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A note on n-clean group rings

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Abstract. Let R be an associative ring with identity. An element $x \in R$ is clean if x can be written as the sum of a unit and an idempotent in R. R is said to be clean if all of its elements are clean. Let n be a positive integer. An element $x \in R$ is n-clean if it can be written as the sum of an idempotent and n units in R. R is said to be n-clean if all of its elements are n-clean. In this paper we obtain conditions which are necessary or sufficient for a group ring to be n-clean.

1. Introduction

Throughout this paper all rings are associative with identity. The notion of clean rings was first introduced by Nicholson in [4]. An element x in a ring R is said to be clean if x can be written as the sum of a unit and an idempotent in R. The ring R is clean if every element in R is clean. In [6], Xiao and Tong generalised clean rings to n-clean rings. For a positive integer n, an element x in a ring R is n-clean if x can be written as the sum of an idempotent and n units in R. A ring R is n-clean if all of its elements are n-clean. Clearly, a 1-clean ring is a clean ring and vice versa.

It is known by work of Chen and Zhou [1], as well as McGovern [3], that for a commutative ring R and an abelian group G, if RG is clean, then G is locally finite. We extend this result to n-clean rings in this paper. We also show that a partial converse of this result is true. That is, we show that if R is a commutative clean ring and G is a locally finite p-group where p is some prime with $p \in J(R)$,

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then RG is n-clean. The notation $J(R)$ as usual denotes the Jacobson radical of the ring R.

2. Some preliminary results

In this section we obtain some results which will be used in the proofs of the main results.

Proposition 2.1. *Let n be a positive integer. Then every homomorphic image of an n-clean ring is n-clean.*

Proof. Let R be an n-clean ring and let $\phi : R \to S$ be a ring epimorphism. Let $x \in S$. Then $x = \phi(y)$ for some $y \in R$. Since R is n-clean, then $y = e + u_1 + \cdots + u_n$ for some idempotent e and units $u_1, \ldots, u_n$ in R. Since ϕ is an epimorphism, we then have that $\phi(e)$ is an idempotent, $\phi(u_i)$ is a unit in S $(i = 1, \ldots, n)$ and $x = \phi(y) = \phi(e) + \phi(u_1) + \cdots + \phi(u_n)$, that is, x is n-clean in S. It follows that $\phi(R) = S$ is n-clean. □

Proposition 2.2. *Let R be a ring and let n be a positive integer. If R is local, then R is n-clean.*

Proof. Since R is local, the only idempotents in R are 0 and 1. Let $x \in R$.

Case 1: x is a unit

Note that

$$x = \begin{cases} 0 + (x + (-x)) + \cdots + (x + (-x)) + x, & \text{if } n \text{ is odd} \\ 1 + (x + (-1)) + (x + (-x)) + \cdots + (x + (-x)), & \text{if } n \text{ is even.} \end{cases}$$

Case 2: x is not a unit

In this case $1 - x$ is a unit. Hence $x - 1$ is also a unit and we have

$$x = \begin{cases} 1 + ((1-x) + (x-1)) + \cdots + ((1-x) + (x-1)) + (x-1), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \text{ is odd} \\ 0 + ((x-1) + 1) + ((1-x) + (x-1)) + \cdots + ((1-x) + (x-1)), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \text{ is even.} \end{cases}$$

In both cases, we have that x can be written as the sum of an idempotent and n units; hence x is n-clean. □

Proposition 2.3. *A clean ring without any nontrivial idempotents is local.*

Proof. Let R be a clean ring with no nontrivial idempotents. Then for any $x \in R$, either x or $1-x$ is a unit. Suppose that R has two distinct maximal right ideals M_1 and M_2. Then there exists an element $a \in M_1$, $a \notin M_2$. Thus for every $r \in R$, $ar \in M_1$ and since M_1 is a maximal right ideal (hence, a proper ideal), it follows that ar is not a unit. Therefore $1-ar$ is a unit for every $r \in R$. Now from $M_2 + aR = R$, we have that $1 - as \in M_2$ for some $s \in R$. But since $1 - as$ is a unit, we have that $M_2 = R$ which contradicts the fact that M_2 is proper. Hence, R only has one maximal right ideal, that is, R must be local. □

