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Approximation by q-parametric operators

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Abstract. We establish sufficient conditions to insure the convergence of a sequence of positive linear operators defined on $C[0,1]$. As applications we obtain quantitative estimates for some q-parametric operators.

1. Introduction

The development of the q-calculus has led to the discovery of q-parametric operators. The first example in this direction was given by A. Lupaş in 1987. He introduced a q-analogue of the well-known Bernstein operators (see [10]), denoted by $R_{n,q}(f,x)$, where $n = 1, 2, \dots,$ $q \in (0,1)$, $f \in C[0,1]$ and $x \in [0,1]$. In [14], Ostrovska defined the limit Lupaş operator $\tilde{R}_{\infty,q}(f,x)$ and proved the convergence of the sequence $\{R_{n,q}(f,x)\}$ to $\tilde{R}_{\infty,q}(f,x)$ as $n \to \infty$, uniformly for $x \in [0,1]$. The case $q \in (1,\infty)$ was also considered in [10] (resp. in [14]).

In 1992, L. Lupaş [11] and independently, in 2000, Trif [17] considered the q-Meyer–König and Zeller operators $M_{n,q}(f,x)$, where $n = 1, 2, \dots,$ $q \in (0,1)$, $f \in C[0,1]$ and $x \in [0,1]$. Wang proved in [19], among others, that the sequence $\{M_{n,q}(f,x)\}$ converges to the limit q-Bernstein operator $B_{\infty,q}(f,x)$ as $n \to \infty$, uniformly for $x \in [0,1]$.

Later, in 1997, Phillips introduced a new generalization of the classical Bernstein operators, based on q-integers (see [15] and [16]). He defined the so-called q-Bernstein operators $B_{n,q}(f,x)$, where $n = 1, 2, \dots,$ $q \in (0,1)$, $f \in C[0,1]$

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and $x \in [0,1]$. From the properties of q-Bernstein operators, we mention the following one established by Il'inskii and Ostrovska [8]: the sequence $\{B_{n,q}(f,x)\}$ converges to $B_{\infty,q}(f,x)$ as $n \to \infty$, uniformly for $x \in [0,1]$.

Further important q-parametric operators were introduced and studied by S. Lewanowicz and P. Wožny [9], M.-M. Derriennic [3], V. Gupta [6], A. Aral [1], V. Gupta and H. Wang [7], A. Aral and V. Gupta [2], and G. Nowak [12], respectively.

Let (L_n) be a sequence of positive linear operators such that $L_n : C[0,1] \to C[0,1]$, $n = 1, 2, \dots$. Motivated by the above convergence results, we propose to obtain sufficient conditions to insure the convergence of the sequence (L_n) to its limit operator denoted by L_∞. Moreover, the rate of approximation $\|L_n f - L_\infty f\|$ will be estimated by the second order Ditzian–Totik modulus of smoothness of $f \in C[0,1]$, defined by

$$\omega_\varphi^2(f,t) = \sup_{0<h\le t} \sup_{x \pm h\varphi(x) \in [0,1]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|, \tag{1}$$

where $\|\cdot\|$ denotes the uniform norm on $C[0,1]$ and φ is an admissible step-weight function on $[0,1]$ (for details see [5]). The corresponding K-functional to (1) is defined for $f \in C[0,1]$ and $t > 0$ as follows:

$$K_{2,\varphi}(f,t) = \inf\{\|f-g\| + t\,\|\varphi^2 g''\| : g \in W^2(\varphi)\},$$

where $W^2(\varphi) = \{g \in C[0,1] : g' \in AC_{loc}[0,1], \varphi^2 g'' \in C[0,1]\}$ and $g' \in AC_{loc}[0,1]$ means that g is differentiable and g' is absolutely continuous on every interval $[a,b] \subset [0,1]$. In view of [5, Theorem 2.1.1] there exists $C > 0$ such that

$$C^{-1}\omega_\varphi^2(f,\sqrt{t}\,) \le K_{2,\varphi}(f,t) \le C\omega_\varphi^2(f,\sqrt{t}\,). \tag{2}$$

Here we mention that C will denote throughout this paper a positive constant which can be different at each occurrence, and it is independent of n, f and x. Further, we use the notations $e_0(x) = 1$, $x \in [0,1]$ and $e_2(x) = x^2$, $x \in [0,1]$.

In Section 2 is established our main theorem, and in Section 3 is applied this theorem for some q-parametric operators.

2. Main result

The next result is our main theorem.

