pi(q(q^2-1)(q^3-\epsilon)) \subseteq \pi_1(G) = \{2,3,5,7\}$.

Let $p_0 = 2$. Then $\mathrm{ord}_{13}(2) = 12$. Since 13 is the only primitive prime of q^3-1 or q^6-1, then 13 is a primitive prime of $p_0^{3\alpha}-1$ or $p_0^{6\alpha}-1$, so $\alpha = 4$ or $\alpha = 2$.

Therefore $q = 16$ or $q = 4$, respectively. But 13 is not the only primitive prime of $16^3 - 1$ and so $q = 4$. Hence $3 \sim 7 = r_{2p}$ in $\Gamma(S)$, which is a contradiction.

Let $p_0 = 3$. Then $q = p_0 = 3$, since 13 is the only primitive prime of $p_0^{3\alpha} - 1$. Since $5 \notin \pi(S)$ and $|\,\mathrm{Out}(S)| = 2$, it follows that $5 \in \pi(K)$. By [4], S contains a Frobenius subgroup with Frobenius complement of order 7 and Frobenius kernel of order 8 and hence $r_{2p} = 7 \sim 5$ in $\Gamma(G)$, which is impossible. If $p_0 \in \{5, 7\}$, then we get a contradiction similarly.

Let $p = 5$. Then $\pi(q^2 - \epsilon q + 1) = \{11\}$ and so 11 is a primitive prime of $q^3 - 1$ or $q^6 - 1$, which is a contradiction, since $3 \nmid (11 - 1)$.

If $S \cong {}^3D_4(q)$, where q is odd; ${}^2G_2(q)$, where $3 \mid q$; ${}^2F_4(q)$, where $q = 2^{2n+1} > 2$; or $F_4(q)$, then we get a contradiction similarly.

Case 5. Let $S \cong E_8(q)$, where $q = p_0^{\alpha}$. Then $t(S) = 11$. Therefore $[3p/4] \leq 11$. Therefore $p \leq 16$. On the other hand, for each q, $31 \in \pi(S) \subseteq \pi(G)$, which implies that $p \geq 17$ and this is a contradiction.

If $S \cong E_6(q)$ or ${}^2E_6(q)$, then we get a contradiction similarly.

Case 6. Let $S \cong {}^2B_2(q)$, where $q = 2^{2m+1} > 2$. Therefore $\pi_1(S) = \{2\}$, $\pi_2(S) = \pi(q-1)$, $\pi_3(S) = \pi(q + \sqrt{2q} + 1)$ and $\pi_4(S) = \pi(q - \sqrt{2q} + 1)$. Also $t(G) = 4$. Which implies that, $[3p/4] \leq 4$ and hence $p \leq 5$.

Let $p = 3$. Then $\pi_2(G) = \{13\}$. If $\pi(q-1) = \{13\}$, then 13 is a primitive prime of $2^{2m+1} - 1$, which is a contradiction, since $\mathrm{ord}_{13}(2) = 12$. If $\pi(q + \sqrt{2q} + 1) = \{13\}$, then 13 is a primitive prime of $q^4 - 1$, which implies that $2m + 1 = 3$, since $\mathrm{ord}_{13}(2) = 12$. So $S \cong {}^2B_2(8)$ and 2 is not adjacent to $5, 7$ in $\Gamma(S)$. On the other hand, 2 is adjacent to $5, 7$ in $\Gamma(G)$. Therefore either 5 and $7 \in \pi(K)$, which implies that $5 \sim 7$, a contradiction; or $2 \in \pi(K)$, since $\pi(\overline{G}/S) \subseteq \{3\}$. Therefore $G/O_2(G) \cong \mathrm{Aut}({}^2B_2(8))$. If $\pi(q - \sqrt{2q} + 1) = \{13\}$, then similarly we have $2m + 1 = 3$ and we get a contradiction, since $\pi(2^3 - 2^2 + 1) \neq \{13\}$.

