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Complete spacelike CMC hypersurfaces in a Lorentzian space form

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Abstract. Let $x : M^n \to \overline{M}_1^{n+1}(c)$ be a complete spacelike hypersurface immersed into a Lorentzian space form, where $\overline{M}_1^{n+1}(c)$ is a Lorentz–Minkowski space $\mathbb{L}^{n+1} = \mathbb{R}_1^{n+1}$, a de Sitter space $\mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}$ or an anti-de Sitter space $\mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$, according to $c = 0$, $c = 1$ or $c = -1$, respectively. Let $\phi = \langle x, a\rangle$ and $\psi = \langle \vec{H}, a\rangle$, where $\vec{H}$ is the mean curvature vector field of M^n and a is a fixed nonzero vector in the corresponding pseudo-Euclidean space. We prove that if M^n has constant mean curvature (CMC), and $\phi = \lambda\psi$, for some real number λ, then M^n is a spacelike isoparametric hypersurface of $\overline{M}_1^{n+1}(c)$. Furthermore, it is either a totally umbilical hypersurface or a hyperbolic cylinder.

1. Introduction

Let $\mathbb{R}_t^{n+2}$ be an $(n+2)$-dimensional pseudo-Euclidean space with index t endowed the indefinite inner product given by with flat semi-Euclidean metric

$$\langle x, y\rangle = -\sum_{i=1}^{t} x_i y_i + \sum_{j=t+1}^{n+2} x_j y_j,$$

where $(x_1, \dots, x_{n+2})$ is a rectangular coordinate system of $\mathbb{R}_t^{n+2}$. The de Sitter space and anti-de Sitter space [9] are defined by $\mathbb{S}_1^{n+1} = \{x \in \mathbb{R}_1^{n+2} | \langle x, x\rangle = 1\}$, $\mathbb{H}_1^{n+1} = \{x \in \mathbb{R}_2^{n+2} | \langle x, x\rangle = -1\}$, respectively, with constant sectional curvature

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$c = 1$ and $c = -1$, respectively. A hypersurface M^n is said to be spacelike if the induced metric on M^n from that of the ambient space is positive definite.

Constant mean curvature (CMC) hypersurfaces are often closely related to either an eigenvalue problem or a differential equation stemming from the Laplacian. At the same time, Maximal and CMC hypersurfaces play a chief role in relativity theory. There are many interesting results in the study of spacelike CMC hypersurfaces.

The study of this kind of hypersurface was inspired, in particular, by a conjecture posed by Goddard [6], stating that every complete spacelike hypersurface with constant mean curvature in $\mathbb{S}_1^{n+1}$ must be totally umbilical. The first result in this direction was obtained by Ramanathan [10] in 1987. He showed that if the constant mean curvature H of a complete spacelike surface in $\mathbb{S}_1^3$ satisfies $H^2 \leq 1$, then the surface is totally umbilical. Independently, and still in 1987, Akutagawa [2] proved Goddard's conjecture for the case $H^2 \leq 1$ if $n = 2$ and for the case $H^2 < 4(n-1)/n^2$ if $n > 2$. On the other hand, Montiel [8] proved the conjecture for the compact case.

In [3], Alías, Brasil and Perdomo studied the quadric constant mean curvature hypersurfaces of spheres and gave a characterization of ones by a linear relation between two functions on the position vector and a Gauss map of ones. Inspired their works, we will investigate complete spacelike CMC hypersurfaces in a Lorentzian space form.

Let $x : M^n \to \overline{M}_1^{n+1}(c)$ be a complete spacelike hypersurface immersed into a Lorentzian space form, where $\overline{M}_1^{n+1}(c)$ is a Lorentz–Minkowski space $\mathbb{L}^{n+1} = \mathbb{R}_1^{n+1}$, a de Sitter space $\mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}$ or an anti-de Sitter space $\mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$ according to $c = 0$, $c = 1$ or $c = -1$, respectively. For some fixed nonzero vector $a \in \mathbb{R}_1^{n+1}$, $\mathbb{R}_1^{n+2}$ or $\mathbb{R}_2^{n+2}$, according to $c = 0$, $c = 1$ or $c = -1$, respectively, let $\phi = \langle x, a\rangle$ and $\psi = \langle \vec{H}, a\rangle$, where $\vec{H}$ is the mean curvature vector field of M^n. In this paper, we will prove that if M^n has constant mean curvature, and $\phi = \lambda\psi$, for some real number λ, then M^n is either a totally umbilical hypersurface or a hyperbolic cylinder. In fact, we prove the following main results.