Theorem 2.1. *Let $(L_n)_{n\ge 1}$ be a sequence of positive linear operators on $C[0,1]$ and let $(\alpha_n)_{n\ge 1}$ be a positive sequence such that $\alpha_n \to 0^+$ as $n\to\infty$. If the positive sequence $(\beta_n)_{n\ge 1}$ satisfies the conditions:*

(i) $\beta_n+\beta_{n+1}+\dots+\beta_{n+p-1}\le C\alpha_n$ *for every* $n,p=1,2,\dots,$

(ii) $\|L_n g - L_{n+1}g\|\le C\beta_n\|\varphi^2 g''\|$ *for every* $g\in W^2(\varphi)$ *and* $n=1,2,\dots,$

then there exists a positive linear operator L_∞ on $C[0,1]$ such that $\|L_n f - L_\infty f\|\to 0$ as $n\to\infty$, for every $f\in C[0,1]$. Furthermore,

$$\|L_n f - L_\infty f\| \le C\,\omega_\varphi^2(f,\sqrt{\alpha_n}\,), \tag{3}$$

where $f\in C[0,1]$ and $n=1,2,\dots$ are arbitrary.

Proof of Theorem 2.1. We mention that $\|L_n f - L_\infty f\|\to 0$ as $n\to\infty$, is a consequence of (3), because $\alpha_n\to 0^+$ as $n\to\infty$.

Furthermore, in view of (ii) and (i), we find for every $g\in W^2(\varphi)$ and $n,p = 1,2,\dots$ that

$$\begin{aligned}\|L_n g - L_{n+p}g\| &\le \|L_n g - L_{n+1}g\| + \|L_{n+1}g - L_{n+2}g\| + \dots + \|L_{n+p-1}g - L_{n+p}g\|\\ &\le C(\beta_n+\beta_{n+1}+\dots+\beta_{n+p-1})\|\varphi^2 g''\| \le C\alpha_n\|\varphi^2 g''\|.\end{aligned} \tag{4}$$

Let $g=e_0$ in (4). Then $L_n e_0 = L_{n+p}e_0$ for all $n,p=1,2,\dots$ In this case the positivity of L_n implies that

$$\begin{aligned}|L_n(f,x)| &\le L_n(|f|,x)\le L_n(\|f\|e_0,x) = \|f\|L_n(e_0,x) = \|f\|L_1(e_0,x)\\ &\le \|f\|\,\|L_1e_0\|\end{aligned}$$

for $x\in[0,1]$, $f\in C[0,1]$ and $n=1,2,\dots$ Hence $\|L_n f\|\le \|L_1 e_0\|\|f\|$ for $f\in C[0,1]$ and $n=1,2,\dots$ This means that

$$\|L_n\|\le \|L_1e_0\| < \infty \tag{5}$$

for all $n=1,2,\dots$

On the other hand, $W^2(\varphi)$ is dense in $C[0,1]$. Then, by the well-known Banach-Steinhaus theorem (see [4]), it is sufficient to prove the convergence of the sequence $(L_n g)_{n\ge 1}$ in $C[0,1]$, for each $g\in W^2(\varphi)$. Because $\alpha_n\to 0^+$ as $n\to\infty$, we get, in view of (4), that $(L_n g)_{n\ge1}$ is a Cauchy-sequence and therefore converges in $C[0,1]$. In conclusion there exists an operator L_∞ on $C[0,1]$ such that $\|L_n f - L_\infty f\| \to 0$ as $n \to \infty$, for all $f \in C[0,1]$. This also implies that L_∞ is a positive linear operator on $C[0,1]$, because L_n are positive linear operators for all $n = 1, 2, \ldots$

Further, by (5),

$$\|L_n f\| \le \|L_n\| \, \|f\| \le \|L_1 e_0\| \, \|f\| \tag{6}$$

for each $f \in C[0,1]$. Because we have the convergence $L_n f \to L_\infty f$ in the uniform norm for all $f \in C[0,1]$, then (6) implies

$$\|L_\infty f\| \le \|L_1 e_0\| \, \|f\| \tag{7}$$

for each $f \in C[0,1]$.

Now let $p \to \infty$ in (4). Then

$$\|L_n g - L_\infty g\| \le C\alpha_n \|\varphi^2 g''\|, \tag{8}$$

where $g \in W^2(\varphi)$ and $n = 1, 2, \ldots$ By combining (6), (7) and (8), we find for every $f \in C[0,1]$, that

$$\begin{aligned} \|L_n f - L_\infty f\| &\le \|L_n f - L_n g\| + \|L_n g - L_\infty g\| + \|L_\infty g - L_\infty f\| \\ &\le 2\|L_1 e_0\| \, \|f - g\| + C\alpha_n \|\varphi^2 g''\| \le C\{\|f - g\| + \alpha_n \|\varphi^2 g''\|\}. \end{aligned}$$

Taking the infimum on the right-hand side over all $g \in W^2(\varphi)$, we get

$$\|L_n f - L_\infty f\| \le C K_{2,\varphi}(f, \alpha_n).$$

Hence, by (2), we obtain the estimate (3). This completes the proof of our theorem. □

We mention that WANG established in [18] a Korovkin-type theorem, which insures for a sequence (L_n) of positive linear operators on $C[0,1]$ that there exists an operator L_∞ on $C[0,1]$ such that $\|L_n f - L_\infty\| \to 0$ as $n \to \infty$, for each $f \in C[0,1]$. Our theorem is different from Wang's result.