Let $p = 5$. Then $\pi_2(G) = \{11\}$. If $\pi(q-1) = \{11\}$, then $\mathrm{ord}_{11}(2) = 2m + 1 = 10$, which is impossible. If $\pi(q - \sqrt{2q} + 1) = \{11\}$, then $\mathrm{ord}_{11}(2) = 4(2m+1) = 10$, which is a contradiction. If $\pi(q + \sqrt{2q} + 1) = \{11\}$, then we get a contradiction similarly.

Case 7. Let $S \cong J_1$. Then $\pi_1(S) = \{2, 3, 5\}$, $\pi_2(S) = \{11\}$, $\pi_3(S) = \{7\}$ and $\pi_4(S) = \{19\}$.

If $\pi_2(S) = \pi((3^p - 1)/2) = \{7\}$, then we get a contradiction, since $6 = \mathrm{ord}_7(3) = p$.

If $\pi_3(S) = \pi((3^p - 1)/2) = \{11\}$, then $p = 5$, which is impossible, since $19 \notin \pi_1(D_6(3))$.

If $\pi_4(S) = \pi((3^p-1)/2) = \{19\}$, then we get a contradiction, since $18 = \mathrm{ord}_{19}(3) = p$.

If S is a sporadic simple group, or $S \cong {}^2A_3(2)$, ${}^2F_4(2)'$, $A_2(2)$, ${}^2A_5(2)$, $A_2(4)$, $E_7(2)$, $E_7(3)$ or ${}^2E_6(2)$, then we get a contradiction, similarly.

Case 8. Let $S \cong C_{p'}(q)$, where $q = 2$ or 3 and p' is a prime. Therefore $\pi_1(S) = \pi(q(q^{p'}+1)\prod_{i=1}^{p'-1}(q^i-1))$ and $\pi_2(S) = \pi((q^{p'}-1)/(2, q-1))$.

Let $q = 2$. By [33, Proposition 2.3], $3 \sim r$ in $\Gamma(S)$, for every $r \in \pi_1(S)$. Therefore $r_{2p} \in \pi(K) \cup \pi(\overline{G}/S)$. Since $\overline{G}/S \le \mathrm{Out}(S)$ and $|\mathrm{Out}(S)| = 1$, we conclude that $r_{2p} \in \pi(K)$. On the other hand, S contains a Frobenius subgroup with Frobenius kernel of order $q^{p'}$ and Frobenius complement of order $q^{p'}-1$, by Lemma 2.4. Therefore $r_{2p} \sim t$, for every $t \in \pi(q^{p'}-1) = \pi((3^p-1)/2)$, which is a contradiction.

Let $q = 3$. Since $\pi((3^{p'}-1)/2) = \pi((3^p-1)/2)$, we have $p' = p$. Hence $S \cong C_p(3)$. If $S < G/K \le \mathrm{Aut}(S)$, then by [26], $s(G/K) = 1$, which is a contradiction. Therefore $S = G/K$. Let $K \ne 1$. Let $k \in \pi(K)$ and $k \ne 3$. We know that $C_p(3)$ contains a Frobenius subgroup with Frobenius kernel of order 3^p and Frobenius complement of order $(3^p-1)/2$. Therefore by Lemma 2.6, $k \sim r_p$, which is a contradiction. Let $k = 3$. We may assume that K is an elementary abelian 3-group. Then S acts unisingularly on K, by [8, Theorem 1.3]. Therefore $3 \sim r_p$, which is impossible. So $K = 1$ and $G \cong C_p(3)$. Using the maximal tori in [33], we can see that, for $p > 5$, $r_{(p-2-\epsilon)/2} \sim r_{(p+4+\epsilon)/2}$ in $\Gamma(D_{p+1}(3))$, but $r_{(p-2-\epsilon)/2} \nsim r_{(p+4+\epsilon)/2}$ in $\Gamma(C_p(3))$, if $p \equiv \epsilon \pmod 4$ and $\epsilon = \pm 1$. Therefore $p \le 5$. If $p = 5$, then $41 \nsim 5$ in $\Gamma(C_5(3))$, a contradiction. For $p = 3$ we know that $\Gamma(D_{p+1}(3)) = \Gamma(C_p(3))$.

