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Classification of minimal Lorentz surfaces in indefinite space forms with arbitrary codimension and arbitrary index

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Abstract. Since J. L. LAGRANGE initiated in [18] the study of minimal surfaces of Euclidean 3-space in 1760, minimal surfaces in real space forms have been studied extensively by many mathematicians during the last two and half centuries. In contrast, so far very few results on minimal Lorentz surfaces in indefinite space forms are known. Hence, in this paper we investigate minimal Lorentz surfaces in arbitrary indefinite space forms. As a consequence, we obtain several classification results for minimal Lorentz surfaces in indefinite space forms. In particular, we completely classify all minimal Lorentz surfaces in a pseudo-Euclidean space $\mathbb{E}^m_s$ with arbitrary dimension m and arbitrary index s.

1. Introduction

Let $\mathbb{E}^m_s$ denote the pseudo-Euclidean m-space with the canonical metric of index s given by

$$g_0 = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{m} dx_j^2, \tag{1.1}$$

where $(x_1, \ldots, x_m)$ is a rectangular coordinate system of $\mathbb{E}^m_s$. The *light cone* $\mathcal{LC}$ of $\mathbb{E}^{m+1}_s$ is defined by $\mathcal{LC} = \{\mathbf{x} \in \mathbb{E}^{m+1}_s : \langle \mathbf{x}, \mathbf{x} \rangle = 0\}$.

We put

$$S^k_s(c) = \{x \in \mathbb{E}^{k+1}_s \mid \langle x, x \rangle = c^{-1} > 0\}, \tag{1.2}$$

$$H^k_s(-c) = \{x \in \mathbb{E}^{k+1}_{s+1} \mid \langle x, x \rangle = -c^{-1} < 0\}, \tag{1.3}$$

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where $\langle\ ,\ \rangle$ denotes the indefinite inner product on $\mathbb{E}^{k+1}_t$. The $S^k_s(c)$ and $H^k_s(-c)$ are complete pseudo-Riemannian manifolds with index s and of constant curvature c and $-c$, respectively.

The $S^k_s(c)$ and $H^k_s(-c)$ are called *pseudo k-sphere* and *pseudo hyperbolic k-space*, respectively. The pseudo-Riemannian manifolds $\mathbb{E}^k_s$, S^k_s and H^k_s are known as the *indefinite space forms*. In particular, $\mathbb{E}^k_1$, S^k_1 and H^k_1 are called Minkowski, de Sitter and anti-de Sitter spacetimes, which play very important roles in relativity theory.

The history of minimal surfaces goes back to J. L. Lagrange (1736–1813) who initiated in 1760 the study of minimal surfaces in Euclidean 3-space (see [18]). Since then minimal surfaces have attracted many mathematician. In particular, minimal surfaces in real space forms have been studied very extensively during the last two and half centuries (see, [4, pages 207–249] and [21], [23] for details).

In [24], [25], L. Verstraelen and M. Pieters studied some families of Lorentz surfaces in 4-dimensional indefinite space forms with index 2. Recently, parallel Lorentz surfaces in indefinite space forms with arbitrary codimension and arbitrary index were completely classified in a series of articles [8]–[14] (see also [1], [15], [16], [19]). Moreover, Lorentz surfaces with parallel mean curvature vector in an arbitrary pseudo-Euclidean space were classified in [7] (see also [17]). Further, minimal Lorentz surfaces in Lorentzian complex space forms $\tilde{M}^2_1(c)$ with complex index one were investigated in [5], [6].

In this paper, we study minimal Lorentz surfaces in indefinite space forms $R^m_s(c)$ with arbitrary codimension and arbitrary index s. In particular, we completely classify minimal Lorentz surfaces in an arbitrary pseudo-Euclidean space in Section 4. In Section 5, we classify minimal Lorentz surfaces of constant curvature one in an arbitrary pseudo m-sphere $S^m_s(1)$. The classification of minimal Lorentz surfaces of constant curvature -1 in a pseudo-hyperbolic m-space $H^m_s(-1)$ are obtained in Section 6. In the last two sections, we provide many explicit examples of minimal Lorentz surfaces in $S^m_s(1)$ and in $H^m_s(-1)$.

2. Basics formulas, equations and definitions

Let $R^m_s(c)$ be an m-dimensional indefinite space form of constant sectional curvature c and with index s. The curvature tensor of $R^m_s(c)$ is given by

$$\tilde{R}(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\}. \tag{2.1}$$

Let $\psi : M^2_1 \to R^m_s(c)$ be an isometric immersion of a Lorentz surface M^2_1 into $R^m_s(c)$. Denote by ∇ and $\tilde{\nabla}$ the Levi–Civita connections on M^2_1 and $\tilde{R}^m_s(c)$, respectively. Let X, Y be vector fields tangent to M_1^2 and ξ normal to M_1^2 in $R_s^m(c)$. The formulas of Gauss and Weingarten are given by (cf. [2], [3], [22]):

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.2}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi. \tag{2.3}$$

These formulas define h, A and D, which are called the second fundamental form, the shape operator and the normal connection, respectively.

For each normal vector $\xi \in T_x^\perp M_1^2$, the shape operator A_ξ at ξ is a symmetric endomorphism of the tangent space $T_x M_1^2$, $x \in M_1^2$. The shape operator and the second fundamental form are related by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle . \tag{2.4}$$

The mean curvature vector H of M_1^2 in $R_s^m(c)$ is defined by

$$H = \frac{1}{2} \operatorname{trace} h. \tag{2.5}$$

A Lorentz surface in an indefinite space form is called *totally geodesic* if its second fundamental form vanishes identically. It is called *minimal* if its mean curvature vector vanishes identically.