3. Applications

In this section we shall apply Theorem 2.1 for some q-parametric operators, namely for the q-Bernstein operator defined by PHILLIPS [15], for the q-Meyer–König and Zeller operator introduced by TRIF [17] (see also [11]) and for a q-analogue of the Bernstein operator considered by LUPAŞ in [10].

1° Let $0 < q \leq 1$. For each non-negative integer k, the q-integers $[k]$ and the q-factorials $[k]!$ are defined by

$$[k] = \begin{cases} 1 + q + \cdots + q^{k-1}, & \text{if } \ k \geq 1 \\ 0, & \text{if } \ k = 0 \end{cases}$$

and

$$[k]! = \begin{cases} [1][2]\ldots[k], & \text{if } \ k \geq 1 \\ 1, & \text{if } \ k = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

In [15], Phillips defined the following generalization of the classical Bernstein operators, based on q-integers. For each $n = 1, 2, \ldots$ and $f \in C[0,1]$, we define the q-Bernstein operators as

$$(B_{n,q}f)(x) \equiv B_{n,q}(f, x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) p_{n,k}(q, x),$$

where $x \in [0,1]$ and

$$p_{n,k}(q, x) = \begin{cases} \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)(1-xq)\ldots(1-xq^{n-k-1}), & \text{if } 0 \leq k \leq n-1 \\ x^n, & \text{if } k = n. \end{cases}$$

For $q = 1$, we recover the well-known Bernstein operators.

By [15, (15)], we have $B_{n,q}(e_2, x) = x^2 + [n]^{-1} x(1-x)$, $x \in [0,1]$. Hence

$$B_{n,q}(e_2, x) - x^2 = \frac{1}{[n]}\, \varphi^2(x), \tag{9}$$

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ and $n = 1, 2, \ldots$

Using [13, (3.2)], we have for $g \in W^2(\varphi)$, $x \in [0,1]$ and $n = 1, 2, \ldots,$

$$B_{n,q}(g, x) - B_{n+1,q}(g, x) = \sum_{k=1}^{n} a_{n,k}(g)\, p_{n+1,k}(q, x), \tag{10}$$

where

$$a_{n,k}(g) = \frac{[n+1-k]}{[n+1]} g\left(\frac{[k]}{[n]}\right) + q^{n+1-k} \frac{[k]}{[n+1]} g\left(\frac{[k-1]}{[n]}\right) - g\left(\frac{[k]}{[n+1]}\right). \tag{11}$$

By Taylor's formula, we find

$$\begin{aligned} g\left(\frac{[k]}{[n]}\right) &= g\left(\frac{[k]}{[n+1]}\right) + \left(\frac{[k]}{[n]} - \frac{[k]}{[n+1]}\right) g'\left(\frac{[k]}{[n+1]}\right) \\ &\quad + \int_{[k]/[n+1]}^{[k]/[n]} \left(\frac{[k]}{[n]} - u\right) g''(u) \end{aligned}$$

and

$$\begin{aligned} g\left(\frac{[k-1]}{[n]}\right) &= g\left(\frac{[k]}{[n+1]}\right) + \left(\frac{[k-1]}{[n]} - \frac{[k]}{[n+1]}\right) g'\left(\frac{[k]}{[n+1]}\right) \\ &\quad + \int_{[k]/[n+1]}^{[k-1]/[n]} \left(\frac{[k-1]}{[n]} - u\right) g''(u)\, du. \end{aligned}$$

Then, in view of (11), we obtain

$$\begin{aligned} a_{n,k}(g) &= \frac{[n+1-k]}{[n+1]}\, g\left(\frac{[k]}{[n]}\right) + q^{n+1-k} \frac{[k]}{[n+1]}\, g\left(\frac{[k-1]}{[n]}\right) \\ &\quad - \frac{[n+1-k] + q^{n+1-k}[k]}{[n+1]}\, g\left(\frac{[k]}{[n+1]}\right) \\ &= \frac{[n+1-k]}{[n+1]} \left(\frac{[k]}{[n]} - \frac{[k]}{[n+1]}\right) g'\left(\frac{[k]}{[n+1]}\right) \\ &\quad + \frac{[n+1-k]}{[n+1]} \int_{[k]/[n+1]}^{[k]/[n]} \left(\frac{[k]}{[n]} - u\right) g''(u)\, du \\ &\quad + \frac{q^{n+1-k}[k]}{[n+1]} \left(\frac{[k-1]}{[n]} - \frac{[k]}{[n+1]}\right) g'\left(\frac{[k]}{[n+1]}\right) \\ &\quad + \frac{q^{n+1-k}[k]}{[n+1]} \int_{[k]/[n+1]}^{[k-1]/[n]} \left(\frac{[k-1]}{[n]} - u\right) g''(u)\, du \\ &= \frac{[n+1-k]}{[n+1]} \int_{[k]/[n+1]}^{[k]/[n]} \left(\frac{[k]}{[n]} - u\right) g''(u)\, du \\ &\quad + \frac{q^{n+1-k}[k]}{[n+1]} \int_{[k]/[n+1]}^{[k-1]/[n]} \left(\frac{[k-1]}{[n]} - u\right) g''(u)\, du, \end{aligned} \tag{12}$$