Theorem 1.1. *Let $x : M^n \to \mathbb{L}^{n+1}$ be a complete spacelike CMC hypersurface immersed into the Lorentz–Minkowski space $\mathbb{L}^{n+1}$. If for some nonzero constant vector $a \in \mathbb{R}_1^{n+1}$ and some real number λ, we have that $\phi = \lambda\psi$, then M^n is one of the following hypersurfaces, up to rigid motions:*

(i) $\mathbb{R}^n = \{x \in \mathbb{R}_1^{n+1} : x_1 = 0\}$;

(ii) $\mathbb{H}^n(\sinh t) = \{x \in \mathbb{R}_1^{n+1} : \|x\|^2 = -\sinh^2 t\}$, *where* $t \in \mathbb{R}$;

(iii) $\mathbb{H}^k(\sinh t) \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^{n+1}_1 : -x_1^2 + x_2^2 + \cdots + x_{k+1}^2 = -\sinh^2 t\}$, *where* $t \in \mathbb{R}$.

Theorem 1.2. *Let* $x : M^n \to \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1$ *be a complete spacelike CMC hypersurface immersed into the de Sitter space* $\mathbb{S}^{n+1}_1$. *If for some nonzero constant vector* $a \in \mathbb{R}^{n+2}_1$ *and some real number* λ, *we have that* $\phi = \lambda\psi$, *then* M^n *is either a totally umbilical hypersurface or a hyperbolic cylinder, i.e.* M^n *is one of the following hypersurfaces, up to rigid motions:*

(i) $\mathbb{R}^n = \{(f(y) + \sinh t, f(y) + \cosh t, y) \in \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : y \in \mathbb{R}^n\}$, *where* $t \in \mathbb{R}$ *and* $f(y) = -(e^t/2)\|y\|^2$;

(ii) $\mathbb{S}^n(\cosh t) = \{(\sinh t, y\cosh t) \in \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : y \in \mathbb{S}^n(1) \subset \mathbb{R}^{n+1}\}$, *where* $t \in \mathbb{R}$;

(iii) $\mathbb{H}^n(\sinh t) = \{(y\sinh t, \cosh t) \in \mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+2}_1 : y \in \mathbb{H}^n(1) \subset \mathbb{R}^{n+1}_1\}$, *where* $t \in (0, +\infty)$;

(iv) $\mathbb{H}^k(\sinh t) \times \mathbb{S}^{n-k}(\cosh t) = \{(y, z) \in \mathbb{R}^{k+1}_1 \times \mathbb{R}^{n-k+1} : \|y\|^2 = -\sinh^2 t$, $\|z\|^2 = \cosh^2 t\}$, *where* $0 < k < n$, $t \in (0, +\infty)$.

Theorem 1.3. *Let* $x : M^n \to \mathbb{H}^{n+1}_1(c) \subset \mathbb{R}^{n+2}_2$ *be a complete spacelike CMC hypersurface immersed into the anti-de Sitter space* $\mathbb{H}^{n+1}_1$. *If for some nonzero constant vector* $a \in \mathbb{R}^{n+2}_2$ *and some real number* λ, *we have that* $\phi = \lambda\psi$, *then* M^n *is either a totally umbilical hypersurface or a hyperbolic cylinder, i.e.* M^n *is one of the following hypersurfaces, up to rigid motions:*

(i) $\mathbb{H}^n(\sin t) = \{(\cos t, y\sin t) \in \mathbb{R}^{n+2}_2 : y \in \mathbb{H}^n \subset \mathbb{R}^{n+1}_1\}$, *where* $t \in (0, \pi/2]$;

(ii) $\mathbb{H}^k(\cos t) \times \mathbb{H}^{n-k}(\sin t) = \{(y, z) \in \mathbb{R}^{k+1}_1 \times \mathbb{R}^{n-k+1}_1 : \|y\|^2 = -\cos^2 t,\ \|z\|^2 = -\sin^2 t\}$, *where* $0 < k < n$, $t \in (0, \pi/2)$.