The equations of Gauss, Codazzi and Ricci are given respectively by

$$R(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} + A_{h(Y,Z)}X - A_{h(X,Z)}Y, \tag{2.6}$$

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z), \tag{2.7}$$

$$\left\langle R^D(X,Y)\xi, \eta \right\rangle = \langle [A_\xi, A_\eta]X, Y \rangle , \tag{2.8}$$

for vector fields X, Y, Z tangent to M_1^2, ξ normal to M_1^2, where $\overline{\nabla} h$ is defined by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2.9}$$

and R^D is the curvature tensor associated to the normal connection D, i.e.,

$$R^D(X,Y)\xi = D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]}\xi. \tag{2.10}$$

A vector v in $R_s^m(c)$ is called *spacelike* (respectively, *timelike*, or *light-like*) if $\langle v,v\rangle > 0$ (respectively, $\langle v,v\rangle < 0$, or $\langle v,v\rangle = 0$ and $v \neq 0$). A curve $z(x)$ in $R_s^m(c)$ is called spacelike (respectively, timelike or null) if its velocity vector $z'(x)$ is spacelike (respectively, timelike or lightlike) at each point.

3. A special coordinate system on a Lorentz surface

Let M_1^2 be a Lorentz surface. We may choose a local coordinate system $\{x, y\}$ on M_1^2 such that the metric tensor is given by

$$g = -E^2(x,y)(dx \otimes dy + dy \otimes dx) \tag{3.1}$$

for some positive function E.

The Levi–Civita connection of g satisfies

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \frac{2E_x}{E}\frac{\partial}{\partial x}, \qquad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0, \qquad \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \frac{2E_y}{E}\frac{\partial}{\partial y} \tag{3.2}$$

and the Gaussian curvature K is given by

$$K = \frac{2EE_{xy} - 2E_xE_y}{E^4}. \tag{3.3}$$

If we put

$$e_1 = \frac{1}{E}\frac{\partial}{\partial x}, \qquad e_2 = \frac{1}{E}\frac{\partial}{\partial y}, \tag{3.4}$$

then $\{e_1, e_2\}$ forms a pseudo-orthonormal frame satisfying

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0, \qquad \langle e_1, e_2\rangle = -1. \tag{3.5}$$

We define the connection 1-form ω by the following equations:

$$\nabla_X e_1 = \omega(X)e_1, \qquad \nabla_X e_2 = -\omega(X)e_2. \tag{3.6}$$

From (3.2) and (3.4) we fin

$$\nabla_{e_1} e_1 = \frac{E_x}{E^2}e_1, \qquad \nabla_{e_2} e_1 = -\frac{E_y}{E^2}e_1, \tag{3.7}$$

$$\nabla_{e_1} e_2 = -\frac{E_x}{E^2}e_2, \qquad \nabla_{e_2} e_2 = \frac{E_y}{E^2}e_2. \tag{3.8}$$

By comparing (3.6) and (3.8), we get

$$\o(e_1) = \frac{E_x}{E^2}, \qquad \o(e_2) = -\frac{E_y}{E^2}. \tag{3.9}$$

Let $\psi : M_1^2 \to R_s^m(c)$ be an isometric immersion of M_1^2 into $R_s^m(c)$. Then it follows from (2.5) and (3.5) that the mean curvature vector of M_1^2 is given by

$$H = -h(e_1, e_2). \tag{3.10}$$

Therefore, M_1^2 is a minimal surface of $R_s^m(c)$ if and only if $h(e_1, e_2) = 0$ holds identically.

4. Minimal Lorentz surfaces in $\mathbb{E}^m_s$

In this section, we completely classify minimal Lorentz surface in an arbitrary pseudo-Euclidean m-space $\mathbb{E}^m_s$. More precisely, we prove the following.

Theorem 4.1. *A Lorentz surface in a pseudo-Euclidean m-space $\mathbb{E}^m_s$ is minimal if and only if locally the immersion takes the form $L(x,y) = z(x) + w(y)$, where z and w are null curves satisfying $\langle z'(x), w'(y)\rangle \neq 0$.*

PROOF. Let $L : M^2_1 \to \mathbb{E}^m_s$ be an isometric immersion of a Lorentz surface M^2_1 into a pseudo-Euclidean m-space $\mathbb{E}^m_s$ with index $s \geq 1$. We choose a local coordinate system $\{x, y\}$ on M^2_1 satisfying

$$g = -E^2(x,y)(dx \otimes dy + dy \otimes dx) \tag{4.1}$$

Then we have (3.2)–(3.10).

If M^2_1 is a minimal surface, it follows from (3.10) that $h(e_1, e_2) = 0$ holds. Hence, we may put

$$h(e_1, e_1) = \xi, \qquad h(e_1, e_2) = 0, \qquad h(e_2, e_2) = \eta \tag{4.2}$$

for some normal vector fields ξ, η. After applying (2.2), (3.2), (3.4), and (4.2), we obtain

$$L_{xx} = \frac{2E_x}{E} L_x + E^2 \xi, \qquad L_{xy} = 0, \qquad L_{yy} = \frac{2E_y}{E} L_y + E^2 \eta. \tag{4.3}$$

After solving the second equation in (4.3), we find

$$L(x,y) = z(x) + w(y) \tag{4.4}$$

for some vector-valued functions $z(x)$, $w(y)$. Thus, by applying (3.1) and (4.4), we obtain $\langle z', z'\rangle = \langle w', w'\rangle = 0$, and $\langle z', w'\rangle = -E^2$. Therefore, z and w are null curves satisfying $\langle z', w'\rangle \neq 0$.

Conversely, if $L : M^2_1 \to \mathbb{E}^m_s$ is an immersion of a Lorentz surface M^2_1 into $\mathbb{E}^4_s$ such that $L = z(x) + w(y)$ for some null curves z, w satisfying $\langle z', w'\rangle \neq 0$, then we obtain $\langle L_x, L_x\rangle = \langle L_y, L_y\rangle = 0$, $\langle L_x, L_y\rangle \neq 0$, and $L_{xy} = 0$. Thus, M^2_1 is surface with induced metric given by $g = F(x,y)(dx \otimes dy + dy \otimes dx)$ for some nonzero function F. Moreover, it follows from (3.10) and $L_{xy} = 0$ that L is a minimal immersion. □

Remark 4.1. Flat minimal Lorentz surfaces in the Lorentzian complex plane $\mathbf{C}^2_1$ have been completely classified in [5]. Moreover, if $m = 3$, this theorem is due to [20, Theorem 3.5].