because

$$\frac{[n+1-k]}{[n+1]} \left(\frac{[k]}{[n]} - \frac{[k]}{[n+1]}\right) + \frac{q^{n+1-k}[k]}{[n+1]} \left(\frac{[k-1]}{[n]} - \frac{[k]}{[n+1]}\right)$$

$$= \frac{[k]}{[n][n+1]^2}\left\{[n+1-k]([n+1]-[n]) + q^{n+1-k}([k-1][n+1]-[k][n])\right\}$$
$$= \frac{[k]}{[n][n+1]^2}\left\{[n+1-k]q^n + q^{n+1-k}(-q^{k-1}[n+1-k])\right\} = 0.$$

Taking into account (12), (9) and the estimate

$$\left|\int_x^t (t-u)g''(u)\,du\right| \le (t-x)^2\varphi^{-2}(x)\|\varphi^2 g''\|, \tag{13}$$

(see [5, Lemma 9.6.1]), we have

$$\begin{aligned}
|a_{n,k}(g)| &\le \frac{[n+1-k]}{[n+1]}\left|\int_{[k]/[n+1]}^{[k]/[n]}\left(\frac{[k]}{[n]}-u\right)g''(u)\,du\right| \\
&\quad + \frac{q^{n+1-k}[k]}{[n+1]}\left|\int_{[k]/[n+1]}^{[k-1]/[n]}\left(\frac{[k-1]}{[n]}-u\right)g''(u)\,du\right| \\
&\le \frac{[n+1-k]}{[n+1]}\left(\frac{[k]}{[n]}-\frac{[k]}{[n+1]}\right)^2\varphi^{-2}\left(\frac{[k]}{[n+1]}\right)\|\varphi^2 g''\| \\
&\quad + \frac{q^{n+1-k}[k]}{[n+1]}\left(\frac{[k-1]}{[n]}-\frac{[k]}{[n+1]}\right)^2\varphi^{-2}\left(\frac{[k]}{[n+1]}\right)\|\varphi^2 g''\| \\
&= \left\{\frac{[n+1-k][k]([n+1]-[n])^2}{[n]^2[n+1]([n+1]-[k])}\right. \\
&\quad \left. + \frac{q^{n+1-k}([k-1][n+1]-[k][n])^2}{[n]^2[n+1]([n+1]-[k])}\right\}\|\varphi^2 g''\| \\
&= \left\{\frac{[n+1-k][k]q^{2n}}{[n]^2[n+1]q^k[n+1-k]}\right. \\
&\quad \left. + \frac{q^{n+1-k}(-q^{k-1}[n+1-k])^2}{[n]^2[n+1]q^k[n+1-k]}\right\}\|\varphi^2 g''\| \\
&= \frac{q^{n-1}}{[n]^2[n+1]}\left\{q^{n+1-k}[k]+[n+1-k]\right\}\|\varphi^2 g''\| = \frac{q^{n-1}}{[n]^2}\|\varphi^2 g''\|.
\end{aligned} \tag{14}$$

Then, by (10), (14) and $B_{n+1,q}(e_0,x)=1$ (see [15, (13)]), we have

$$\begin{aligned}
|B_{n,q}(g,x)-B_{n+1,q}(g,x)| &\le \sum_{k=1}^{n}|a_{n,k}(g)|\,p_{n+1,k}(q,x) \\
&\le \sum_{k=1}^{n}\frac{q^{n-1}}{[n]^2}\|\varphi^2 g''\|p_{n+1,k}(q,x) \\
&\le \frac{q^{n-1}}{[n]^2}B_{n+1,q}(e_0,x)\|\varphi^2 g''\| = \frac{q^{n-1}}{[n]^2}\|\varphi^2 g''\|.
\end{aligned} \tag{15}$$

Let $\beta_n = q^{n-1}/[n]^2$, $n = 1, 2, \dots$, where $0 < q < 1$. Then

$$\begin{aligned}\beta_n + \beta_{n+1} + \cdots + \beta_{n+p-1} &\le \frac{q^{n-1}}{[n]^2}\left(1 + q + \cdots + q^{p-1}\right)\\ &\le \frac{1}{[n]^2}\frac{q^{n-1}}{1-q} \le \frac{q^{n-1}}{(1-q^n)^2}\end{aligned}$$

for all $n, p = 1, 2, \dots$, i.e. we obtain the condition (i) of Theorem 2.1 with $\alpha_n = q^{n-1}/(1-q^n)^2$, $n = 1, 2, \dots$. Obviously $\alpha_n \to 0^+$ as $n \to \infty$.