2. Preliminaries and auxiliary results

In this section, we give some formulas and notions of submanifolds in the space forms by using the method of moving frames. Let $x : M \longrightarrow \overline{M}^{n+1}_1(c) \subset \mathbb{R}^{n+2}_t$ be a isometric immersion from Riemannian manifold M^n to Lorentz space forms $\overline{M}^{n+1}_1(c)$ with constant sectional curvature c. Let ∇, $\overline{\nabla}$, $\widetilde{\nabla}$ be the Levi-Civita connection on M^n, $\overline{M}^{n+1}_1(c)$ and $\mathbb{R}^{n+2}_t$.

For any $p \in M$, we can choose a local orthonormal frame fields $e_1, \ldots, e_{n+2}$ ($e_{n+2} = x$ when $c = \pm 1$) in a neighborhood U of M such that $e_1, \ldots, e_n$ are tangential to M, e_{n+1} is a unit timelike normal vector field of M^n. In the following we shall make use of the following convention on the ranges of indices:

$$1 \le A, B, C \le n+2, \quad 1 \le i, j, k, l \le n, \quad n+1 \le \alpha, \beta, \gamma \le n+2.$$

Let ω_A be the corresponding dual frame. The smooth connection 1-forms are denoted by ω_{AB}. Then we have the structure equations of $\mathbb{R}^{n+2}_t$

$$\begin{cases} dx = \sum_A \varepsilon_A \omega_A e_A, \\ de_A = \sum_B \varepsilon_B \omega_{AB} e_B, \ \omega_{AB} + \omega_{BA} = 0, \\ d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \\ d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB}, \end{cases} \tag{2.1}$$

where $\varepsilon_A = \langle e_A, e_A \rangle$, $\varepsilon_i = 1$, $\varepsilon_{n+1} = -1$, $\varepsilon_{n+2} = c$. A well-known argument shows that the forms ω_{in+1} may be expressed as $\omega_{in+1} = \sum_j h_{ij}\omega_j$, $h_{ij} = h_{ji}$. From (2.1) we obtain structure equations of M in $M^{n+1}_1(c)$

$$\begin{cases} dx = \sum_i \omega_i e_i, \\ de_i = \sum_j \omega_{ij} e_j - \sum_j h_{ij}\omega_j e_{n+1} - c\omega_i x, \\ de_{n+1} = -\sum_{i,j} h_{ij}\omega_j e_i. \end{cases} \tag{2.2}$$

The second fundamental form is defined

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j, \tag{2.3}$$

and the square of the length of h is given by $S = |h|^2 = \sum_{i,j} h_{ij}^2$.

The Gauss and Codazzi equations are

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.4}$$

$$h_{ijk} = h_{ikj}, \tag{2.5}$$

where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}. \tag{2.6}$$

Associated to the shape operator A of M one has n invariants S_r, $1 \leq r \leq n$, given by the equality

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k}.$$

If $p \in M$ and e_k is basis of T_pM formed by eigenvectors of the shape operator A_p, with corresponding eigenvalues λ_k, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[x_1, \dots, x_n]$ is the r-th elementary symmetric polynomial on the indeterminates $x_1, \dots, x_n$. The r-th mean curvature of M is given by

$$H_r = \frac{1}{\binom{n}{r}} S_r.$$

In particular, when $r = 1$

$$H_1 = \frac{1}{n} \sum_i \lambda_i = \frac{1}{n} S_1 = H$$

is nothing but the mean curvature of M.