If $S \cong B_{p'}(3)$, where p' is a prime, then similarly we get a contradiction for $p > 3$ and we have $\Gamma(D_{p+1}(3)) = \Gamma(B_p(3))$ for $p = 3$.

Also we conclude that S is not isomorphic to $C_n(q)$ and $B_n(q)$, where $n = 2^m \ge 2$; or $D_{p'}(q)$, for every prime p', similarly to the above discussion.

Case 9. Let $S \cong D_{p'+1}(q)$, where $q = 2$ or 3 and p' is a prime. Therefore $\pi_1(S) = \pi(q(q^{p'}+1)\prod_{i=1}^{p'-1}(q^i-1))$ and $\pi_2(S) = \pi((q^{p'}-1)/(2, q-1))$.

Let $q = 2$. By Lemma 2.12, $3 \sim r$ in $\Gamma(S)$, for every $r \in \pi_1(S)$. Therefore $r_{2p} \in \pi(K) \cup \pi(\overline{G}/S)$. Since $\overline{G}/S \le \mathrm{Out}(S)$ and $|\mathrm{Out}(S)| = 2$, we conclude that $r_{2p} \in \pi(K)$. On the other hand, S contains a Frobenius subgroup with Frobenius kernel of order $q^{p'(p'-1)/2}$ and Frobenius complement of order $u_{p'}$, by Lemma 2.4. Therefore $r_{2p} \sim u_{p'}$, which is a contradiction.

Let $q = 3$. Since $\pi((3^{p'}-1)/2) = \pi((3^p-1)/2)$, we have $p' = p$. Hence $S \cong D_{p+1}(3)$. If $S < G/K \le \mathrm{Aut}(S)$, then by [26], $s(G/K) = 1$, which is a contradiction. Therefore $S = G/K$. Let $K \neq 1$. Let $k \in \pi(K)$ and $k \neq 3$. We know that $D_{p+1}(3)$ contains a Frobenius subgroup with Frobenius kernel of order $3^{p(p-1)/2}$ and Frobenius complement of order r_p. Therefore by Lemma 2.6, $k \sim r_p$, which is a contradiction. Let $k = 3$. We may assume that K is an elementary abelian 3-group. Then S acts unisingularly on K, by [8, Theorem 1.3]. Therefore $3 \sim r_p$, which is impossible. So $K = 1$ and $G \cong D_{p+1}(3)$.

Also similarly to these cases we conclude that $S \ncong {}^2D_n(q)$.

Now the proof of this theorem is completed and it follows that $D_{p+1}(3)$ is recognizable by prime graph. □

Theorem 3.3. *Let G be a finite group such that $\Gamma(G) = \Gamma(D_p(3))$, where p is an odd prime. If $p > 3$, then G is quasirecognizable by prime graph. Moreover $D_p(3) \leq G/O_3(G) \leq \mathrm{Aut}(D_p(3))$. If $p = 3$, then either $G \cong {}^2F_4(2)'$; $G \cong {}^2F_4(2)$; $A_1(25) \leq G/O_2(G) \leq \mathrm{Aut}(A_1(25))$ or $D_3(3) \leq G/O_3(G) \leq \mathrm{Aut}(D_3(3))$.*

Proof. Similarly to the proof of Theorem 3.2, there exists a finite nonabelian simple group S such that $S \leq G/K \leq \mathrm{Aut}(S)$ for the maximal normal solvable subgroup K of G, and K is nilpotent. Also similarly to the proof of Theorem 3.2, we can prove that S is not isomorphic to simple exceptional groups of Lie type, alternating groups, sporadic groups, except ${}^2F_4(2)'$. For convenience we omit the proof of these cases. So we only consider finite simple classical groups. Let r_i and u_i be similar to Theorem 3.2.

Case 1. Let $S \cong A_{n-1}(q)$, where $q = p_0^\alpha$, $n \in \{p', p'+1\}$ and p' is an odd prime.

