Publ. Math. Debrecen
78/3-4 (2011), 505–512
DOI: 10.5486/PMD.2011.4447

The length of curvature tensor for Riemannian manifold with parallel Ricci curvature tensor

By JIAN-FENG ZHANG (Lishui)

Abstract. In this paper, we study compact Riemannian manifolds with ∇^2Ric=0, and establish a pinching theorem for the square of the length of Riemannian curvature tensor. When $n \geq 10$, the pinching constant is sharp.

0. Introduction

Let (M^n, g) be a closed (i.e. compact, without boundary) Riemannian manifold of dimension $n(n \geq 3)$. $e_1, \dots, e_n$ are local orthonormal frames. The components of Riemannian curvature tensor, the components of Ricci curvature tensor, the components of the scalar curvature, and Ricci curvature tensor of M^n, are denoted by R_{ijkl}, R_{ij}, R and Ric respectively. If $\nabla \operatorname{Ric} = 0$, we say that (M^n, g) has a parallel Ricci cuvature tensor, that is,

$$R_{ij,k} = 0, \forall i, j, k.$$

There are many Riemannian manifolds whose Ricci curvature tensors are parallel, for instance, Einstein manifolds and symmetry spaces.

Li and Zhao studied the Riemannian manifolds whose Ricci curvature tensors are parallel in [1], they obtained the following two theorems:

Mathematics Subject Classification: 53C40.
Key words and phrases: Ricci curvature tensor, Riemannian curvature tensor, Einstein space, Weyl conformal curvature tensor.
Project supported partially by National Natural Science Foundation of China (Grant No. 10901147) and ZheJiang provincial natural science foundation of China (Grant No. Y6100218).

Theorem A. *Suppose M^n is a closed manifold with parallel Ricci curvature tensor, and scalar curvature $R = n(n-1)$. If*

$$2n(n-1) \le \sum R_{ijkl}^2 \le 2n(n-1) + \left(\frac{n}{3+\sqrt{n-2}}\right)^2,$$

then M is a space form.

Theorem B. *Suppose M^n is a closed Einstein manifold, and Scalar curvature $R = n(n-1)$. If*

$$0 \le \sum W_{ijkl}^2 < \frac{4}{9}n(n-1),$$

where W_{ijkl} is Weyl conformal curvature tensor, then M is a space form.

Li and Zhao proposed the following problems in [1]: Find the best pinching constant with respect to the length of curvature tensor for Riemannian manifold with parallel Ricci curvature tensor and determine the Riemannian manifold with this constant. Chen studied this problem in [2] and obtained the following theorem:

Theorem C. *Suppose M^n is a closed manifold with parallel Ricci curvature tensor, and Scalar curvature $R = n(n-1)$. If M isn't an Einstein manifold, and*

$$2n(n-1) \le \sum R_{ijkl}^2 \le \frac{2n^2(n-1)}{n-2},$$

then $\sum R_{ijkl}^2 = \frac{2n^2(n-1)}{n-2}$, and M is an isometric quotient of $M^{n-1}\left(\frac{n}{n-2}\right) \times M^1$, where M^1 is a 1-dimension curve, $M^{n-1}\left(\frac{n}{n-2}\right)$ is an (n-1)-dimension space form with constant section curvature $\frac{n}{n-2}$.

Combining Theorem B with Theorem C, the above problem in [1] have been solved when $n \ge 12$.

In this paper, we study the analogous problem in [1] and we obtain:

Main Theorem. *Suppose M is an n-dimensional closed Riemannian manifold with $\nabla^2 \operatorname{Ric} = 0$, then the scalar curvature R is a constant. Now we suppose $n \ge 10$ and $R = n(n-1)$. If*

$$2n(n-1) \le \sum R_{ijkl}^2 \le \frac{2n^2(n-1)}{n-2},$$

then $\sum R_{ijkl}^2$ is a constant. Moreover, M must be one of the following two cases:

i) *$\sum R_{ijkl}^2 = 2n(n-1)$, and M is a space form;*

ii) *$\sum R_{ijkl}^2 = \frac{2n^2(n-1)}{n-2}$, and $M = M^{n-1}\left(\frac{n}{n-2}\right) \times M^1$, where M^1 is a 1-dimension curve, $M^{n-1}\left(\frac{n}{n-2}\right)$ is an $(n-1)$-dimension space form with constant section curvature $\frac{n}{n-2}$.*