In particular, if M_1^2 is a flat Lorentz surface, we have the following.

Corollary 4.1. *A flat Lorentz surface in a pseudo-Euclidean m-space $\mathbb{E}_s^m$ is minimal if and only if locally the immersion takes the form*

$$L(x,y) = z(x) + w(y), \tag{4.5}$$

where z and w are null curves satisfying $\langle z', w'\rangle = \text{constant} \neq 0$.

Proof. Let $L : M_1^2 \to \mathbb{E}_s^m$ be an isometric immersion of a flat Lorentz surface M_1^2 into a pseudo-Euclidean m-space $\mathbb{E}_s^m$. Then we may choose a local coordinate system $\{x, y\}$ on M_1^2 satisfying

$$g = -(dx \otimes dy + dy \otimes dx) \tag{4.6}$$

Then we find from (4.3) that the immersion L satisfies

$$L_{xx} = \xi, \quad L_{xy} = 0, \quad L_{yy} = \eta \tag{4.7}$$

for some normal vector fields ξ, η. After solving the second equation in (4.7) we find

$$L(x,y) = z(x) + w(y) \tag{4.8}$$

for some vector functions z, w. Thus, by applying (4.6), we find $\langle z', z'\rangle = \langle w', w'\rangle = 0$ and $\langle z', w'\rangle = -1$. Consequently, z and w are null curves satisfying $\langle z', w'\rangle = -1$.

Conversely, consider a map L defined by $L(x,y) = z(x) + w(y)$, where z and w are null curves satisfying $\langle z', w'\rangle = \text{constant} \neq 0$. Then we have

$$\langle L_x, L_x\rangle = \langle L_y, L_y\rangle = 0, \quad \langle L_x, L_y\rangle = \text{constant} \neq 0.$$

Thus, with respect to the induced metric, (4.5) defines an isometric immersion of a flat Lorentz surface M_1^2 into $\mathbb{E}_s^m$. The remaining follows from Theorem 4.1. □

5. Minimal Lorentz surfaces in $S_s^m(1)$

Let $\psi : M_1^2 \to S_s^m(1)$ be an isometric immersion of a Lorentz surface into $S_s^m(1)$. Denote by $L = \iota \circ \psi : M_1^2 \to \mathbb{E}_s^{m+1}$ the composition of ψ and the inclusion $\iota : S_s^m(1) \subset \mathbb{E}_s^{m+1}$ via (1.2).

Obviously, every totally geodesic Lorentz surface in an indefinite space form $R_s^m(c)$ is of constant curvature c. A natural question is the following:

Question. *Besides totally geodesic ones how many minimal Lorentz surfaces of constant curvature c in $R^m_s(c)$ are there?*

Theorem 5.1 of [5] provides the answer to this basic question for $c = 0$.

In this section, we give an answer to this question for $c > 0$. More precisely, we classify all minimal Lorentz surfaces of constant curvature one in the pseudo-sphere $S^m_s(1)$ with arbitrary m and arbitrary index s.

Theorem 5.1. *Let M^2_1 be a Lorentz surface of constant curvature one. Then an isometric immersion $\psi : M^2_1 \to S^m_s(1)$ is minimal if and only if one of the following three cases occurs:*

(a) *M^2_1 is an open portion of a totally geodesic $S^2_1(1) \subset S^m_s(1)$.*

(b) *The immersion $L = \iota \circ \psi : M^2_1 \to S^m_s(1) \subset \mathbb{E}^{m+1}_s$ is locally given by*

$$L(x, y) = \frac{z(x)}{x+y} - \frac{z'(x)}{2}, \tag{5.1}$$

where $z(x)$ is a spacelike curve with constant speed 2 lying in the light cone $\mathcal{LC}$ satisfying $\langle z'', z''\rangle = 0$ and $z''' \neq 0$.

(c) *The immersion $L = \iota \circ \psi : M^2_1 \to S^m_s(1) \subset \mathbb{E}^{m+1}_s$ is locally given by*

$$L(x, y) = \frac{z(x)+w(y)}{x+y} - \frac{z'(x)+w'(y)}{2}, \tag{5.2}$$

where z and w are curves in $\mathbb{E}^{m+1}_s$ satisfying

$$\left\langle \frac{z(x)+w(y)}{x+y} - \frac{z'(x)+w'(y)}{2}, \frac{z(x)+w(y)}{x+y} - \frac{z'(x)+w'(y)}{2} \right\rangle = 1, \tag{c.1}$$

$$2\langle z+w, z'''\rangle = (x+y)\langle z'+w', z'''\rangle, \tag{c.2}$$

$$2\langle z+w, w'''\rangle = (x+y)\langle z'+w', w'''\rangle. \tag{c.3}$$

Proof. Assume that $\psi : M^2_1 \to S^m_s(1)$ is an isometric immersion of a Lorentz surface M^2_1 of constant curvature one into $S^m_s(1)$. If M^2_1 is totally geodesic in $S^m_s(1)$, we obtain case (a). Hence, let us assume that M^2_1 is non-totally geodesic in $S^m_s(1)$.