Due to (15), we get $\|B_{n,q}g - B_{n+1,q}g\| \le \frac{q^{n-1}}{[n]^2}\|\varphi^2 g''\|$, which is the condition (ii) of Theorem 2.1 for the q-Bernstein operator. Applying Theorem 2.1, we have the following statement.

Let $q \in (0,1)$ be given. Then there exists a positive linear operator $L_{\infty,q}$ on $C[0,1]$ such that

$$\|B_{n,q}f - L_{\infty,q}f\| \le C\omega_\varphi^2\big(f, \sqrt{q^{n-1}}/(1-q^n)\big)$$

for all $f \in C[0,1]$ and $n = 1, 2, \dots$. The operator $L_{\infty,q}$ coincides with the limit q-Bernstein operator $B_{\infty,q}$, because in [8] it is proved that $\{B_{n,q}f\}$ converges to $B_{\infty,q}f$ as $n \to \infty$, uniformly on $[0,1]$.

2° Let $0 < q \le 1$. For each $n = 1, 2, \dots$ and $f \in C[0,1]$, we define the q-Meyer–König and Zeller operators [17] as follows.

$$\begin{aligned}(M_{n,q}f)(x) &\equiv M_{n,q}(f,x)\\ &= \begin{cases}\displaystyle\prod_{s=0}^{n}(1-xq^s)\sum_{k=0}^{\infty} f\left(\frac{[k]}{[n+k]}\right)\begin{bmatrix} n+k \\ k\end{bmatrix} x^k, & \text{if } \ 0 \le x < 1\\ f(1), & \text{if } \ x = 1.\end{cases}\end{aligned}$$

For $q = 1$, we recover the well-known Meyer–König and Zeller operators.

By [17, (2.3)–(2.4)], we have

$$\begin{aligned}|M_{n,q}(e_2,x) - x^2| &\le \frac{1}{[n-1]}x(1-x)(1-xq^n)\\ &\quad + \frac{[2]q^{n-1}}{[n-1][n-2]}x(1-x)(1-xq)(1-xq^n)\\ &\le \frac{1}{[n-1]}\varphi^2(x) + \frac{1}{[n-1]}\varphi^2(x) \le \frac{4}{[n]}\varphi^2(x), \qquad (16)\end{aligned}$$

where $n \ge 4$ and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$.

On the other hand, by [17, Theorem 3.3], we have for $x \in [0,1]$, $n = 1, 2, \ldots$ and $g \in W^2(\varphi)$,

$$\begin{aligned}
&M_{n,q}(g,x) - M_{n+1,q}(g,x) \\
&\quad = x \prod_{s=0}^{n}(1 - xq^s) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k \left\{ q^{n+1} \frac{[n+k+1]}{[n+1]} g\left(\frac{[k]}{[n+k+1]}\right) \right. \\
&\quad - q^{n+1} \frac{[n+k+1]}{[n+1]} g\left(\frac{[k+1]}{[n+k+2]}\right) - \frac{[n+k+1]}{[k+1]} g\left(\frac{[k+1]}{[n+k+2]}\right) \\
&\quad \left. + \frac{[n+k+1]}{[k+1]} g\left(\frac{[k+1]}{[n+k+1]}\right) \right\}. \qquad (17)
\end{aligned}$$

By Taylor's formula, we obtain

$$\begin{aligned}
&g\left(\frac{[k]}{[n+k+1]}\right) \\
&\quad = g\left(\frac{[k+1]}{[n+k+2]}\right) + \left(\frac{[k]}{[n+k+1]} - \frac{[k+1]}{[n+k+2]}\right) g'\left(\frac{[k+1]}{[n+k+2]}\right) \\
&\quad + \int_{[k+1]/[n+k+2]}^{[k]/[n+k+1]} \left(\frac{[k]}{[n+k+1]} - u\right) g''(u)\, du \\
&\quad = g\left(\frac{[k+1]}{[n+k+2]}\right) - \frac{q^k[n+1]}{[n+k+1][n+k+2]} g'\left(\frac{[k+1]}{[n+k+2]}\right) \\
&\quad + \int_{[k+1]/[n+k+2]}^{[k]/[n+k+1]} \left(\frac{[k]}{[n+k+1]} - u\right) g''(u)\, du \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
&g\left(\frac{[k+1]}{[n+k+1]}\right) \\
&\quad = g\left(\frac{[k+1]}{[n+k+2]}\right) + \left(\frac{[k+1]}{[n+k+1]} - \frac{[k+1]}{[n+k+2]}\right) g'\left(\frac{[k+1]}{[n+k+2]}\right) \\
&\quad + \int_{[k+1]/[n+k+2]}^{[k+1]/[n+k+1]} \left(\frac{[k+1]}{[n+k+1]} - u\right) g''(u) du \\
&\quad = g\left(\frac{[k+1]}{[n+k+2]}\right) + \frac{q^k[n+1]}{[n+k+1][n+k+2]} g'\left(\frac{[k+1]}{[n+k+2]}\right) \\
&\quad + \int_{[k+1]/[n+k+2]}^{[k+1]/[n+k+1]} \left(\frac{[k+1]}{[n+k+1]} - u\right) g''(u) du, \qquad (19)
\end{aligned}$$