The classical Newton transformations $P_r : \mathcal{X}(M) \to \mathcal{X}(M)$ are defined inductively from the shape operator A by

$$P_r = \begin{cases} I, & r = 0, \\ S_r I - A \circ P_{r-1}, & r = 1, \dots, n, \end{cases} \tag{2.7}$$

where I denotes the identity transformation in $\mathcal{X}(M)$. Equivalently,

$$P_r = \sum_{j=0}^{r} (-1)^j S_{r-j} A^j. \tag{2.8}$$

Thus $P_n = 0$ by Cayley–Hamilton Theorem. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of T_pM, diagonalizing A at $p \in M$, also diagonalize all of the P_r at p. Let $\{e_k\}$ be such a basis. Denoting by A_i the restriction of A to $\mathrm{span}\{e_i\}^\perp \subset T_pM$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k}, \tag{2.9}$$

where

$$S_k(A_i) = \sum_{\substack{1\le j_1<\cdots<j_k\le n\\ j_1,\ldots,j_k\neq i}} \lambda_{j_1}\ldots\lambda_{j_k}.$$

With the above notions, it is also immediate to check that $P_r(e_i) = \sum_j T^r_{ij} e_j = S_r(A_i)e_i$, where $T^r_{ij} = \langle P_r(e_i), e_j\rangle$. Thus, according to [4], [5], [11], [12], we have the following lemma.

Lemma 2.1. *For each* $1 \le r \le n-2$

(a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$;

(b) $\mathrm{tr}(P_r) = (n-r)S_r$;

(c) $\mathrm{tr}(A\circ P_r) = (r+1)S_{r+1}$;

(d) $\mathrm{tr}(A^2\circ P_r) = S_1S_{r+1} - (r+2)S_{r+2}$;

(e) $\mathrm{tr}(P_r\circ\nabla_X A) = \langle\nabla S_{r+1}, X\rangle$ *for* $X\in\mathcal{X}(M)$.

Associated to each Newton transformation P_r, we consider the second-order linear differential operator $L_r : C^\infty(M)\to C^\infty(M)$ given by

$$L_r(f) = \mathrm{tr}(P_r\circ\nabla^2 f),$$

where $\nabla^2 f : \mathcal{X}(M)\to\mathcal{X}(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and given by

$$\langle\nabla^2 f(X), Y\rangle = \langle\nabla_X(\nabla f), Y\rangle,\quad X, Y\in\mathcal{X}(M).$$

Using that P_r is a symmetrical operator, we have

$$L_r(fg) = fL_rg + gL_rf + 2\langle P_r(\nabla f), \nabla g\rangle \tag{2.10}$$

for every $f, g\in C^\infty(M)$.

Since $\vec{H}$ is parallel to e_{n+1}, then $\vec{H} = He_{n+1}$. Furthermore,

$$\phi = \langle x, a\rangle,\ \psi = H\langle e_{n+1}, a\rangle,$$

where a is a fixed vector in $\mathbb{R}^{n+2}_1$ if $c = 1$; or a is a fixed vector $\mathbb{R}^{n+2}_2$ if $c = -1$. Then we have that

Proposition 2.2. *If* M^n *is a spacelike hypersurface of* $\overline{M}^{n+1}_1(c)$ *with non-zero mean curvature vector field. Then the gradients of functions* ϕ *and* ψ *are give by*

$$\nabla\phi = a^\top,\quad \nabla\psi = \nabla(\ln|H|)\psi - HA(a^\top), \tag{2.11}$$

where $a^\top$ *denotes the tangential component of* a.

Proof. Since

$$\langle \nabla\phi, e_i\rangle = e_i(\phi) = \langle \widetilde{\nabla}_{e_i} x, a\rangle = \langle dx(e_i), a\rangle = \langle e_i, a\rangle,$$

and

$$\begin{aligned}\langle \nabla\langle e_{n+1}, a\rangle, e_i\rangle &= \langle \widetilde{\nabla}_{e_i} e_{n+1}, a\rangle = \langle de_{n+1}(e_i), a\rangle \\ &= -\Big\langle \sum_j h_{ij} e_j, a\Big\rangle = -\langle Ae_i, a\rangle = -\langle e_i, A(a^\top)\rangle,\end{aligned}$$

thus

$$\begin{aligned}\nabla\phi &= \sum_i \langle e_i, a\rangle e_i = a^\top, \ \nabla\langle e_{n+1}, a\rangle = -A(a^\top) \\ \nabla\psi &= \nabla(H\langle e_{n+1}, a\rangle) = \nabla H\langle e_{n+1}, a\rangle + H\nabla\langle e_{n+1}, a\rangle \\ &= \nabla H\langle e_{n+1}, a\rangle - HA(a^\top) = \nabla(\ln|H|)\psi - HA(a^\top).\end{aligned}$$

□

Now, we have the following result.