Let $p = 3$. Then $p_0 \in \pi_1(S) \subseteq \{2, 3, 5\}$. Since 13 is the only primitive prime of $p_0^{p'\alpha} - 1$, we have $q = p_0 = 3$ and $p' = 3$. So either $S \cong A_3(3) \cong D_3(3)$ or $S \cong A_2(3)$. If $S \cong A_2(3)$, then $5 \in \pi(K)$ and $\langle \phi|_{\langle x \rangle}, 1|_{\langle x \rangle} \rangle > 0$, for every Brauer character of $A_2(3)$ in characteristic 5, where x is an element of order 3 in $A_2(3)$, by [4] and [13, Theorem 15.13]. Therefore we get a contradiction, since by Lemma 2.7, $3 \sim 5$. Hence $S \cong D_3(3)$. Let $p = 5$. So $\pi_1(S) \subseteq \{2, 3, 5, 7, 13, 41\}$ and 11 is the only primitive prime of $q^{p'} - 1$. So by easy calculation we get a contradiction. Therefore $p > 5$.

We claim that $\pi(q + 1) \subseteq \{2, 3, 5\}$. Let $2 \neq x \in \pi(q + 1)$. By [33, Propositions 2.1, 3.1], we have $x \sim t$, for every $t \in \pi_1(S)$. Let $A = \{r_{2(p-1)}, r_{2(p-2)}, r_{p-2}\}$ which is an independent set of $\Gamma(G)$. By Lemma 2.2, we have $|A \setminus \pi(S)| \leq 1$. Let $r, s \in A \cap \pi_1(S)$, for $r \neq s$. Then $x \sim r$ and $x \sim s$. If $r = r_{2(p-1)}$ and $s = r_{2(p-2)}$, then using the orders of maximal tori of $D_p(3)$, we see that there is only one torus T of $D_p(3)$ such that $r_{2(p-1)} \mid |T|$. On the other hand, there is a maximal torus T' of $D_p(3)$ such that $r_{2(p-2)} \mid |T'|$ and so $x \mid (|T|, |T'|)$, which implies that $x \in \{2,5\}$. Similarly for every $r,s \in A$, we have $x \in \{2,3,5\}$. So $\pi(q+1) \subseteq \{2,3,5\}$.

Since $p_0 \in \pi(S)$, it follows that $p_0 \in \pi(G)$. Let p_0 be a primitive prime of 3^t-1, where $t \leq 2(p-1)$. We claim that $t \notin \{2,3,4,6,8\}$. Otherwise, for example if $t=4$, then $q=5^\alpha$ and $\pi(q+1) \subseteq \{2,3,5\}$, which implies that $q=5$. So $S \cong A_{n-1}(5)$. If $n \geq 5$, then $31 \in \pi(5^3-1) \subseteq \pi_1(S)$ and by [33], $t(31,S) \leq 4$. On the other hand, by the orders of maximal tori in $\Gamma(G)$, $t(31,G) \geq 7$, since 31 is a primitive prime of $3^{30}-1$ and so $p \geq 17$. Therefore by Lemma 2.2, we get a contradiction. So $S \cong A_2(5)$ or $S \cong A_3(5)$ and hence $\pi((3^p-1)/2) = \pi((5^3-1)/4)$, which is a contradiction.

By [33], we have $\rho(p_0,S) = \{p_0, u_{n-1}, u_n\}$. Let $t \geq 5$ be odd. Then $\{p_0, r_p, r_{2(p-1)}, r_{2(p-2)}, r_{2(p-3)}\} \subseteq \rho(p_0,G)$. So we get a contradiction by Lemma 2.2.

Let $t \geq 10$ be even and $t/2 \equiv \epsilon \pmod 2$, where $\epsilon \in \{0,1\}$. Then $\{p_0, r_p, r_{2(p-2+\epsilon)}, r_{p-2}, r_{p-4}\} \subseteq \rho(p_0,G)$, and so we get a contradiction by Lemma 2.2.

Therefore $p_0 = 3$ and so the set of primitive primes of $3^{\alpha p'}-1$ is equal to the set of primitive primes of $(3^p-1)/2$, which implies that $\alpha p' = p$. Then $\alpha = 1$ and $p' = p$. Therefore $\{r_{2(p-1)}, r_{2(p-2)}\} \subseteq \pi_1(G) \setminus \pi_1(S)$ and we get a contradiction, since $\pi(\overline{G}/S) \subseteq \{2\}$, K is nilpotent and $r_{2(p-1)} \nsim r_{2(p-2)}$ in $\Gamma(G)$.

