Publ. Math. Debrecen
78/3-4 (2011), 527–533
DOI: 10.5486/PMD.2011.4579

On measurable solutions of a general functional equation on topological groups

By TOMASZ KOCHANEK and MICHAŁ LEWICKI (Katowice)

Abstract. We establish a theorem of the type "measurability implies continuity" for solutions f of the functional equation

$$\Gamma(f(x), f(y)) = \Phi(x, y, f(\alpha_1 x + \beta_1 y), \dots, f(\alpha_n x + \beta_n y))$$

under reasonable conditions upon the integers α_i, β_i and the mappings Γ, Φ.

1. Introduction

The theory of regularity properties of functional equations is widely developed and has been perfectly described in the monograph of Antal Járai [5]. However, while considering the following simple-looking equation

$$|f(x) - f(y)| = \min\{f(x+y), f(x-y)\}, \tag{1}$$

and trying to prove the continuity of, let's say, Haar measurable solutions $f : G \to \mathbb{R}$ defined on a locally compact group G, we notice that none of the very general results presented in [5, Chapter III.8] is directly applicable. The reason is that equality (1) does not give any explicit formula of the form

$$f(x) = h(x, y, f(g_1(x, y)), \dots)$$

Mathematics Subject Classification: 39B52, 22B99, 43A05.
Key words and phrases: regularity properties of functional equations, measurable function.
This research has been supported by the scholarship from the UPGOW project co-financed by the European Social Fund – the first author, and by the University of Silesia Mathematics Department, Iterative Functional Equations and Real Analysis program – the second author.

with a continuous function h.

Equation (1) was introduced by A. Simon and P. Volkmann [7] who were considering the problem of characterizing functions being of the form $|a(x)|$ with an additive mapping a. However, as they observed, there exist solutions of (1) which are not of such a form; the question about general solution remains open. During his talk [8] P. Volkmann showed that a function $f:\mathbb{R}\to\mathbb{R}$ is a continuous (at a point) solution of (1) if and only if either $f(x)=c|x|$ for $x\in\mathbb{R}$ and for some $c\geq 0$, or $f(x)=c|x|$ for $|x|\leq p$ and $f(x+2p)=f(x)$ for $x\in\mathbb{R}$, with some $c\geq 0$, $p>0$. K. Baron [1] proved that if G is a metrizable topological group, then the continuity of a solution $f:G\to\mathbb{R}$ of (1) follows actually from the Baire measurability of a restriction $f|_M$ where $M\subset G$ fulfils, for instance, one of the two following conditions:

(i) M is a second category set with the property of Baire and $f^{-1}(\{0\})\subset M$,

(ii) M contains an open neighborhood of a point from $f^{-1}(\{0\})$.

Attempting to prove that Haar measurable solutions of (1) are necessarily continuous we arrive at a more general regularity problem for the following functional equation:

$$\Gamma(f(x),f(y))=\Phi(x,y,f(\alpha_1x+\beta_1y),\dots,f(\alpha_nx+\beta_ny)),\quad x,y\in G,\qquad (2)$$

which will be dealt with in the case where G is a metrizable locally compact group and α_i, β_i are integers.

To avoid undesirable behavior of the combinations $\alpha_ix+\beta_iy$ we need some special assumptions. A reasonable condition was proposed by A. Járai in [2, Proposition, p. 32]; see also [5, pp. 238–239]. Unfortunately, we need a little stronger hypothesis:

(H) There exists a compact zero neighborhood (i.e. a compact set which contains an open neighborhood of zero) $W\subset G$ such that for any compact subset $K\subset W$ with positive right Haar measure the set $\{\alpha_ix : x\in K\}$ has a positive right Haar measure as well.

The difference is that Járai's condition requires W only to be a compact set with positive Haar measure which automatically follows from (H), since any non-empty open set has a positive Haar measure. Nevertheless, the reasoning presented on [5, p. 238] shows that hypothesis (H) holds true in any Lie group G provided, of course, $\alpha_i\neq 0$.

Another difference between our main result and results of Járai is that in general we are not able to obtain the continuity of f at an arbitrary point of G. Actually, we show only the continuity at zero, leaving the problem of the global continuity for separate considerations. In many particular cases of equation (2) one can easily deduce the continuity of f having already its continuity at zero and it is like that in the case of equation (1).