Proposition 2.3. *If M^n is a spacelike hypersurface of $\overline{M}_1^{n+1}(c)$ with nonzero mean curvature vector field. For each $1 \le r \le n-2$, we have*

$$L_r\phi = -c(n-r)S_r\phi - H^{-1}(r+1)S_{r+1}\psi, \tag{2.12}$$

$$\begin{aligned}L_r\psi = {}& c(r+1)S_{r+1}H\phi + (H^{-1}L_rH + S_1S_{r+1} - (r+2)S_{r+2})\psi \\ & - H\langle\nabla S_{r+1}, a\rangle - 2\langle A\circ P_r(\nabla H), a\rangle.\end{aligned} \tag{2.13}$$

Proof. Since

$$\phi_i = e_i\langle x, a\rangle = \langle e_i, a\rangle, \tag{2.14}$$

then

$$\begin{aligned}\sum_j \phi_{ij}\omega_j &= d\phi_i + \sum_j \phi_j\omega_{ji} = \langle de_i, a\rangle - \sum_j \langle e_j, a\rangle\omega_{ij} \\ &= \Big\langle \sum_j \omega_{ij}e_j - \sum_j h_{ij}\omega_j e_{n+1} - c\omega_i x, a\Big\rangle - \sum_j \langle e_j, a\rangle\omega_{ij} \\ &= -c\phi\sum_j \delta_{ij}\omega_j - H^{-1}\psi\sum_j h_{ij}\omega_j.\end{aligned}$$

Thus

$$\phi_{ij} = -c\phi\delta_{ij} - H^{-1}\psi h_{ij}. \tag{2.15}$$

With that in mind we calculate

$$\begin{aligned}L_r\phi &= \sum_{i,j} T^r_{ij}\phi_{ij} = -c\phi\sum_{i,j} T^r_{ij}\delta_{ij} - H^{-1}\psi\sum_{i,j} T^r_{ij}h_{ij} \\ &= -c\phi\operatorname{tr}(P_r) - H^{-1}\psi\operatorname{tr}(A\circ P_r) = -c(n-r)S_r\phi - (r+1)S_{r+1}H^{-1}\psi.\end{aligned}$$

Next we let $\eta = \langle e_{n+1}, a\rangle$

$$\eta_i = e_i\langle e_{n+1}, a\rangle = \langle de_{n+1}(e_i), a\rangle = -\sum_j h_{ij}\langle e_j, a\rangle, \tag{2.16}$$

thus we can make the suggestive calculation

$$\begin{aligned}\sum_j \eta_{ij}\omega_j &= d\eta_i + \sum_j \eta_j\omega_{ji} = -\sum_j dh_{ij}\langle e_j, a\rangle - \sum_j h_{ij}\langle de_j, a\rangle - \sum_j \eta_j\omega_{ij}\\ &= -\sum_{j,k}(h_{ijk}\omega_k - h_{kj}\omega_{ki} - h_{ik}\omega_{kj})\langle e_j, a\rangle\\ &\quad - \Big\langle \sum_{j,k} h_{ij}\omega_{jk}e_k - \sum_{j,k} h_{ij}h_{jk}\omega_k e_{n+1} - c\sum_j h_{ij}\omega_j x, a\Big\rangle + \sum_{j,k} h_{jk}\langle e_k, a\rangle\omega_{ij}\\ &= -\sum_{j,k} h_{ijk}\omega_j\langle e_k, a\rangle + c\phi\sum_j h_{ij}\omega_j + \eta\sum_{j,k} h_{ik}h_{jk}\omega_j.\end{aligned}$$

This shows that

$$\eta_{ij} = -h_{ijk}\langle e_k, a\rangle + c\phi h_{ij} + \eta\sum_k h_{ik}h_{jk}. \tag{2.17}$$