respectively. Then, in view of (17), (18), (19), (16), (13) and $M_{n,q}(e_0,x)=1$, $x\in[0,1]$ (see [17, (2.1)]), we have

$$\begin{aligned}
&|M_{n,q}(g,x)-M_{n+1,q}(g,x)| \le x\prod_{s=0}^{n}(1-xq^s)\sum_{k=0}^{\infty}\begin{bmatrix} n+k\\ k\end{bmatrix}x^k\\
&\quad\times\left|\,q^{n+1}\frac{[n+k+1]}{[n+1]}\left(\frac{[k]}{[n+k+1]}-\frac{[k+1]}{[n+k+2]}\right)+\frac{[n+k+1]}{[k+1]}\right.\\
&\quad\times\left.\left(\frac{[k+1]}{[n+k+1]}-\frac{[k+1]}{[n+k+2]}\right)\right|\left|\,g'\left(\frac{[k+1]}{[n+k+2]}\right)\right|\\
&\quad+\; x\prod_{s=0}^{n}(1-xq^s)\sum_{k=0}^{\infty}\begin{bmatrix} n+k\\ k\end{bmatrix}x^k\\
&\quad\times\left\{q^{n+1}\frac{[n+k+1]}{[n+1]}\left|\int_{[k+1]/[n+k+2]}^{[k]/[n+k+1]}\left(\frac{[k]}{[n+k+1]}-u\right)g''(u)du\right|\right.\\
&\quad+\left.\frac{[n+k+1]}{[k+1]}\left|\int_{[k+1]/[n+k+2]}^{[k+1]/[n+k+1]}\left(\frac{[k+1]}{[n+k+1]}-u\right)g''(u)\,du\right|\right\}\\
&=x\prod_{s=0}^{n}(1-xq^s)\sum_{k=0}^{\infty}\begin{bmatrix} n+k\\ k\end{bmatrix}x^k\\
&\quad\times\left\{q^{n+1}\frac{[n+k+1]}{[n+1]}\left|\int_{[k+1]/[n+k+2]}^{[k]/[n+k+1]}\left(\frac{[k]}{[n+k+1]}-u\right)g''(u)du\right|\right.\\
&\quad+\left.\frac{[n+k+1]}{[k+1]}\left|\int_{[k+1]/[n+k+2]}^{[k+1]/[n+k+1]}\left(\frac{[k+1]}{[n+k+1]}-u\right)g''(u)\,du\right|\right\}\\
&\le x\prod_{s=0}^{n}(1-xq^s)\sum_{k=0}^{\infty}\begin{bmatrix} n+k\\ k\end{bmatrix}x^k\\
&\quad\times\left\{q^{n+1}\frac{[n+k+1]}{[n+1]}\left(\frac{[k]}{[n+k+1]}-\frac{[k+1]}{[n+k+2]}\right)^2+\frac{[n+k+1]}{[k+1]}\right.\\
&\quad\times\left.\left(\frac{[k+1]}{[n+k+1]}-\frac{[k+1]}{[n+k+2]}\right)^2\right\}\varphi^{-2}\left(\frac{[k+1]}{[n+k+2]}\right)\|\varphi^2 g''\|\\
&=x\prod_{s=0}^{n}(1-xq^s)\sum_{k=0}^{\infty}\begin{bmatrix} n+k\\ k\end{bmatrix}x^k\\
&\quad\times\left\{q^{n+1}\frac{[n+k+1]}{[n+1]}\,\frac{q^{2k}[n+1]^2}{[n+k+1]^2[n+k+2]^2}\,\frac{[n+k+2]}{[k+1]}\right.
\end{aligned}$$

$$\times \frac{[n+k+2]}{[n+k+2]-[k+1]} + \frac{[n+k+1]}{[k+1]} \frac{q^{2n+2k+2}[k+1]^2}{[n+k+1]^2[n+k+2]^2}$$
$$\times \left.\frac{[n+k+2]}{[k+1]} \frac{[n+k+2]}{[n+k+2]-[k+1]}\right\} \|\varphi^2 g''\|$$
$$= x\prod_{s=0}^{n}(1-xq^s)\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix} x^k \frac{q^n}{[n+1]} \frac{q^k}{[k+1]} \frac{[n+k+2]}{[n+k+1]} \|\varphi^2 g''\|$$
$$\le \frac{2q^n}{[n+1]} x\prod_{s=0}^{n}(1-xq^s)\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix} x^k \|\varphi^2 g''\|$$
$$\le \frac{2q^n}{[n+1]} M_{n,q}(e_0,x)\|\varphi^2 g''\| = \frac{2q^n}{[n+1]} \|\varphi^2 g''\|.$$