Since M^2_1 is of constant curvature one, we may choose local coordinates $\{x, y\}$ such that the metric tensor is given by

$$g = \frac{-2}{(x+y)^2}(dx \otimes dy + dy \otimes dx). \tag{5.3}$$

Hence the Levi–Civita connection satisfies

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}=\frac{-2}{x+y}\frac{\partial}{\partial x},\qquad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}=0,\qquad \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}=\frac{-2}{x+y}\frac{\partial}{\partial y}. \tag{5.4}$$

Let us put

$$\frac{\partial}{\partial x}=\frac{\sqrt{2}e_1}{x+y},\qquad \frac{\partial}{\partial y}=\frac{\sqrt{2}e_2}{x+y}. \tag{5.5}$$

Then we get

$$\langle e_1,e_1\rangle=\langle e_2,e_2\rangle=0,\qquad \langle e_1,e_2\rangle=-1. \tag{5.6}$$

Because M_1^2 is minimal in $S_s^m(1)$, it follows from (3.10) and (5.3) that $h(e_1,e_2)$=0. Hence, we may put

$$h(e_1,e_1)=\xi,\quad h(e_1,e_2)=0,\quad h(e_2,e_2)=\eta \tag{5.7}$$

for some normal vector fields ξ, η. Without loss of generality, we may assume $\xi\neq 0$. Since $K=1$, it follows from the equation (2.6) of Gauss and (5.3) that $\langle\xi,\eta\rangle=0$.

Case (i): $\eta=0$. From formula (2.2) of Gauss, (5.3)–(5.5), and (5.7), we get

$$L_{xx}=\frac{2\xi}{(x+y)^2}-\frac{2L_x}{x+y},\qquad L_{xy}=\frac{2L}{(x+y)^2},\qquad L_{yy}=-\frac{2L_y}{x+y}. \tag{5.8}$$

After solving the last two equations in (5.8) we obtain

$$L(x,y)=\frac{z(x)}{x+y}-\frac{z'(x)}{2} \tag{5.9}$$

for some $\mathbb{E}_s^{m+1}$-valued function $z(x)$. Since the metric tensor is given by (5.3), one finds

$$\langle L_x,L_x\rangle=\langle L_y,L_y\rangle=0\quad\text{and}\quad\langle L_x,L_y\rangle=-\frac{2}{(x+y)^2}.$$

Thus, it follows from (5.8), (5.9) and $\langle L,L\rangle=1$ that $z(x)$ satisfies

$$\langle z,z\rangle=\langle z'',z''\rangle=0\quad\text{and}\quad\langle z',z'\rangle=4.$$

Moreover, by substituting (5.9) into the first equation (5.8) we find

$$\xi=-\frac{(x+y)^2z'''(x)}{4}. \tag{5.10}$$

Combining this with $\xi \neq 0$ gives $z'''(x) \neq 0$. Consequently, we obtain case (b).

Conversely, suppose that L is given by (5.1), where $z(x)$ is a spacelike curve with constant speed 2 lying in the light cone $\mathcal{LC} \subset \mathbb{E}_s^{m+1}$ satisfying $\langle z'', z'' \rangle = 0$ and $z''' \neq 0$. Then L satisfies (5.9) with ξ given by (5.10). From the assumption, we have

$$\langle z, z \rangle = \langle z, z' \rangle = \langle z'', z'' \rangle = 0, \qquad \langle z', z' \rangle = -\langle z, z'' \rangle = 4. \tag{5.11}$$

By using (5.9) and (5.11) we know that $\langle L, L \rangle = 1$ and the induced metric tensor is given by (5.3). Moreover, the second equation in (5.8) shows that the second fundamental form of ψ satisfies $h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 0$. Consequently, the immersion ψ is a minimal immersion.

Case (ii): $\eta \neq 0$. After applying formula (2.2) of Gauss, (5.3)–(5.5), and (5.7), we obtain

$$L_{xx} = \frac{2\xi}{(x+y)^2} - \frac{2L_x}{x+y}, \quad L_{xy} = \frac{2L}{(x+y)^2}, \quad L_{yy} = \frac{2\eta}{(x+y)^2} - \frac{2L_y}{x+y}. \tag{5.12}$$

The compatibility conditions of (5.12) are given by

$$\tilde{\nabla}_{\frac{\partial}{\partial y}} \xi = \frac{2\xi}{x+y}, \qquad \tilde{\nabla}_{\frac{\partial}{\partial x}} \eta = \frac{2\eta}{x+y}. \tag{5.13}$$

Solving (5.13) gives

$$\xi = (x+y)^2 A(x), \qquad \eta = (x+y)^2 B(y) \tag{5.14}$$

for some $\mathbb{E}_s^{m+1}$-valued functions $A(x)$, $B(y)$. Substituting (5.14) into (5.12) yields

$$L_{xx} = A(x) - \frac{2L_x}{x+y}, \qquad L_{xy} = \frac{2L}{(x+y)^2}, \qquad L_{yy} = B(y) - \frac{2L_y}{x+y}. \tag{5.15}$$

After solving system (5.15), we obtain

$$L(x,y) = \frac{z(x) + w(y)}{x+y} - \frac{z'(x) + w'(y)}{2}, \tag{5.16}$$

where $z(x)$, $w(y)$ are $\mathbb{E}_s^{m+1}$-valued functions satisfying

$$z'''(x) = -4A(x), \qquad w'''(y) = -4B(y). \tag{5.17}$$

From $\langle L, L \rangle = 1$ and (5.16), we obtain condition (c.1) in Theorem 5.1.

By combining (5.15) and (5.17), we obtain

$$L_{xx} = -\frac{z'''}{4} - \frac{2L_x}{x+y}, \quad L_{xy} = \frac{2L}{(x+y)^2}, \quad L_{yy} = -\frac{w'''(y)}{4} - \frac{2L_y}{x+y}. \tag{5.18}$$

Since the metric tensor of M_1^2 is given by (5.3), we find

$$\langle L_x, L_x \rangle = \langle L_y, L_y \rangle = 0, \qquad \langle L_x, L_y \rangle = -\frac{2}{(x+y)^2}. \tag{5.19}$$

Because $\langle L, L \rangle = 1$, we have $\langle L_{xx}, L \rangle = -\langle L_x, L_x \rangle = 0$. Thus, we obtain condition (c.2) from (5.16), (5.18) and (5.19)

Similarly, due to $\langle L_{yy}, L \rangle = -\langle L_y, L_y \rangle = 0$, we may also derive condition (c.3) from (5.16) and (5.18).