With this, then we have

$$\begin{aligned}L_r\eta &= \sum_{i,j} T^r_{ij}\eta_{ij} = -\sum_{i,j,k} T^r_{ij}h_{ijk}\langle e_k, a\rangle + c\phi\sum_{i,j} T^r_{ij}h_{ij} + \eta\sum_{i,j,k} T^r_{ij}h_{ik}h_{jk}\\ &= -\sum_{i,j,k} T^r_{ij}h_{ijk}\langle e_k, a\rangle + c\phi\operatorname{tr}(A\circ P_r) + \eta\operatorname{tr}(A^2\circ P_r)\\ &= -\langle\nabla S_{r+1}, a\rangle + c\phi(r+1)S_{r+1} + \eta(S_1S_{r+1} - (r+2)S_{r+2}).\end{aligned}$$

Thus

$$\begin{aligned}L_r\psi &= L_r(H\eta) = HL_r\eta + 2\langle P_r(\nabla H), \nabla\eta\rangle + \eta L_rH\\ &= H(-\langle\nabla S_{r+1}, a\rangle + c\phi(r+1)S_{r+1} + \eta(S_1S_{r+1} - (r+2)S_{r+2}))\\ &\quad + 2\langle P_r(\nabla H), -A(a^\top)\rangle + H^{-1}L_rH\psi\\ &= c(r+1)S_{r+1}H\phi + (H^{-1}L_rH + S_1S_{r+1} - (r+2)S_{r+2})\psi\\ &\quad - 2\langle A\circ P_r(\nabla H), a\rangle - H\langle\nabla S_{r+1}, a\rangle.\end{aligned}$$

□

3. Proof of the main theorems

In this section, using Proposition 2.3, we will prove Theorem 1.1, Theorem 1.2 and Theorem 1.3 in a union form.

Since $\phi = \lambda\psi$, if $H = 0$ or $\lambda = 0$, then $\phi = 0$ and M^n is a totally geodesic submanifold in $\overline{M}_1^{n+1}(c)$. In the following ,we suppose $\lambda H \neq 0$. From Proposition 2.3, we obtain

$$\begin{aligned} L_r\phi &= -c(n-r)S_r\phi - (r+1)S_{r+1}H^{-1}\psi \\ &= -[c\lambda(n-r)S_r + (r+1)S_{r+1}H^{-1}]\psi \\ &= \lambda\{c(r+1)S_{r+1}H\phi + (H^{-1}L_rH + S_1S_{r+1} - (r+2)S_{r+2})\psi \\ &\quad - H\langle\nabla S_{r+1}, a\rangle - 2\langle A \circ P_r(\nabla H), a\rangle\} \\ &= -H\langle\nabla S_{r+1}, a\rangle\lambda + [c(r+1)S_{r+1}H\lambda^2 + (S_1S_{r+1} - (r+2)S_{r+2})\lambda]\psi. \end{aligned}$$

This imply

$$\begin{aligned} \{c(r+1)S_{r+1}H\lambda^2 + [S_1S_{r+1} - (r+2)S_{r+2} + c(n-r)S_r]\lambda + (r+1)S_{r+1}H^{-1}\}\psi \\ = \lambda\langle\nabla S_{r+1}, a\rangle H. \end{aligned}$$

If $\psi = 0$, then $\phi = 0$ and M^n is a totally geodesic submanifold in $\overline{M}_1^{n+1}(c)$. There is nothing to prove. Therefore we can assume $\psi \neq 0$, then

$$\begin{aligned} c(r+1)S_{r+1}H\lambda^2 + [S_1S_{r+1} - (r+2)S_{r+2} + c(n-r)S_r]\lambda + (r+1)S_{r+1}H^{-1} \\ = \lambda\psi^{-1}\langle\nabla S_{r+1}, a\rangle H. \end{aligned} \tag{3.1}$$

Taking $r = 0$, and since H is constant, then

$$cnH^2\lambda^2 + (S_1^2 - 2S_2 + cn)\lambda + n = 0.$$

Thus

$$S_2 = \frac{n}{2\lambda}[cH^2\lambda^2 + (nH^2 + c)\lambda + 1]$$

is constant.