1. Preliminaries

Let (M^n, g) be a closed Riemannian manifold of dimension $n(n \geq 3)$. $\omega_1, \dots \omega_n$ are local orthonormal frames. We have structure equations:

$$\omega_i = -\sum_j \omega_{ij} \Lambda \omega_j, \omega_{ij} + \omega_{ji} = 0. \tag{1.1}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \Lambda \omega_{kj} + \Omega_{ij}, \tag{1.2}$$

where $\Omega_{ij} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \Lambda \omega_l, R_{ijkl}$ are the components of Riemannian curvature tensor of M^n. The indexes range from 1 to n.

Ricci curvature are defined by $R_{ij} = \sum_l R_{iljl}$. The covariant derivative are defined by

$$\sum_k R_{ij,k} \omega_k = dR_{ij} - \sum_m R_{mj} \omega_{mi} - \sum_m R_{im} \omega_{mj}, \tag{1.3}$$

$$\sum_k R_{ij,kl} \omega_l = dR_{ij,k} - \sum_m R_{mj,k} \omega_{mi} - \sum_m R_{im,k} \omega_{mj} - \sum_m R_{ij,m} \omega_{mk}, \tag{1.4}$$

We denote $\nabla \operatorname{Ric} = 0$ and $\nabla^2 \operatorname{Ric} = 0$ by $R_{ij,k} = 0$ and $R_{ij,kl} = 0$ respectively, where Ric denotes Ricci curvature tensor of M^n.

Now we suppose M^n is a closed Einstein manifold with parallel Ricci curvature tensor, and its scalar curvature is $R = n(n-1)$. We define

$$D_{ijkl} = R_{ijkl} - (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{1.5}$$

One can derive the following identities:

$$D_{ijkl} = -D_{jikl} = -D_{ijlk} = D_{klij}, \tag{1.6}$$

$$D_{ijkl} + D_{iljk} + D_{iklj} = 0, \tag{1.7}$$

$$D_{ijkl,m} + D_{ijmk,l} + D_{ijlm,k} = 0, \tag{1.8}$$

$$\sum_i D_{ijil} = R_{jl} - (n-1)\delta_{jl}, \tag{1.9}$$

$$\sum_i D_{ijil,k} = 0, \tag{1.10}$$

$$\|D\|^2 = \sum_{ijkl} D_{ijkl}^2 = \sum_{ijkl} R_{ijkl}^2 - 2n(n-1). \tag{1.11}$$

From (1.11) we have $\Delta \|D\|^2 = \Delta\big(\sum_{ijkl} R_{ijkl}^2\big)$. Now we compute $\Delta\|D\|^2$. Since

$$D_{ijkl,m} = D_{ijkm,l} + D_{ijml,k},$$

We have

$$\frac{1}{2}\Delta\|D\|^2 = \sum_{ijkl,m} D^2_{ijkl,m} + \sum_{ijkl,m} D_{ijkl}D_{ijkl,mm}$$
$$= \|\nabla D\|^2 + 2\sum_{ijkl,m} D_{ijkl}D_{ijkm,lm}$$
$$= \|\nabla D\|^2 + 2\sum_{ijklhm} D_{ijkl}(D_{hjkm}R_{hilm} + D_{ihkm}R_{hjlm} + D_{ijhm}R_{hklm} + D_{ijkh}R_{hl}),$$

where we have used (1.10) and Ricci identity

$$D_{ijkm,lm} - D_{ijkm,ml} = D_{hjkm}R_{hilm} + D_{ihkm}R_{hjlm} + D_{ijhm}R_{hklm} + D_{ijkh}R_{hl}.$$

By a straight-forward computation, we get

$$\frac{1}{4}\Delta\|D\|^2 = \frac{1}{2}\|\nabla D\|^2 + \sum_{ijklhm} D_{ijkl}(D_{ihkm}D_{hjlm} - D_{jhkm}D_{hilm})$$
$$- \frac{1}{2}\sum_{ijklhm} D_{ijkl}D_{klhm}D_{hmij} + (n-1)\|D\|^2. \tag{1.12}$$

Here we have used the identity $\sum_i D_{ijik} = 0$ (M is an Einstein manifold).