If $S \cong {}^2A_{n-1}(q)$, then we get a contradiction, similarly.

Case 2. Let $S \cong A_1(q)$, where $q = p_0^\alpha$. Since $t(S) = 3$, it follows that $t(G) = [(3p+1)/4] \leq 4$, and so $p \in \{3,5\}$.

Let $p=3$. Then $\pi_2(G) = \{13\}$, $\pi_1(G) = \{2,3,5\}$ and $3 \sim 2 \sim 5$ and $3 \nsim 5$ in $\Gamma(G)$.

If $\pi(q) = \{13\}$, then $\pi(q^2-1) \subseteq \{2,3,5\}$, which is a contradiction.

Let $\pi((q+\epsilon)/(2,q-1)) = \{13\}$, where $\epsilon = \pm 1$ and $4 \nmid (q+\epsilon)$. Then $\pi(p_0(q-\epsilon)) \subseteq \{2,3,5\}$. By easy calculation we can see that $q=25$. So $S \cong A_1(25)$. Let $3, 5$ or $13 \in \pi(K)$. Then by [14] and Lemma 2.7, we have $3 \sim 13$, $5 \sim 13$ or $2 \sim 13$, respectively which is a contradiction. So K is a 2-group. Note that $\overline{G}/S \leq \mathrm{Out}(A_1(25))$. If a diagonal automorphism is a generator of $\overline{G}/S$, then $2 \sim 13$ in $\overline{G}$, a contradiction. If a diagonal-field automorphism is a generator of $\overline{G}/S$, then $\Gamma(A_1(25)) = \Gamma(\overline{G})$ and if a field automorphism is a generator of $\overline{G}/S$, then $\Gamma(D_3(3)) = \Gamma(\overline{G})$.

Let $p=5$. Therefore $\pi_1(G) = \{2,3,5,7,13,41\}$, and $\pi_2(G) = \{11\}$.

Let $\pi(q) = \{11\}$. Then $q = 11^\alpha$ and $\pi(q^2-1) \subseteq \pi_1(G)$. By Lemma 2.3 and easy calculation we conclude that $q = 11$. Therefore $7, 13, 41 \notin \pi(S)$, which implies that $7, 13, 41 \in \pi(K)$. So $13 \sim 41$, since K is nilpotent, which is a contradiction.

Let $\pi((q+\epsilon)/(2,q-1)) = \{11\}$, where $\epsilon = \pm 1$ and $4 \nmid (q+\epsilon)$. Then

$\pi(p_0(q-\epsilon)) \subseteq \{2,3,5,7,13,41\}$. By easy calculation and Lemma 2.3 we get a contradiction.

Case 3. Let $S \cong {}^2F_4(2)'$. Then $\{13\} = \pi_2(S) = \pi((3^p-1)/2) = \pi_2(G)$ and $\pi_1(S) = \{2,3,5\} \subseteq \pi_1(G)$. So $p = 3$. We know that $\Gamma(S) = \Gamma(D_3(3))$. If 2, 3 or $5 \in \pi(K)$, then by [14] and Lemma 2.7, we can see that $2 \sim 13$, $3 \sim 13$ or $5 \sim 13$, a contradiction. So $K = 1$. Note that $|\operatorname{Out}(S)| = 2$ and $G/S \leq \operatorname{Out}(S)$. If $|G/S| = 2$, then $G =^2 F_4(2)$, by [26] and $\Gamma(G) = \Gamma(D_3(3))$. So $G = {}^2F_4(2)$ or $G = {}^2F_4(2)'$.

Case 4. Let $S \cong {}^2D_n(q)$, where $n = 2^m \geq 4$ and $q = p_0^\alpha$.

Let $p = 3$. Then $p_0 \in \pi_1(S) \subseteq \{2,3,5\}$ and 13 is the only primitive prime of $q^{2n}-1$. By easy calculation, we have $q = 5$ and $n = 2$, which is a contradiction by assumption. If $p = 5$, then by easy calculation we get a contradiction. Therefore $p > 5$.