Conversely, assume that L is defined by

$$L(x,y) = \frac{z(x) + w(y)}{x+y} - \frac{z'(x) + w'(y)}{2}, \tag{5.20}$$

where $z(x), w(y)$ are curves satisfying conditions (c1), (c.2) and (c.3). Then it follows from (5.20) that L satisfies system (5.18). Also, it follows from (5.20) and condition (c.1) that $\langle L, L \rangle = 1$. Thus, we have

$$\langle L, L_x \rangle = \langle L, L_y \rangle = 0, \tag{5.21}$$

which implies that

$$\langle L_x, L_x \rangle = -\langle L, L_{xx} \rangle, \quad \langle L_x, L_y \rangle = -\langle L, L_{xy} \rangle, \quad \langle L_y, L_y \rangle = -\langle L, L_{yy} \rangle. \tag{5.22}$$

By applying (5.22), (c.1) and the first equation in (5.22), we obtain

$$\langle L_x, L_x \rangle = -\langle L, L_{xx} \rangle = \frac{2}{(x+y)^2} \langle L_x, L_x \rangle, \tag{5.23}$$

which shows that $\langle L_x, L_x \rangle = 0$.

Similarly, from (5.22) and (c.3) we find $\langle L_y, L_y \rangle = 0$. Also, after applying (c.1), (5.22) and the second equation in (5.18), we find $\langle L_x, L_y \rangle = -2/(x+y)^2$. Consequently, the induced metric tensor via L is given by (5.3). Finally, it follows from (3.10) and the second equation in (5.18) that $\psi : M_1^2 \to S_s^m(1)$ is a minimal immersion. □

6. Minimal Lorentz surfaces in $H_s^m(-1)$

Let $\psi : M_1^2 \to H_s^m(-1)$ be an isometric immersion of a Lorentz surface into $H_s^m(-1)$. Denote by $L = \iota \circ \psi : M_1^2 \to \mathbb{E}_{s+1}^{m+1}$ the composition of ψ and the inclusion $\iota : H_s^m(-1) \subset \mathbb{E}_{s+1}^{m+1}$ via (1.2).

In this section, we provide the following answer to the basic question proposed in Section 5 for $c < 0$.

Theorem 6.1. *Let M_1^2 be a Lorentz surface of constant Gauss curvature -1. Then an isometric immersion $\psi : M_1^2 \to H_s^m(-1)$ is a minimal immersion if and only if one of the following three cases occurs:*

(i) *M_1^2 is an open portion of a totally geodesic $H_1^2(-1) \subset H_s^m(-1)$.*

(ii) *The immersion $L = \iota \circ \psi : M_1^2 \to H_s^m(-1) \subset \mathbb{E}_{s+1}^{m+1}$ is locally given by*

$$L(x,y) = z(x) \tanh\left(\frac{x+y}{\sqrt{2}}\right) - \frac{z'(x)}{\sqrt{2}}, \tag{6.1}$$

where $z(x)$ is a timelike curve with constant speed $\sqrt{2}$ lying in the light cone $\mathcal{LC} \subset \mathbb{E}_{s+1}^{m+1}$ satisfying $\langle z'', z'' \rangle = 4$ and $z''' \neq 2z'$.

(iii) *The immersion $L = \iota \circ \psi : M_1^2 \to H_s^m(-1) \subset \mathbb{E}_{s+1}^{m+1}$ is locally given by*

$$L(x,y) = (z(x) + w(y)) \tanh\left(\frac{x+y}{\sqrt{2}}\right) - \frac{z'(x) + w'(y)}{\sqrt{2}}, \tag{6.2}$$

where z and w are curves satisfying

$$\left\langle (z+w)\tanh\left(\frac{x+y}{\sqrt{2}}\right) - \frac{z'+w'}{\sqrt{2}}, (z+w)\tanh\left(\frac{x+y}{\sqrt{2}}\right) - \frac{z'+w'}{\sqrt{2}} \right\rangle = -1, \tag{iii.1}$$

$$\sqrt{2}\,\langle z+w, 2z' - z''' \rangle \tanh\left(\frac{x+y}{\sqrt{2}}\right) = \langle z' + w', 2z' - z''' \rangle, \tag{iii.2}$$

$$\sqrt{2}\,\langle z+w, 2w' - w''' \rangle \tanh\left(\frac{x+y}{\sqrt{2}}\right) = \langle z' + w', 2w' - w''' \rangle. \tag{iii.3}$$

Proof. Assume that $\psi : M_1^2 \to H_s^m(-1)$ is an isometric immersion of a Lorentz surface M_1^2 of constant curvature -1 into $H_s^m(-1)$. If M is totally geodesic in $H_s^m(-1)$, we obtain (i). Hence, let us assume that M_1^2 is non-totally geodesic.

Since M_1^2 is assumed to be of constant curvature -1, we may choose local coordinates $\{x, y\}$ such that the metric tensor is given by

$$g = -\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)(dx \otimes dy + dy \otimes dx). \tag{6.3}$$

Hence, the Levi–Civita connection satisfies

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} &= -\sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\frac{\partial}{\partial x}, \\ \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} &= 0, \\ \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} &= -\sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\frac{\partial}{\partial y}. \end{aligned} \tag{6.4}$$

Let us put

$$\frac{\partial}{\partial x} = \operatorname{sech}\left(\frac{x+y}{\sqrt{2}}\right)e_1, \qquad \frac{\partial}{\partial y} = \operatorname{sech}\left(\frac{x+y}{\sqrt{2}}\right)e_2. \tag{6.5}$$

Then we get

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0, \qquad \langle e_1, e_2\rangle = -1. \tag{6.6}$$

Because M_1^2 is minimal, it follows from (3.10) and (6.3) that $h(e_1, e_2) = 0$ holds. Hence, we may put

$$h(e_1, e_1) = \xi,\ h(e_1, e_2) = 0, \qquad h(e_2, e_2) = \eta \tag{6.7}$$

for some normal vector fields ξ, η. Without loss of generality, we may assume $\xi \neq 0$. Since M_1^2 is of curvature -1, the equation of Gauss and (6.7) imply that $\langle \xi, \eta\rangle = 0$.