We need these following lemmas:

Lemma 1.1 ([4]). *Let $a_1, \ldots, a_n (n \geq 2)$ be real numbers satisfying $\sum_i a_i = 0$. Then*

$$\Big|\sum_i a_i^3\Big| \leq \frac{n-2}{\sqrt{n(n-1)}}\Big(\sum_i a_i^2\Big)^{\frac{3}{2}},$$

where the equality holds if and only if at least $n-1$ numbers of the $a_i's$ are same with each other.

Lemma 1.2 ([5]). *Let A, B be anti-symmetric matrixes, then*

$$\big\|[A,B]\big\| \leq \sqrt{2}\,\|A\|\,\|B\|,$$

where $[A,B] = AB - BA, \|A\|^2 = (A,A)$, and $(A,B) = -\frac{1}{2}tr(AB)$.

Lemma 1.3 ([5]). *Let $\sigma = (s_{ij})$ be a symmetric $n \times n$ matrix with $s_{ij} \geq 0$ for any i, j and $s_{ii} = 0$ for any i. If* $\mathrm{trace}\,(\sigma^2) = n(n-1)$, *then*

$$tr(\sigma^3) \leq n(n-1)(n-2).$$

The equality holds if and only if $s_{ij} = 1 - \delta_{ij}$.

Let A be a $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ matrix,

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n-1} \\ \dots & \dots & \dots & \dots \\ A_{n-11} & A_{n-12} & \dots & A_{n-1n-1} \end{pmatrix}$$

where the submatrix A_{ik} is given by

$$A_{ik} = \begin{pmatrix} D_{ii+1kk+1} & D_{ii+1kk+2} & \dots & D_{ii+1kn} \\ D_{ii+2kk+1} & D_{ii+2kk+2} & \dots & D_{ii+2kn} \\ \dots & \dots & \dots & \dots \\ D_{inkk+1} & D_{inkk+2} & \dots & D_{inkn} \end{pmatrix}$$

It is easy to know that A_{ik} is an $(n-i) \times (n-k)$ matrix, and A is a symmetric matrix. In fact, the element $D_{ijkl}(i<j, k<l)$ of A lies on the place of $\big(\frac{1}{2}(i-1)(2n-i)+j-i, \frac{1}{2}(k-1)(2n-k)+l-k\big)$, and the element of A on the place of $\big(\frac{1}{2}(k-1)(2n-k)+l-k, \frac{1}{2}(i-1)(2n-i)+j-i\big)$is D_{klij}.

By the knowledge of matrix, there exists a matrix P such that PAP^{-1} is a diagonal matrix,that is,

$$PAP^{-1} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix}$$

where $N = \frac{1}{2}n(n-1)$, and $\lambda_1, \lambda_2, \dots, \lambda_N$ are real numbers.

It follows that

$$tr(A^k) = \sum_{i=1}^{N} \lambda_i^k, \quad k = 1, 2, 3, \dots$$

Since M is a Einstein manifold, we have

$$\sum_{i=1}^{N} \lambda_i = trA = \sum_{i<j} D_{ijij} = \frac{1}{2} \sum_{i,j=1}^{n} D_{ijij} = 0,$$

$$\sum_{i=1}^{N} \lambda_i^2 = tr(A^2) = \sum_{i<j,k<l} D_{ijkl}^2 = \frac{1}{4} \sum_{i,j,k,l=1}^{n} D_{ijkl}^2 = \frac{1}{4}\|D\|^2,$$

$$\sum_{i=1}^{N} \lambda_i^3 = tr(A^3) = \sum_{i<j,k<l,h<m} D_{ijkl} D_{klhm} D_{hmij} = \frac{1}{8} \sum_{i,j,k,l=1}^{n} D_{ijkl} D_{klhm} D_{hmij}.$$