By [33, Proposition 3.1] and Lemma 2.12, for every $2 \neq t \in \pi(q^2-1)$ and $s \in \pi_1(S)$, we have $t \sim s$. Similarly to Case 1, using Lemma 2.2 and considering $A = \{r_{2(p-1)}, r_{2(p-2)}, r_{p-2}\}$, we have $\pi(q^2-1) \subseteq \{2,3,5\}$.

Since $p_0 \in \pi(S)$, it follows that $p_0 \in \pi(G)$. Let p_0 be a primitive prime of 3^t-1, where $t \leq 2(p-1)$. We claim that $t \notin \{2,3,4,5,6,8,10,12\}$. Otherwise, for example if $t = 3$, then $q = 13^\alpha$ and $\pi(q^2-1) \subseteq \{2,3,5\}$, which is a contradiction.

By [33], we have $\rho(p_0, S) = \{p_0, u_{n-1}, u_{2(n-1)}, u_{2n}\}$. Let $t \geq 7$ be odd. So $p > 7$ and $\{p_0, r_p, r_{2(p-1)}, r_{2(p-2)}, r_{2(p-3)}, r_{2(p-4)}\} \subseteq \rho(p_0, G)$. So we get a contradiction by Lemma 2.2.

Let $t \geq 14$ be even and $t/2 \equiv \epsilon \pmod 2$, where $\epsilon \in \{0,1\}$. So $p > 7$ and $\{p_0, r_p, r_{2(p-2+\epsilon)}, r_{p-2}, r_{2(p-4+\epsilon)}, r_{p-4}\} \subseteq \rho(p_0, G)$. Therefore we get a contradiction by Lemma 2.2.

Therefore $p_0 = 3$ and so the set of primitive primes of $3^{2n\alpha}-1$ is equal to the set of primitive primes of $(3^p-1)/2$, which implies that $2n\alpha = p$ and we get a contradiction.

If $S \cong B_n(q)$, where $n = 2^m$; $C_n(q)$, where $n = 2^m$; ${}^2D_n(q)$, where $n = 2^m+1$, or ${}^2D_{p'}(3)$, where $p' \neq 2^m+1$ is prime, then we get a contradiction similarly.

Case 5. Let $S \cong B_{p'}(q)$ or $C_{p'}(q)$. Then similar to the proof of Theorem 3.2, we have $p = p'$ and $q = 3$. If $S \cong C_p(3)$ or $B_p(3)$, then $(3^p+1) \mid |S|$ and so there exists a primitive prime u of $3^{2p}-1$ such that $u \in \pi(S) \setminus \pi(G)$, which is impossible.

Case 6. Let $S \cong D_n(q)$. Then similar to the proof of Theorem 3.2, we have $n = p$ and $q = 3$. Let $K \neq 1$. Let $k \in \pi(K)$ and $k \neq 3$. We know that $D_p(3)$ contains a Frobenius subgroup with Frobenius kernel of order $3^{p(p-1)/2}$

and Frobenius complement of order r_p. Therefore by Lemma 2.6, $k \sim r_p$ in $\Gamma(G)$, which is a contradiction. So K is a 3-group. □

Remark 3.4. W. Shi and J. Bi in [30] put forward the following conjecture:

Conjecture. Let G be a group and M be a finite simple group. Then $G \cong M$ if and only if

(i) $|G| = |M|$, and,

(ii) $\pi_e(G) = \pi_e(M)$.

As a corollary of the main theorem of this paper we prove a generalization of Shi–Bi conjecture for $D_n(3)$, where $n \in \{p, p+1\}$.