Case (i): $\eta = 0$. By applying formula (2.2) of Gauss, (6.3)–(6.5), and (6.7), we obtain

$$\begin{aligned} L_{xx} &= \operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)\xi - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_x, \\ L_{xy} &= -\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)L, \\ L_{yy} &= -\sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_y. \end{aligned} \tag{6.8}$$

After solving the last two equations in (6.8) we have

$$L(x,y) = z(x)\tanh\left(\frac{x+y}{\sqrt{2}}\right) - \frac{z'(x)}{\sqrt{2}} \tag{6.9}$$

for some $\mathbb{E}^{m+1}_{s+1}$-valued function z. It follows from (6.3), (6.8), (6.9) and $\langle L, L\rangle = -1$ that $z(x)$ satisfies $\langle z, z\rangle = 0$, $\langle z', z'\rangle = -2$, and $\langle z'', z''\rangle = 4$. Moreover, substituting (6.9) into the first equation (6.8) yields

$$\xi = \left(\sqrt{2}z'(x) - \frac{z'''(x)}{\sqrt{2}}\right)\cosh\left(\frac{x+y}{\sqrt{2}}\right). \tag{6.10}$$

Combining this with $\xi \neq 0$ gives $z'''(x) \neq 2z'(x)$. Consequently, we obtain (ii).

Conversely, suppose that L is given by (6.2), where $z(x)$ is a timelike curve with constant speed $\sqrt{2}$ lying in the light cone $\mathcal{L}C$ satisfying $\langle z'', z''\rangle = 4$ and $z''' \neq 2z'$. Then, L satisfies (6.8) with ξ given by (6.10). Moreover, from the assumption, we have

$$\langle z, z\rangle = \langle z, z'\rangle = 0, \qquad \langle z, z''\rangle = -\langle z', z'\rangle = 2, \qquad \langle z'', z''\rangle = 4. \tag{6.11}$$

Hence, we know from (6.9) and (6.11) that the induced metric tensor is given by (6.3). Consequently, we see from (6.8) that the second fundamental form of ψ satisfies $h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 0$. Therefore, the immersion ψ is minimal.

Case (ii): $\eta \neq 0$. By applying formula (2.2) of Gauss, (6.3)-(6.5) and (6.7), we obtain

$$\begin{aligned}
L_{xx} &= \operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)\xi - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_x,\\
L_{xy} &= -\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)L,\\
L_{yy} &= \operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)\eta - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_y.
\end{aligned} \tag{6.12}$$

The compatibility conditions of (6.12) are given by

$$\tilde{\nabla}_{\frac{\partial}{\partial y}}\xi = \sqrt{2}\xi\tanh\left(\frac{x+y}{\sqrt{2}}\right), \qquad \tilde{\nabla}_{\frac{\partial}{\partial x}}\eta = \sqrt{2}\eta\tanh\left(\frac{x+y}{\sqrt{2}}\right). \tag{6.13}$$

Solving (6.13) gives

$$\xi = A(x)\cosh^2\left(\frac{x+y}{\sqrt{2}}\right), \qquad \eta = B(y)\cosh^2\left(\frac{x+y}{\sqrt{2}}\right)$$

for some $\mathbb{E}^{m+1}_s$-valued functions $A(x)$, $B(y)$ satisfying $\langle A, B\rangle = 0$. Substituting these into (6.12) yields

$$\begin{aligned}
L_{xx} &= A(x) - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_x,\\
L_{xy} &= -\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)L,\\
L_{yy} &= B(y) - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_y.
\end{aligned} \tag{6.14}$$

After solving system (6.14) we obtain

$$L(x,y) = (z(x)+w(y))\tanh\left(\frac{x+y}{\sqrt{2}}\right) - \frac{z'(x)+w'(y)}{\sqrt{2}}, \tag{6.15}$$

$$A(x) = \sqrt{2}z'(x) - \frac{z'''(x)}{\sqrt{2}}, \quad B(y) = \sqrt{2}w'(y) - \frac{w'''(y)}{\sqrt{2}} \tag{6.16}$$

for some $\mathbb{E}_s^{m+1}$-valued functions z, w. From (6.15) and $\langle L, L\rangle = -1$, we obtain condition (iii.1) of the theorem.

After differentiating (6.15) we get

$$\begin{aligned} L_x &= z'(x)\tanh\left(\frac{x+y}{\sqrt{2}}\right) + \frac{z(x)+w(y)}{\sqrt{2}}\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) - \frac{z''(x)}{\sqrt{2}}, \\ L_y &= w'(y)\tanh\left(\frac{x+y}{\sqrt{2}}\right) + \frac{z(x)+w(y)}{\sqrt{2}}\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right) - \frac{w''(y)}{\sqrt{2}}. \end{aligned} \tag{6.17}$$

Since the metric tensor of M_1^2 is given by (6.3), we find

$$\langle L_x, L_x\rangle = \langle L_y, L_y\rangle = 0, \qquad \langle L_x, L_y\rangle = -\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right). \tag{6.18}$$

From (6.14) and (6.16), we obtain

$$\begin{aligned} L_{xx} &= \sqrt{2}z'(x) - \frac{z'''(x)}{\sqrt{2}} - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_x, \\ L_{xy} &= -\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)L, \\ L_{yy} &= \sqrt{2}w'(y) - \frac{w'''(y)}{\sqrt{2}} - \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)L_y. \end{aligned} \tag{6.19}$$

Because $\langle L, L\rangle = -1$, we have $\langle L_{xx}, L\rangle = -\langle L_x, L_x\rangle = 0$. Thus, we derive from (6.18) and (6.19) that

$$\sqrt{2}\,\langle z+w, 2z' - z'''\rangle \tanh\left(\frac{x+y}{\sqrt{2}}\right) = \langle z'+w', 2z' - z'''\rangle. \tag{6.20}$$

Similarly, from $\langle L_{yy}, L\rangle = -\langle L_y, L_y\rangle = 0$, we have

$$\sqrt{2}\,\langle z+w, 2w' - w'''\rangle \tanh\left(\frac{x+y}{\sqrt{2}}\right) = \langle z'+w', 2w' - w'''\rangle. \tag{6.21}$$

These give conditions (iii.2) and (iii.3). Therefore, we obtain case (iii).