2. Main result

In the sequel we will use the additive notation in the group G although we do not assume its commutativity. Let us recall that if λ is a Haar measure in G, X is a topological space, then a function $f : G \to X$ is called *Lusin λ-measurable* if and only if for every measurable set $A \subset G$ with $\lambda(A) < \infty$ and every $\varepsilon > 0$ there is a compact set $K \subset A$ such that $\lambda(A \setminus K) < \varepsilon$ and $f|_K$ is continuous.

For completeness we quote here several theorems which we will apply in the proof of our main result.

Theorem A (A. Járai, 1979). *Let G be a locally compact topological group and let λ be the right Haar measure in G. If $\alpha_i \in \mathbb{Z}$ satisfies* (H) *then for every $\varepsilon > 0$ there is $\delta > 0$ such that for every measurable set $A \subset W$ with $\lambda(A) \geq \varepsilon$ we have $\lambda\{\alpha_i a : a \in A\} \geq \delta$.*

A similar statement appeared first in the proof of [2, Proposition, pp. 32–33]. Precisely the same assertion as the one above is shown on [5, p. 239] for $\alpha_i = 2$. For any other $\alpha_i \in \mathbb{Z} \setminus \{0\}$ the proof is the same.

Theorem B (T. Krausz, 1980). *Let T, Y and X_i $(i = 0, 1, \dots, n)$ be locally compact Hausdorff spaces and let ν and μ_i be finite Radon measures on T and X_i, respectively. Given continuous functions $g_i : T \times Y \to X_i$ assume, with the notation*

$$g_{i,y}(t) = g_i(t, y),$$

that for every $\varepsilon > 0$ there is $\delta > 0$ such that for every set $B \subset T$ with $\nu(B) \geq \varepsilon$ we have $\mu_i(g_{i,y}(B)) \geq \delta$ whenever $y \in Y$ and $0 \leq i \leq n$. Then for arbitrary μ_i-measurable sets $A_i \subset X_i$ the function

$$\varphi(y) = \nu\left(\bigcap_{i=0}^{n} g_{i,y}^{-1}(A_i)\right) \tag{3}$$

is continuous on Y.

Such a generalization of the Steinhaus–Weil theorem appeared in this form in the paper of T. Krausz [6]. It was further generalized by A. Járai; see [3, Corollary 1].

Theorem C (A. JÁRAI, 2003)**.** *Let Y be topological space, T, Z be Hausdorff spaces and let ν and μ be finite Radon measures on T and Z, respectively. Given a continuous function $\varphi : T \times Y \to Z$ assume, with the notation*

$$\varphi_y(t) = \varphi(t, y),$$

that for every $\varepsilon > 0$ there is $\delta > 0$ such that for every set $B \subset T$ with $\nu(B) \geq \varepsilon$ we have $\mu(\varphi_y(B)) \geq \delta$ whenever $y \in Y$. Suppose, moreover, that $y_0 \in Y$ and f is a Lusin μ-measurable function on Z taking values in a topological space. Then the following condition holds true:

(R) *For each sequence $y_m \longrightarrow y_0$ there exists a subsequence (y_{m_k}) such that for ν-almost all $t \in T$ we have*

$$f(\varphi(t, y_{m_k})) \longrightarrow f(\varphi(t, y_0)).$$

The above statement is just a part of a much wider theorem of A. JÁRAI [4, Theorem 2.6].

We are going to apply the above results only in the case where all the underlying measures are simply Haar measures on compact subsets of a topological group. For a precise definition of Radon measure see [5, §2.5]. Haar measure is a special case of Radon measure, in particular – it is defined on all subsets of an underlying group G. Not all of them are measurable (only those which satisfy the Carathéodory condition), but for each $A \subset G$ one may find a Borel set $B \subset G$ such that $A \subset B$ and Haar measures of A and B are equal. This means that Haar measure is *Borel regular* (see [5, §2.5 and §2.8]).