By Lemma 1.1, we have

$$\Big|\sum_{i,j,k,l=1}^{n} D_{ijkl}D_{klhm}D_{hmij}\Big| = 8\sum_{i=1}^{N}\lambda_i^3 \le 8\frac{N-2}{\sqrt{N(N-1)}}\left(\sum_{i=1}^{N}\lambda_i^2\right)^{\frac{3}{2}}$$
$$= \frac{N-2}{\sqrt{N(N-1)}}\|D\|^3, \tag{1.13}$$

Let $\hat{D}_{ij} := (D_{lmij})$, then $\hat{D}_{ij}$ is an anti-symmetric matrix for any i, j. Now we have

$$\sum_{ijklhm} D_{ijkl}(D_{ihkm}D_{hjlm} - D_{jhkm}D_{hilm}) = 2\sum_{klm}\big(\hat{D}_{kl}, [\hat{D}_{km}, \hat{D}_{lm}]\big), \tag{1.14}$$

By using lemma 1.2, we have

$$2\Big|\sum_{klm}\big(\hat{D}_{kl}, [\hat{D}_{km}, \hat{D}_{lm}]\big)\Big| \le 2\sum_{klm}\big|\big(\hat{D}_{kl}, [\hat{D}_{km}, \hat{D}_{lm}]\big)\big| \le 2\sum_{klm}\|\hat{D}_{kl}\|\cdot\big\|[\hat{D}_{km}, \hat{D}_{lm}]\big\|$$
$$\le 2\sqrt{2}\sum_{klm}\|\hat{D}_{kl}\|\cdot\|\hat{D}_{km}\|\cdot\|\hat{D}_{lm}\| = \sum_{klm}s_{km}s_{ml}s_{lk},$$

where $s_{ij} := \sqrt{2}\|\hat{D}_{ij}\|$. Let $\sigma = (s_{ij}), t = \frac{\sqrt{n(n-1)}}{\|D\|}$, then

$$tr((t\sigma)^2) = t^2 Tr(\sigma^2) = 2t^2\sum_{kl}\|\hat{D}_{kl}\|^2 = t^2\sum_{ijkl}D_{ijkl}^2 = t^2\|D\|^2 = n(n-1),$$

By using lemma 1.3, we get

$$Tr((t\sigma)^3) \le n(n-1)(n-2),$$

and

$$\sum_{ijklhm} D_{ijkl}(D_{ihkm}D_{hjlm} - D_{jhkm}D_{hilm}) \le 2\Big|\sum_{klm}\big(\hat{D}_{kl}, [\hat{D}_{km}, \hat{D}_{lm}]\big)\Big|$$
$$\le \frac{n-2}{\sqrt{n(n-1)}}\|D\|^3,$$

Replacing (1.13) and (1.14) into (1.12), we have

$$\frac{1}{4}\Delta\|D\|^2 \ge \frac{1}{2}\|\nabla D\|^2 - \left(\frac{1}{2}\frac{N-2}{\sqrt{N(N-1)}} + \frac{n-2}{\sqrt{n(n-1)}}\right)\|D\|^3 + (n-1)\|D\|^2$$
$$\ge (n-1)\|D\|^2\left(1 - \frac{S}{n-1}\|D\|\right),$$

where $S = \frac{1}{2}\frac{N-2}{\sqrt{N(N-1)}} + \frac{n-2}{\sqrt{n(n-1)}}$.

2. Proof of Main Theorem

For any function f on M^n, the Laplacian of f is defined by $\triangle f = f_{,kk}$. Then we have:

Lemma 2.1. *Suppose M^n is a closed manifold, then $\nabla^2 \operatorname{Ric} = 0$ if and only if $\nabla \operatorname{Ric} = 0$.*

Proof. We only need to prove the condition $\nabla^2 \operatorname{Ric} = 0$ is the sufficient condition of $\nabla \operatorname{Ric} = 0$.

Suppose $\nabla^2 \operatorname{Ric} = 0$, that is, $R_{ij,kl} = 0$ for any i, j, k, l. Let $f = \sum_{ij} R_{ij}^2$, we have

$$\frac{1}{2}\triangle\Big(\sum_{ij} R_{ij}^2\Big) = \sum_{ij,k} R_{ij,k}^2 + \sum_{ij,k} R_{ij}R_{ij,kk} = \sum_{ij,k} R_{ij,k}^2 \geq 0.$$

Since M^n is a closed manifold, we obtain $\sum_{ij} R_{ij}^2 = const.$, hence $R_{ij,k} = 0$. $\square$

Remark. The first part of main theorem (scalar curvature R is a constant when $\nabla^2 \operatorname{Ric} = 0$) has been got by Xu in [3], but they hasn't got Lemma 2.1. In fact, we can prove that: If M^n is closed, $\nabla^2 T = 0$ implies $\nabla T = 0$ for any tensor T.