Conversely, if L is given by (6.2) such that $z(x), w(y)$ satisfy conditions (iii.1), (iii.2) and (iii.3), then we know from (6.2) that L satisfies (6.19). Also, it follows from (6.2) and (iii.1) that $\langle L, L\rangle = -1$. Thus, we have

$$\langle L, L_x\rangle = \langle L, L_y\rangle = 0, \tag{6.22}$$

which implies that

$$\langle L_x, L_x\rangle = -\langle L, L_{xx}\rangle, \ \ \langle L_x, L_y\rangle = -\langle L, L_{xy}\rangle, \ \ \langle L_y, L_y\rangle = -\langle L, L_{yy}\rangle. \tag{6.23}$$

By applying (6.19), (iii.1) and the first equation in (6.23), we obtain

$$\langle L_x, L_x\rangle = -\langle L, L_{xx}\rangle = \sqrt{2}\tanh\left(\frac{x+y}{\sqrt{2}}\right)\langle L_x, L_x\rangle, \tag{6.24}$$

which yields $\langle L_x, L_x\rangle = 0$. Similarly, from (6.19) and (iii.3) we find $\langle L_y, L_y\rangle = 0$. Also, after applying (iii.1), (6.19) and the second equation in (6.23), we obtain $\langle L_x, L_y\rangle = -\operatorname{sech}^2\left(\frac{x+y}{\sqrt{2}}\right)$. Consequently, the induced metric tensor via L is given by (6.3). Therefore, it follows from (3.10) and the second equation in (6.19) that the immersion $\psi : M_1^2 \to H_s^m(-1)$ is minimal. □

7. Explicit examples of minimal Lorentz surfaces in $S_s^m(1)$

There exist infinitely many spacelike curves with constant speed 2 lying in the light cone $\mathcal{LC} \subset \mathbb{E}_s^{m+1}$ satisfying $\langle z'', z''\rangle = 0$ and $z''' \neq 0$.

Example 7.1. Consider the curve $z = z(x)$ in $\mathbb{E}_3^7$ defined by

$$\begin{aligned} z(x) = \Bigg(& a\cosh px, \frac{\sqrt{4r^2 + a^2p^2(p^2 - r^2)}}{q\sqrt{r^2 - q^2}}\cosh qx, \frac{\sqrt{4q^2 + a^2p^2(p^2 - q^2)}}{r\sqrt{r^2 - q^2}}\sinh rx, \\ & a\sinh px, \frac{\sqrt{4r^2 + a^2p^2(p^2 - r^2)}}{q\sqrt{r^2 - q^2}}\sinh qx, \frac{\sqrt{4q^2 + a^2p^2(p^2 - q^2)}}{r\sqrt{r^2 - q^2}}\cosh rx, \\ & \frac{\sqrt{4(q^2 + r^2) + a^2(p^2 - r^2)(p^2 - q^2)}}{qr}\Bigg), \end{aligned}$$

where a, p, q, r are real numbers satisfying $p > r > q > 0$. It is easy to verify that z is a spacelike curve of constant speed 2 lying in $\mathcal{LC}$ satisfying $\langle z'', z''\rangle = 0, z''' \neq 0$.

It is direct to verify that the immersion defined by

$$L(x, y) = \frac{z(x)}{x+y} - \frac{z'(x)}{2}$$

gives rise to minimal Lorentz surfaces of constant curvature one in $S_3^6(1)$. Thus, there exist infinitely many minimal Lorentz surfaces of type (b) of Theorem 5.1.

There exist infinitely many pairs (z, w) of curves satisfying conditions (c.1), (c.2) and (c.3) of Theorem 5.1. Here we provide some examples of such pairs.

Example 7.2. Let p, q, r be positive numbers satisfying

$$\frac{315}{4}p^2 > 80 + 189r^2 - 64q^2 > 35p^2.$$

Consider curves $z(x)$ and $w(y)$ in $\mathbb{E}^{14}_6$ defined by

$$\begin{aligned} z(x) = \Bigg(& \frac{\sqrt{256q^2+369r^2}}{4\sqrt{15}}\cosh 2x, \frac{\sqrt{16q^2+609r^2}}{4\sqrt{15}}\sinh 4x, r\cosh 5x, 0, 0, 0, \\ & \frac{\sqrt{256q^2+369r^2}}{4\sqrt{15}}\sinh 2x, \frac{\sqrt{16q^2+609r^2}}{4\sqrt{15}}\cosh 4x, r\sinh 5x, 0, 0, 0, q, 0\Bigg), \\ w(y) = \Bigg(& 0, 0, 0, p\cosh\left(\frac{3y}{2}\right), \frac{\sqrt{320+225p^2+756r^2-256q^2}}{8\sqrt{15}}\sinh 2y, \\ & \frac{\sqrt{315p^2+1024q^2-3024r^2-1280}}{4\sqrt{15}}\sinh y, 0, 0, 0, p\sinh\left(\frac{3y}{2}\right), \\ & \frac{\sqrt{320+225p^2+756r^2-256q^2}}{8\sqrt{15}}\cosh 2y, \\ & \frac{\sqrt{315p^2+1024q^2-3024r^2-1280}}{4\sqrt{15}}\cosh y, \\ & 0, \frac{\sqrt{320+756r^2-35p^2-256q^2}}{8}\Bigg). \end{aligned}$$

Then z, w are constant speed curves lying in the light cone $\mathcal{LC} \subset \mathbb{E}^{14}_6$ satisfying

$$\langle z', z'\rangle = \frac{64q^2-189r^2}{20}, \qquad \langle w', w'\rangle = \frac{80+189r^2-64q^2}{20},$$
$$\langle z, w\rangle = \langle z, z'''\rangle = \langle z', z'''\rangle = \langle z'', z''\rangle = \langle w, w'''\rangle = \langle w', w'''\rangle = \langle w'', w''\rangle = 0.$$

Moreover, it is easy to see that conditions (c.1), (c.2) and (c.3) are satisfied. It is straightforward to verify that the immersion:

$$L(x, y) = \frac{z(x)+w(y)}{x+y} - \frac{z'(x)+w'(y)}{2}$$

via (z, w) defines a minimal Lorentz surface of constant curvature one in $S^{13}_6(1)$.