Hence
$$\|M_{n,q}g - M_{n+1,q}g\| \le \frac{2q^n}{[n+1]} \|\varphi^2 g''\| \tag{20}$$
for all $n = 1,2,\dots$ and $g \in W^2(\varphi)$.

In this case we consider $\beta_n = q^n/[n+1]$, $n = 1,2,\dots$ and $0 < q < 1$. Then
$$\beta_n + \beta_{n+1} + \cdots + \beta_{n+p-1} \le \frac{q^n}{[n+1]}(1+q+\cdots+q^{p-1}) \le \frac{q^n}{1-q^{n+1}}$$
for all $n, p = 1,2,\dots$ We set $\alpha_n = q^n/(1-q^{n+1})$, $n = 1,2,\dots$ Then $\alpha_n \to 0^+$ as $n \to \infty$. In conclusion, in view of (20), we can apply Theorem 2.1.

Let $q \in (0,1)$ be given. Then there exists a positive linear operator $L_{\infty,q}$ on $C[0,1]$ such that
$$\|M_{n,q}f - L_{\infty,q}f\| \le C\omega_\varphi^2(f, \sqrt{q^n/(1-q^{n+1})})$$
for all $f \in C[0,1]$ and $n = 4,5,\dots$

In this case $L_{\infty,q}$ is identical with $B_{\infty,q}$, because in [19] it is proved that $\{M_{n,q}f\}$ converges to $B_{\infty,q}f$ as $n \to \infty$, uniformly on $[0,1]$.

3° Let $0 < q \le 1$. Following [10], the positive linear operators $R_{n,q} : C[0,1] \to C[0,1]$, defined by
$$(R_{n,q}f)(x) \equiv R_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right)\begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k-1)/2}x^k(1-x)^{n-k}}{(1-x+xq)\dots(1-x+xq^{n-1})}$$
are called the q-analogue of the Bernstein operators. For $q = 1$, we recover the well-known Bernstein operators. Due to [14, Lemma 1], we have
$$R_{n,q}(e_0,x) = 1 \tag{21}$$

and

$$|R_{n,q}(e_2,x)-x^2| = \frac{1}{[n]}x(1-x)\cdot\frac{1-x+xq^n}{1-x+xq} \le \frac{1}{[n]}x(1-x). \tag{22}$$

Thus we set $\varphi(x)=\sqrt{x(1-x)}$, $x\in[0,1]$. Taking into account [10, (2)], we have for $g\in W^2(\varphi)$,

$$R_{n+1,q}(g,x)-R_{n,q}(g,x) = \frac{x(1-x)}{(1-x+xq)\dots(1-x+xq^n)}\sum_{k=0}^{n-1}\begin{bmatrix} n-1\\ k\end{bmatrix} q^{k(k-1)/2}x^k$$
$$\times (1-x)^{n-1-k}\left\{\frac{q^k[n]}{[k+1]}\,g\left(\frac{[k+1]}{[n]}\right)-\frac{q^k[n][n+1]}{[k+1][n-k]}\,g\left(\frac{[k+1]}{[n+1]}\right)\right.$$
$$\left.+\frac{q^n[n]}{[n-k]}\,g\left(\frac{[k]}{[n]}\right)\right\}. \tag{23}$$

By Taylor's formula, we find

$$g\left(\frac{[k+1]}{[n]}\right) = g\left(\frac{[k+1]}{[n+1]}\right)+\left(\frac{[k+1]}{[n]}-\frac{[k+1]}{[n+1]}\right)g'\left(\frac{[k+1]}{[n+1]}\right)$$
$$+\int_{[k+1]/[n+1]}^{[k+1]/[n]}\left(\frac{[k+1]}{[n]}-u\right)g''(u)\,du \tag{24}$$

and

$$g\left(\frac{[k]}{[n]}\right) = g\left(\frac{[k+1]}{[n+1]}\right)+\left(\frac{[k]}{[n]}-\frac{[k+1]}{[n+1]}\right)g'\left(\frac{[k+1]}{[n+1]}\right)$$
$$+\int_{[k+1]/[n+1]}^{[k]/[n]}\left(\frac{[k]}{[n]}-u\right)g''(u)\,du, \tag{25}$$