Using (3.1), by inductive method, we show that S_r is constant for every $1 \leq r \leq n$. This means that M^n is a complete spacelike isoparametric hypersurface of $\overline{M}_1^{n+1}(c)$. According to Theorem 1 and Theorem 2 in [7] or by the congruence theorem of Abe, Koike and Yamaguchi [1], we conclude that M^n is either a totally umbilical hypersurface or a hyperbolic cylinder, i.e. M^n is one of the following hypersurfaces, up to rigid motions:

(i) $\mathbb{R}^n$, $\mathbb{H}^n(\sinh t)$, $\mathbb{H}^k(\sinh t) \times \mathbb{R}^{n-k}$ in $\mathbb{L}^{n+1}$;

(ii) $\mathbb{R}^n$, $\mathbb{S}^n(\cosh t)$, $\mathbb{H}^n(\sinh t)$, $\mathbb{H}^k(\sinh t) \times \mathbb{S}^{n-k}(\cosh t)$ in $\mathbb{S}_1^{n+1}$;

(iii) $\mathbb{H}^n(\sin t)$, $\mathbb{H}^k(\cos t) \times \mathbb{H}^{n-k}(\sin t)$ in $\mathbb{H}_1^{n+1}$.

4. Some examples

In this section, we give some examples for hypersurfaces appearing in the main theorems and verify further our results.

Example 4.1. let $f : \mathbb{L}^{n+1} \to \mathbb{R}$ be a real function defined by

$$f(x_0, \dots, x_n) = \delta_1(-x_0^2 + x_2^2 + \cdots + x_k^2) + x_{k+1}^2 + \delta_2(x_{k+2}^2 + \cdots + x_n^2), \quad (4.1)$$

where $\delta_1,\ \delta_2 \in \{0, 1\}$ and $\delta_1^2 + \delta_2^2 \neq 0$. Taking $r > 0$ and $\epsilon = \pm 1$, the set $M^n = f^{-1}(\epsilon r^2)$ is a hypersurface of $\mathbb{L}^{n+1}$ provided $(\delta_1, \delta_2, \epsilon) \neq (0, 1, -1)$.

A straightforward computation shows the unit normal vector field is written as

$$e_{n+1} = \frac{1}{r}(\delta_1 x_0, \dots, \delta_1 x_k, x_{k+1}, \delta_2 x_{k+2}, \dots, \delta_2 x_n). \quad (4.2)$$

Moreover, the principal curvatures of M^n are given by

$$\lambda_1 = \cdots = \lambda_k = -\frac{\delta_1}{r},$$

$$\lambda_{k+1} = \cdots = \lambda_n = -\frac{\delta_2}{r},$$

and we also have that

$$H = -\frac{k\delta_1 + (n-k)\delta_2}{nr}.$$

Thus

(1) when $(\delta_1, \delta_2, \epsilon) = (0, 1, 1)$, $M^n = \mathbb{L}^k \times \mathbb{S}^{n-k}(r)$, and if we take $a = (0, \dots, 0, a_{k+1}, \dots, a_{n+1}) \in \mathbb{R}_1^{n+1}$, then we have $\phi = -\frac{nr^2}{n-k}\psi$;

(2) when $(\delta_1, \delta_2, \epsilon) = (1, 0, -1)$, $M^n = \mathbb{H}^k(r) \times \mathbb{R}^{n-k}$, and if we take $a = (a_1, \dots, a_{k+1}, 0, \dots, 0) \in \mathbb{R}_1^{n+1}$, then we have $\phi = -\frac{nr^2}{k}\psi$;

(3) when $(\delta_1, \delta_2, \epsilon) = (1, 1, -1)$, $M^n = \mathbb{H}^n(r)$, and if we take

$$a = (a_1, \dots, a_{n+1}) \in \mathbb{R}_1^{n+1},$$

then we have $\phi = -r^2\psi$.