Theorem. *Let G be a metrizable locally compact group with the right Haar measure λ and let α_i, β_i be integers such that α_i fulfils condition* (H) *for $i = 1, \ldots, n$. Let also X be a Hausdorff topological space and $\Gamma : X \times X \to X$, $\Phi : G \times G \times X^n \to X$ be mappings which satisfy the following conditions:*

(i) *for all $x, y \in X$ if $\Gamma(x, x) = \Gamma(x, y) = x$ then $x = y$,*

(ii) *for all $y_1, y_2 \in X$, $y_1 \neq y_2$ there is at most one $x \in X$ with $\Gamma(x, y_1) = \Gamma(x, y_2)$,*

(iii) *for all $x \in X$ the function $\Gamma(x, \cdot)$ is continuous,*

(iv) *for all $x \in X$ and every sequence (x_n) in X, if there exists a limit $\lim_{n\to\infty} \Gamma(x, x_n)$ then (x_n) has a convergent subsequence,*

(v) *for all $x, y \in G$, $z \in X$ we have $\Phi(x, y, z, z, \ldots, z) = z$,*

(vi) *for all $x \in G$, $\boldsymbol{y} \in X^n$ the function $\Phi(x, \cdot, *)$ is continuous at the point $(0, \boldsymbol{y})$.*

Then every Lusin λ-measurable solution $f : G \to X$ of equation (2) *is continuous at zero.*

Proof. Let $W \subset G$ be a compact zero neighborhood such that condition (H) holds true for every $i = 1, \dots, n$. Fix arbitrarily a sequence $y_m \longrightarrow 0$ in G and suppose on the contrary that there is no subsequence (y_{m_k}) such that $f(y_{m_k}) \longrightarrow f(0)$. We may assume that $y_m \in W$ for all $m \in \mathbb{N}$.

Let $\alpha_0 = 1$, $\beta_0 = 0$, $g_0(x, y) = x$ and $g_i(x, y) = \alpha_i x + \beta_i y$ for $i = 1, \dots, n$. All these functions g_i are continuous and map $W \times W$ to a compact set $g_i(W \times W)$. If $y \in W$ and $A \subset W$ is λ-measurable then $g_{i,y}(A) = \alpha_i A + \beta_i y$ and hence

$$\lambda(g_{i,y}(A)) = \lambda\{\alpha_i a : a \in A\}.$$

Consequently, in view of Theorem A, for every $\varepsilon > 0$ there is $\delta > 0$ such that for every measurable set $A \subset W$ with $\lambda(A) \geq \varepsilon$ we have $\lambda(g_{i,y}(A)) \geq \delta$ whenever $y \in W$ and $i = 0, 1, \dots, n$. Since λ is a Borel regular measure, the above condition remains valid if we replace A by an arbitrary, not necessarily measurable, set $B \subset W$ with $\lambda(B) \geq \varepsilon$. Applying Theorem C consecutively for the functions $g_1, \dots, g_n$ we get a subsequence (y_{m_k}) such that $f(g_i(x, y_{m_k})) \longrightarrow f(g_i(x, 0)) = f(\alpha_i x)$ for $i = 1, \dots, n$ and almost all $x \in W$. By virtue of equation (2) and condition (vi), we obtain

$$\Gamma(f(x), f(y_{m_k})) \longrightarrow \Phi(x, 0, f(\alpha_1 x), \dots, f(\alpha_n x)) \tag{4}$$

for almost all $x \in W$.

Therefore, fixing for a moment any $x \in W$ for which (4) holds true and using condition (iv), we infer that there exists a subsequence $(y_{m_{k_j}})$ such that $f(y_{m_{k_j}})$ tends to an element z_0 of X. By condition (iii), we may pass to the limit in (4) to obtain that

$$\Gamma(f(x), z_0) = \Phi(x, 0, f(\alpha_1 x), \dots, f(\alpha_n x)) = \Gamma(f(x), f(0))$$

for almost all $x \in W$. According to our supposition we have $z_0 \neq f(0)$. Hence assumption (ii) implies that f is constant almost everywhere on W, i.e. $f(x) = z$ for some $z \in X$ and every $x \in W \setminus W_0$, where $\lambda(W_0) = 0$.