Corollary 3.5. *Let G be a finite group satisfying $|G| = |D_n(3)|$ and $\Gamma(G) = \Gamma(D_n(3))$, where $n \in \{p, p+1\}$, for some prime number p. Then $G \cong D_n(3)$.*

Proof. If $\Gamma(G) = \Gamma(D_n(3))$, where $n \in \{p, p+1\}$ and $p > 3$, then by Theorems 3.2 and 3.3, it follows that G has a composition factor isomorphic to $S \cong D_n(3)$ and since $|G| = |D_n(3)|$, we conclude that $S \cong G$. If $\Gamma(G) = \Gamma(D_4(3))$, then by Theorem 3.2, we have either $G \cong D_4(3)$; $C_3(3)$, $B_3(3)$ or $G/O_2(G) \cong \mathrm{Aut}(^2B_2(8))$. Since $|B_3(3)| = |C_3(3)| \neq |D_4(3)|$ and $3 \| |G|$, where $G/O_2(G) \cong \mathrm{Aut}(^2B_2(8))$, then $G \cong D_4(3)$. If $\Gamma(G) = \Gamma(D_3(3))$, then by Theorem 3.3, G has a composition factor isomorphic to $G \cong D_3(3)$; $A_1(25)$; $^2F_4(2)'$ or $^2F_4(2)$. On the other hand 25 divides the order of $A_1(25)$, $^2F_4(2)'$ and $^2F_4(2)$ and 25 does not divide the order of $D_3(3)$, which implies that $G \cong D_3(3)$. □

References

[1] Z. Akhlaghi, M. Khatami and B. Khosravi, Quasirecognition by prime graph of the simple group $^2F_4(q)$, *Acta Math. Hungar.* **122**(4) (2009), 387–397.

[2] Z. Akhlaghi, B. Khosravi and M. Khatami, Characterization by prime graph of $PGL(2, p^k)$ where p and $k > 1$ are odd, *Int. J. Algebra and Computation* **20**(7) (2010), 847–873.

[3] A. Babai, B. Khosravi and N. Hasani, Quasirecognition by prime graph of $^2D_p(3)$ where $p = 2^n + 1 \geq 5$ is a prime, *Bull. Malaysian Math. Sci. Soc.* (2) **32**(3) (2009), 343–350.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, Atlas of Finite Groups, *Oxford University Press*, *Oxford*, 1985.

[5] P. Crescenzo, A diophantine equation which arises in the theory of finite groups, *Advances in Math.* **17**(1) (1975), 25–29.

[6] M. Foroudi Ghasemabadi and A. Iranmanesh, Quasirecognition by the prime graph of the group $C_n(2)$, where $n \neq 3$ is odd, *Bull. Malaysian Math. Sci. Soc.* (2) (*to appear*).

[7] D. Gorenstein, Finite Groups, *Harper and Row*, *New York*, 1968.

[8] R. M. Guralnick and P. H. Tiep, Finite simple unisingular groups of Lie type, *J. Group Theory* **6** (2003), 271–310.

[9] M. Hagie, The prime graph of a sporadic simple group, *Comm. Algebra* **31**(9) (2003), 4405–4424.

[10] H. He and W. Shi, Recognition of some finite simple groups of type $D_n(q)$ by spectrum, *Internat. J. Algebra Comput.* **19**(5) (2009), 681–698.

[11] B. Huppert, Endliche Gruppen I, *Springer-Verlag*, 1967.

[12] K. Ireland and M. Rosen, A Classical Introduction to Modern Number Theory, Second edition, *Springer-Verlag, New York*, 1990.

[13] I. M. Issacs, Character Theory of Finite Groups, *Academic Press*, *New York*, 1976.

[14] C. Jansen, K. Lux, R. Parker and R. Wilson, An Atlas of Brauer Characters, *Clarendon press*, *Oxford*, 1995.

[15] M. Khatami, B. Khosravi and Z. Akhlaghi, NCF-distinguishability by prime graph of $PGL(2,p)$, where p is a prime, *Rocky Mountain J. Math.* (*to appear*).

[16] A. Khosravi and B. Khosravi, Quasirecognition by prime graph of the simple group ${}^2G_2(q)$, *Sib. Math. J.* **48**(3) (2007), 570–577.

[17] B. Khosravi, n-Recognition by prime graph of the simple group $PSL(2,q)$, *J. Algebra Appl.* **7**(6) (2008), 735–748.

[18] B. Khosravi, Quasirecognition by prime graph of $L_{10}(2)$, *Siberian Math. J.* **50**(2) (2009), 355–359.