Moreover, we have:

Lemma 2.2. *Suppose M^n is a closed Einstein manifold, and Scalar curvature $R = n(n-1)$. If*

$$0 \leq \sum W_{ijkl}^2 < \frac{(n-1)^2}{S^2},$$

where W_{ijkl} is Weyl conformal curvature tensor, constant $S = \frac{1}{2}\frac{N-2}{\sqrt{N(N-1)}} + \frac{n-2}{\sqrt{n(n-1)}}$, and $N = \frac{n(n-1)}{2}$, then M is a space form.

Proof. The Weyl conformal curvature tensor is

$$\begin{aligned} W_{ijkl} = R_{ijkl} &- \frac{1}{n-2}(\delta_{ik}R_{jl} + \delta_{jl}R_{ik} - \delta_{il}R_{jk} - \delta_{jk}R_{il}) \\ &+ \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \end{aligned}$$

Since M^n is an Einstein manifold , we have $R_{jl} = (n-1)\delta_{jl}$. A straight-forward computation gives

$$\|W\|^2 = \sum_{i,j,k,l=1} R_{ijkl}^2 - 2n(n-1) = \|D\|^2, \tag{2.1}$$

By the condition $0 \le \sum W_{ijkl}^2 < \frac{(n-1)^2}{S^2}$, and (1.16) we get:

$$\frac{1}{4}\Delta\|D\|^2 \ge (n-1)\|D\|^2\left(1 - \frac{S}{n-1}\|D\|\right) \ge 0.$$

Since M^n is a closed manifold, we obtain $\|D\| = 0$ by Hopf Theorem, so M is a space form. □

Proof of Main Theorem. It is easy to know that the Scalar curvature R is a constant by Lemma 2.1. Moreover, Lemma 2.2 and Theorem C still hold under the condition "$\nabla^2 \operatorname{Ric} = 0$" by Lemma 2.1.

If M is not an Einstein manifold, main theorem is true by Theorem C .

Now we suppose M is an Einstein manifold.

From (2.1), we know the condition $2n(n-1) \le \sum R_{ijkl}^2 \le \frac{2n^2(n-1)}{n-2}$ is equivalent to the condition $0 \le \sum W_{ijkl}^2 \le \frac{4n(n-1)}{n-2}$.

When $n \ge 10$, $\|W\|^2 \le \frac{4n(n-1)}{n-2} < \frac{(n-1)^2}{S^2}$, we know M is a space form by Lemma 2.2, and main theorem also holds □

References

[1] An-Min Li and Guo-Song Zhao, Isolation phenoment on Riemannian manifold whose Ricci curvature tensor are parallel, *Acta Math. Sinica* **37** (1994), 19–24 (in *Chinese*).

[2] Jian-Hua Chen, The length of curvature tensor for Riemannian manifold with parallel ricci curvature, *Acta Math. Sinica* **39** (1996), 345–348 (in *Chinese*).

[3] Sen-Lin Xu and Jia-Qiang Mei, Rigidity theorems of Riemannian manifold with $\bigtriangledown^2 \operatorname{Ric} = 0$, *J. Math. Res. Exposition* **18** (1998), 1–10.

[4] M. Okumura, Hypersurfaces and a pinching problem on the second fundamental tensor, *Amer. J. Math.,* **96** (1974), 207–213.

[5] Jean-Pierre Bourguignon and H. Blaine Lawson, Stability and isolation phenomena for Yang–Mills fields, *Math. Phys.* **79** (1981), 189–230.

JIAN-FENG ZHANG
DEPARTMENT OF MATHEMATICS
LISHUI UNIVERSITY
LISHUI 323000
P.R. CHINA

E-mail: zjf7212@163.com

(Received October 27, 2008; revised November 15, 2010)