Example 4.2. Given any integer $k \in \{1, \dots, n-1\}$ and any real number $r \in (0, 1)$, let

$$\begin{aligned} M^n &= \{(x, y) \in \mathbb{L}^{k+1} \times \mathbb{R}^{n-k+1} : \|x\|^2 = -r^2 \text{ and } \|y\|^2 = 1 + r^2\} \\ &= \mathbb{H}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1+r^2}\,). \end{aligned}$$

It is not difficult to see that for any $(x, y) \in M^n$ one gets

$$T_{(x,y)}M^n = \{(v, w) \in \mathbb{L}^{k+1} \times \mathbb{R}^{n-k+1} : \langle x, v\rangle = 0 \text{ and } \langle y, w\rangle = 0\}.$$

Therefore, the unit timelike normal vector field e_{n+1} is given by

$$e_{n+1}(x, y) = \left(\frac{\sqrt{1 + r^2}}{r}x, -\frac{r}{\sqrt{1 + r^2}}y\right).$$

Moreover, the principal curvatures of M^n are given by

$$\lambda_1 = \cdots = \lambda_k = -\frac{\sqrt{1 + r^2}}{r},$$
$$\lambda_{k+1} = \cdots = \lambda_n = \frac{r}{\sqrt{1 + r^2}},$$

and we also have that

$$H = \frac{(n - 2k)r^2 - k}{nr\sqrt{1 + r^2}}.$$

Thus, when $H \neq 0$, i.e. $n \leq 2k$, or $n > 2k$ and $r \neq \sqrt{\frac{k}{n-2k}}$,

(i) if we take $a = (a_1, \ldots, a_{k+1}, 0, \ldots, 0) \in \mathbb{R}^{n+2}_1$, then we have that

$$\phi = \frac{nr^2}{(n - 2k)r^2 - k}\psi; \tag{4.3}$$

(ii) if we take $a = (0, \ldots, 0, a_{k+2}, \ldots, a_{n+2}) \in \mathbb{R}^{n+2}_1$, then we have that

$$\phi = -\frac{n(1 + r^2)}{(n - 2k)r^2 - k}\psi. \tag{4.4}$$

Similarly, we have the following example.

Example 4.3. Given any integer $k \in \{1, \ldots, n - 1\}$ and any real number $r \in (0, 1)$, let

$$\begin{aligned} M^n &= \{(x, y) \in \mathbb{L}^{k+1} \times \mathbb{L}^{n-k+1} : \|x\|^2 = -r^2 \text{ and } \|y\|^2 = -1 + r^2\} \\ &= \mathbb{H}^k(r) \times \mathbb{H}^{n-k}(\sqrt{1 - r^2}\,). \end{aligned}$$

It is not difficult to see that for any $(x, y) \in M^n$ one gets

$$T_{(x,y)}M^n = \{(v, w) \in \mathbb{L}^{k+1} \times \mathbb{L}^{n-k+1} : \langle x, v\rangle = 0 \text{ and } \langle y, w\rangle = 0\}.$$

Therefore, the unit timelike normal vector field e_{n+1} is given by

$$e_{n+1}(x,y)=\left(\frac{\sqrt{1-r^2}}{r}x,-\frac{r}{\sqrt{1-r^2}}y\right).$$

Moreover, the principal curvatures of M^n are given by

$$\lambda_1=\cdots=\lambda_k=-\frac{\sqrt{1-r^2}}{r},$$
$$\lambda_{k+1}=\cdots=\lambda_n=\frac{r}{\sqrt{1-r^2}},$$

and we also have that

$$H=\frac{nr^2-k}{nr\sqrt{1-r^2}}.$$

Thus, when $H\neq 0$, i.e. $r\neq\sqrt{\frac{k}{n}}$,

(i) if we take $a=(a_1,\dots,a_{k+1},0,\dots,0)\in\mathbb{R}^{n+2}_1$, then we have that

$$\phi=\frac{nr^2}{nr^2-k}\psi; \tag{4.5}$$

(ii) if we take $a=(0,\dots,0,a_{k+2},\dots,a_{n+2})\in\mathbb{R}^{n+2}_1$, then we have that

$$\phi=-\frac{n(1-r^2)}{nr^2-k}\psi. \tag{4.6}$$

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