8. Explicit examples of minimal Lorentz surfaces in $H_s^m(-1)$

There exist infinitely many timelike curves with constant speed $\sqrt{2}$ lying in the light cone $\mathcal{LC} \subset \mathbb{E}_{s+1}^{m+1}$ satisfying $\langle z'', z'' \rangle = 4$ and $z''' \neq 2z'$.

Example 8.1. Let a, b, p, q be positive numbers satisfying

$$p^2 < \frac{4+a^2}{2+a^2} < q^2 \quad \text{and} \quad b^2 > \frac{a^2(q^2-1)(1-p^2) + 2(p^2+q^2-2)}{p^2q^2}.$$

Consider the curve $z = z(x)$ in $\mathbb{E}_4^8$ defined by

$$z(x) = \Bigg(b, a\cosh x, \frac{\sqrt{q^2(2+a^2)-(4+a^2)}}{p\sqrt{q^2-p^2}} \sinh px, \frac{\sqrt{4+a^2-p^2(2+a^2)}}{q\sqrt{q^2-p^2}} \sinh qx,$$
$$a\sinh px, \frac{\sqrt{q^2(2+a^2)-(4+a^2)}}{p\sqrt{q^2-p^2}} \cosh px, \frac{\sqrt{4+a^2-p^2(2+a^2)}}{q\sqrt{q^2-p^2}} \cosh qx$$
$$\frac{\sqrt{b^2p^2q^2 - a^2(q^2-1)(1-p^2) - 2(p^2+q^2-2)}}{pq}\Bigg).$$

It is easy to see that z is curve lying in $\mathcal{LC}$ satisfying $\langle z', z' \rangle = -2$, $\langle z'', z'' \rangle = 4$ and $z''' \neq 2z'$. A direct computation shows that

$$L(x,y) = z(x)\tanh\left(\frac{x+y}{\sqrt{2}}\right) - \frac{z'(x)}{\sqrt{2}},$$

defines a minimal Lorentz surfaces of constant curvature -1 in $H_3^7(-1)$. Hence, there exist infinitely many minimal Lorentz surfaces of type (ii) of Theorem 6.1.

There are many pairs (z, w) of curves satisfying conditions (iii.1)–(iii.3) of Theorem 6.1. Here we provide infinitely many examples of such pair of curves.

Example 8.2. Let a, b, p, q, r, s be positive numbers satisfying

$$a, b < 1, \qquad p, q, r, s > 1, \qquad p^2 < \frac{1}{1-b^2} < q^2, \qquad r^2 < \frac{1}{1-a^2} < s^2,$$
$$b^2 < \frac{p^2+q^2-2}{p^2q^2}, \qquad a^2 < \frac{r^2+s^2-2}{r^2s^2}. \tag{8.1}$$

Consider curves $z(x)$ and $w(y)$ in $\mathbb{E}_8^{14}$ defined by

$$z(x) = \Bigg(b, \frac{\sqrt{p^2+q^2-b^2p^2q^2-2}}{\sqrt{(p^2-1)(q^2-1)}} \cosh x, \frac{\sqrt{1-p^2(1-b^2)}}{\sqrt{(q^2-p^2)(q^2-1)}} \sinh qx,$$

$$\frac{\sqrt{q^2(1-b^2)-1}}{\sqrt{(q^2-p^2)(p^2-1)}}\sinh px, 0,0,0,0, \frac{\sqrt{p^2+q^2-b^2p^2q^2-2}}{\sqrt{(p^2-1)(q^2-1)}}\sinh x,$$
$$\left.\frac{\sqrt{1-p^2(1-b^2)}}{\sqrt{(q^2-p^2)(q^2-1)}}\cosh qx, \frac{\sqrt{q^2(1-b^2)-1}}{\sqrt{(q^2-p^2)(p^2-1)}}\cosh px, 0,0,0\right),$$
$$w(y)=\left(0,0,0,0,a, \frac{\sqrt{r^2+s^2-a^2r^2s^2-2}}{\sqrt{(r^2-1)(s^2-1)}}\cosh y, \frac{\sqrt{1-r^2(1-a^2)}}{\sqrt{(s^2-r^2)(s^2-1)}}\sinh sy,\right.$$
$$\frac{\sqrt{s^2(1-a^2)-1}}{\sqrt{(s^2-r^2)(r^2-1)}}\sinh ry, 0,0,0, \frac{\sqrt{r^2+s^2-a^2r^2s^2-2}}{\sqrt{(r^2-1)(s^2-1)}}\sinh y,$$
$$\left.\frac{\sqrt{1-r^2(1-a^2)}}{\sqrt{(s^2-r^2)(s^2-1)}}\cosh qy, \frac{\sqrt{s^2(1-a^2)-1}}{\sqrt{(s^2-r^2)(r^2-1)}}\cosh py\right).$$

It is easy to verify that z and w satisfy conditions (iii.1), (iii.2) and (iii.3). The associated map

$$L(x,y)=(z(x)+w(y))\tanh\left(\frac{x+y}{\sqrt{2}}\right)-\frac{z'(x)+w'(y)}{\sqrt{2}}$$

defines a minimal Lorentz surface of constant curvature -1 in $H^{13}_7(-1)$.

Remark 8.1. There exist many positive numbers a, b, p, q, r, s satisfying the conditions given in (8.2). For instance, $a=b=1/\sqrt{2}, p=r=1.1$ and $q=s=1.5$ satisfy all conditions given in (8.2).

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