[19] B. Khosravi, Some characterizations of $L_9(2)$ related to its prime graph, *Publ. Math. Debrecen* **75** (2009), 375–385.

[20] B. Khosravi and A. Babai, Quasirecognition by prime graph of $F_4(q)$, where $q = 2^n > 2$, *Monatsh. Math.*, DOI: 10.1007//s00605-009-0155-6 (*to appear*).

[21] B. Khosravi, B. Khosravi and B. Khosravi, Groups with the same prime graph as a CIT simple group, *Houston J. Math.* **33**(4) (2007), 967–977.

[22] B. Khosravi, B. Khosravi and B. Khosravi, On the prime graph of $PSL(2,p)$ where $p > 3$ is a prime number, *Acta Math. Hungar.* **116**(4) (2007), 295–307.

[23] B. Khosravi, B. Khosravi and B. Khosravi, 2-Recognizability of $PSL(2,p^2)$ by the prime graph, *Sib. Math. J.* **49**(4) (2008), 749–757.

[24] B. Khosravi, B. Khosravi and B. Khosravi, A characterization of the finite simple group $L_{16}(2)$ by its prime graph, *Manuscripta Math.* **126** (2008), 49–58.

[25] B. Khosravi and H. Moradi, Quasirecognition by prime graph of finite simple groups $L_n(2)$ and $U_n(2)$, *Acta Math. Hungar.* (*to appear*).

[26] M. S. Lucido, Prime graph components of finite almost simple groups, *Rend. Sem. Mat. Univ. Padova* **102** (1999), 1–14.

[27] M. S. Lucido and A. R. Moghaddamfar, Groups in which all the connected components of their prime graphs are complete, *J. Group Theory* **7**(3) (2004), 373–384.

[28] V. D. Mazurov, Characterizations of finite groups by sets of their element orders, *Algebra and Logic* **36**(1) (1997), 23–32.

[29] C. E. Praeger and W. Shi, A characterization of some alternating and symmetric groups, *Comm. Algebra* **22**(5) (1994), 1507–1530.

[30] W. Shi and J. Bi, A characteristic property for each finite projective special linear group, *Lecture Notes in Math.* **1456** (1990), 171–180.

[31] A. V. Vasiliev, On connection between the structure of a finite group and properties of its prime graph, *Sibirsk. Mat. Zh.* **46**(3) (2005), 511–522.

[32] A. V. Vasiliev and M. A. Grechkoseeva, On recognition by spectrum of finite simple linear groups over fields of characteristic 2, *Sib. Math. J.* **46**(4) (2005), 593–600.

[33] A. V. Vasiliev and E. P. Vdovin, An adjacency criterion for the prime graph of a finite simple group, *Algebra and Logic* **44**(6) (2005), 381–406.

[34] J. S. Williams, Prime graph components of finite groups, *J. Algebra* **69**(2) (1981), 487–513.

[35] A. V. Zavarnitsin, Recognition of finite groups by the prime graph, *Algebra and Logic* **43**(4) (2006), 220–231.

[36] K. Zsigmondy, Zur theorie der potenzreste, *Monatsh. Math. Phys.* **3**(1) (1892), 265–284.

B. KHOSRAVI
DEPARTMENT OF PURE MATHEMATICS
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
AMIRKABIR UNIVERSITY OF TECHNOLOGY (TEHRAN POLYTECHNIC)
424, HAFEZ AVE., TEHRAN 15914
IRAN

E-mail: khosravibbb@yahoo.com

Z. AKHLAGHI
DEPARTMENT OF PURE MATHEMATICS
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
AMIRKABIR UNIVERSITY OF TECHNOLOGY (TEHRAN POLYTECHNIC)
424, HAFEZ AVE., TEHRAN 15914
IRAN

E-mail: zeinab_akhlaghi@yahoo.com

M. KHATAMI
DEPARTMENT OF PURE MATHEMATICS
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
AMIRKABIR UNIVERSITY OF TECHNOLOGY (TEHRAN POLYTECHNIC)
424, HAFEZ AVE., TEHRAN 15914
IRAN

E-mail: maryam_khatami81@yahoo.com

(Received February 23, 2010; revised September 16, 2010)