respectively. Because

$$\frac{q^k[n]}{[k+1]}+\frac{q^n[n]}{[n-k]} = \frac{q^k[n][n+1]}{[k+1][n-k]}$$

and

$$\frac{q^k[n]}{[k+1]}\left(\frac{[k+1]}{[n]}-\frac{[k+1]}{[n+1]}\right)+\frac{q^n[n]}{[n-k]}\left(\frac{[k]}{[n]}-\frac{[k+1]}{[n+1]}\right)=0,$$

by combining (23), (24), (25), (22), (13) and (21), we obtain

$$|R_{n+1,q}(g,x)-R_{n,q}(g,x)| \le \frac{x(1-x)}{(1-x+xq)\dots(1-x+xq^n)}\sum_{k=0}^{n-1}\begin{bmatrix} n-1\\ k\end{bmatrix} q^{k(k-1)/2}$$
$$\times x^k(1-x)^{n-1-k}\left\{\frac{q^k[n]}{[k+1]}\left|\int_{[k+1]/[n+1]}^{[k+1]/[n]}\left(\frac{[k+1]}{[n]}-u\right)g''(u)\,du\right|\right.$$
$$\left.+\frac{q^n[n]}{[n-k]}\left|\int_{[k+1]/[n+1]}^{[k]/[n]}\left(\frac{[k]}{[n]}-u\right)g''(u)\,du\right|\right\}$$

$$
\begin{aligned}
&\leq \frac{x(1-x)}{(1-x+xq)\dots(1-x+xq^n)} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{k(k-1)/2} x^k (1-x)^{n-1-k} \\
&\quad \times \left\{ \frac{q^k[n]}{[k+1]} \left(\frac{[k+1]}{[n]} - \frac{[k+1]}{[n+1]} \right)^2 \varphi^{-2}\left(\frac{[k+1]}{[n+1]} \right) \|\varphi^2 g''\| \right. \\
&\quad \left. + \frac{q^n[n]}{[n-k]} \left(\frac{[k]}{[n]} - \frac{[k+1]}{[n+1]} \right)^2 \varphi^{-2}\left(\frac{[k+1]}{[n+1]} \right) \|\varphi^2 g''\| \right\} \\
&= \frac{x(1-x)}{(1-x+xq)\dots(1-x+xq^n)} \|\varphi^2 g''\| \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{k(k-1)/2} \\
&\quad \times x^k (1-x)^{n-1-k} \left\{ \frac{q^k[n]}{[k+1]} \frac{[k+1]^2 q^{2n}}{[n]^2[n+1]^2} \frac{[n+1]^2}{[k+1]([n+1]-[k+1])} \right. \\
&\quad \left. + \frac{q^n[n]}{[n-k]} \frac{q^{2k}[n-k]^2}{[n]^2[n+1]^2} \frac{[n+1]^2}{[k+1]([n+1]-[k+1])} \right\} \\
&= \frac{x(1-x)}{(1-x+xq)\dots(1-x+xq^n)} \|\varphi^2 g''\| \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{k(k-1)/2} \\
&\quad \times x^k (1-x)^{n-1-k} \frac{q^{n-1+k}[n+1]}{[n][n-k][k+1]} \\
&= \frac{\|\varphi^2 g''\|}{(1-x+xq)\dots(1-x+xq^n)} \sum_{k=0}^{n-1} \frac{q^{n-1}}{[n]^2} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} q^{(k+1)k/2} x^{k+1} (1-x)^{n-k} \\
&= \frac{q^{n-1}}{[n]^2} \|\varphi^2 g''\| \sum_{k=1}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix} \frac{q^{k(k-1)/2} x^k (1-x)^{n+1-k}}{(1-x+xq)\dots(1-x+xq^n)} \\
&\leq \frac{q^{n-1}}{[n]^2} \|\varphi^2 g''\| R_{n+1,q}(e_0, x) = \frac{q^{n-1}}{[n]^2} \|\varphi^2 g''\|.
\end{aligned}
$$

Similarly to the case **1**°, we can choose the sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ as $\alpha_n = q^{n-1}/(1-q^n)^2$ and $\beta_n = q^{n-1}/[n]^2$, where $n = 1, 2, \dots$ and $0 < q < 1$. By Theorem 2.1, we have the following statement.

Let $q \in (0,1)$ be given. Then there exists a positive linear operator $L_{\infty,q}$ on $C[0,1]$ such that

$$\|R_{n,q}f - L_{\infty,q}f\| \leq C\omega_\varphi^2\Big(f, \sqrt{q^{n-1}}/(1-q^n)\Big)$$

for all $f \in C[0,1]$ and $n = 1, 2, \dots$

The operator $L_{\infty,q}$ is identical with the limit q-analogue of the Bernstein operator, denoted by $\tilde{R}_{\infty,q}$, because in [14] it is proved that $\{R_{n,q}f\}$ converges to $\tilde{R}_{\infty,q}f$ as $n \to \infty$, uniformly on $[0,1]$.

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