In the light of Theorem B, the function φ, given by formula (3) with $A_i = W \setminus W_0$ and $\nu = \lambda$, is continuous on W. We have

$$\bigcap_{i=0}^{n} g_{i,0}^{-1}(W \setminus W_0) = \bigcap_{i=0}^{n} \{x \in W : \alpha_i x \in W\} \setminus \bigcup_{i=0}^{n} \{x \in W : \alpha_i x \in W_0\}.$$

If the set $\{x \in W : \alpha_i x \in W_0\}$ had positive measure for any $i = 0, 1, \dots, n$, it would contain a compact subset K being of positive measure as well. Then

$\{\alpha_i x : x \in K\} \subset W_0$ which is of measure zero. This contradicts hypothesis (H). Therefore

$$\lambda\left(\bigcap_{i=0}^{n} g_{i,0}^{-1}(W\setminus W_0)\right) = \lambda\left(\bigcap_{i=0}^{n}\{x\in W : \alpha_i x \in W\}\right). \tag{5}$$

If $U \subset G$ is any symmetric zero neighborhood such that

$$\underbrace{U+U+\dots+U}_{\alpha} \subset W, \quad \text{where } \alpha := \max_{1\le i\le n}|\alpha_i|,$$

then

$$U \subset \bigcap_{i=0}^{n}\{x\in W : \alpha_i x\in W\},$$

hence equality (5) yields $\varphi(0) \geq \lambda(U) > 0$.

By the continuity of φ, there is a zero neighborhood $V \subset W$ such that $\varphi|_V > 0$, which implies that for every $y \in V$ there exists $x \in W$ with $g_i(x,y) \in W\setminus W_0$ for $i = 0,1,\dots,n$. For such x and y equation (2), jointly with condition (v), yields

$$\Gamma(z, f(y)) = \Phi(x, y, z, z, \dots, z) = z.$$

Taking, in particular, any $y \in V\cap(W\setminus W_0)$ we get $\Gamma(z,z) = z$. Hence, by virtue of condition (i), we have $f(y) = z$ for every $y \in V$, which completes the proof. □

Remark. The above proof works with no essential changes in the case where f is assumed to be Lusin λ-measurable only on a certain zero neighborhood. By virtue of Lusin's theorem (see, e.g., [5, §2.7]), if the space X is supposed to be second countable and $f : G \to X$ is Haar measurable on some measurable set $E \subset G$ then f is Lusin λ-measurable on E. Consequently, in our Theorem it may be assumed that f is Haar measurable on some zero neighborhood.

It is easily seen that every solution $f : G \to \mathbb{R}$ of equation (1) which is continuous at zero has to be continuous everywhere. As a consequence of our Theorem and the above remark we obtain the following corollary.

Corollary. *Let G be a metrizable locally compact group. If $f : G \to \mathbb{R}$ is a solution of equation* (1) *and is Haar measurable on some zero neighborhood then f is continuous.*

Acknowledgment. The authors would like to express their gratitude to Professor Karol Baron for fruitful discussions.

References

[1] K. Baron, On Baire measurable solutions of some functional equations, *Cent. Eur. J. Math.* **7** (2009), 804–808.

[2] A. Járai, On measurable solutions of functional equations, *Publ. Math. Debrecen* **26** (1979), 17–35.

[3] A. Járai, A Steinhaus type theorem, *Publ. Math. Debrecen* **47** (1995), 1–13.

[4] A. Járai, Measurability implies continuity for solutions of functional equations – even with few variables, *Aequationes Math.* **65** (2003), 236–266.

[5] A. Járai, Regularity Properties of Functional Equations in Several Variables, Advances in Mathematics, Vol. 8, *Springer*, 2005.

[6] T. Krausz, Diákköri dolgozat, *KLTE Mat. Int.*, *Debrecen*, 1980.

[7] A. Simon and P. Volkmann, Caractérisation du module d'une fonction additive à l'aide d'une équation fonctionnelle, *Aequationes Math.* **47** (1994), 60–68.

[8] P. Volkmann, On the functional equation $\min\{f(x+y), f(x-y)\} = |f(x) - f(y)|$, talk at the Seminar on Functional Equations and Inequalities in Several Variables in the Silesian University Mathematics Department, January 19, 2009.

TOMASZ KOCHANEK
INSTITUTE OF MATHEMATICS
UNIVERSITY OF SILESIA
BANKOWA 14
40-007 KATOWICE
POLAND

E-mail: tkochanek@math.us.edu.pl

MICHAŁ LEWICKI
INSTITUTE OF MATHEMATICS
UNIVERSITY OF SILESIA
BANKOWA 14
40-007 KATOWICE
POLAND

E-mail: mlewicki@math.us.edu.pl

(Received March 31, 2009; revised November 17, 2010)