Proposition 2.4. *Let R be a ring, let G be a group and let n be a positive integer. If $x \in RG$ is not n-clean, then there exists a proper ideal J of R such that R/J does not have any nontrivial central idempotent and $x + JG \in RG/JG$ is not n-clean.*

Proof. Suppose that $x \in RG$ is not n-clean. Let $\mathcal{C} = \{I \triangleleft R \mid I \neq R,\ x + IG \in RG/IG \text{ is not } n\text{-clean}\}$. Then $\mathcal{C} \neq \emptyset$ because $\{0\} \in \mathcal{C}$. Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_j \subseteq I_{j+1} \subseteq \dots$ be a chain of ideals in $\mathcal{C}$ and let $I = \cup I_i$. Then I is a proper ideal of R and $\bar{x} = x + IG \in RG/IG$ is not n-clean. Indeed, if $\bar{x}$ is n-clean, then $x + I_tG \in RG/I_tG$ is n-clean for some $t \in \mathbb{N}$; a contradiction. Therefore $I \in \mathcal{C}$. By Zorn's Lemma, $\mathcal{C}$ has a maximal element, say J.

If R/J has a nontrivial central idempotent $e + J$, then

$$R/J = (e+J)R/J \oplus ((1-e)+J)R/J \cong I_1/J \times I_2/J$$

for some ideals I_1, I_2 of R properly containing J. Then

$$RG/JG \cong (R/J)\,G \cong (I_1/J)G \times (I_2/J)G \cong I_1G/JG \times I_2G/JG. \tag{1}$$

Note that

$$RG/I_1G \cong (RG/JG)\,/\,(I_1G/JG) \cong I_2G/JG \tag{2}$$

and

$$RG/I_2G \cong (RG/JG)\,/\,(I_2G/JG) \cong I_1G/JG. \tag{3}$$

Let $(x_1 + JG,\ x_2 + JG) \in I_1G/JG \times I_2G/JG$ be the image of $x + JG$ under the isomorphism in (1). By the maximality of J in $\mathcal{C}$, $x + I_tG$ is n-clean in RG/I_tG ($t = 1, 2$). Hence, by (2) and (3), $x_t + JG$ is n-clean in I_tG/JG ($t = 1,2$). It follows by (1) that $x + JG \in RG/JG$ is n-clean. This is a contradiction since $J \in \mathcal{C}$. Hence, R/J does not have any nontrivial central idempotent. □

Corollary 2.5. *Let R be a commutative clean ring, let G be a group and let n be a positive integer. If $x \in RG$ is not n-clean, then there exists a proper ideal J of R such that R/J is local and $x + JG \in RG/JG$ is not n-clean.*

Proof. By Proposition 2.4, there exists a proper ideal J of R such that R/J does not have any nontrivial idempotent and $x + JG \in RG/JG$ is not n-clean. Since R/J is clean (by Proposition 2.1), it follows by Proposition 2.3 that R/J is local. □

In [5], Nicholson obtained sufficient conditions for a group ring to be local as follows:

Proposition 2.6. *Let R be a local ring and let G be a locally finite p-group where p is some prime with $p \in J(R)$. Then RG is local.*

3. Main results

For a ring R, let $Id(R)$ and $U(R)$ denote the set of all idempotents and the set of all units of R, respectively. We first extend an idea in [1] to obtain necessary conditions for a commutative group ring to be n-clean.

Theorem 3.1. *Let R be a commutative ring, let G be an abelian group and let n be a positive integer. If RG is n-clean, then R is n-clean and G is locally finite.*

Proof. Since $RG/\Delta \cong R$ where Δ is the augmentation ideal of RG, it follows readily by Proposition 2.1 that R is n-clean. Suppose that G is not locally finite. Then G is not torsion; hence $G/t(G)$ is nontrivial and torsion-free, where $t(G)$ is the torsion subgroup of G. Since $R(G/t(G)) \cong RG/R(t(G))$ is a homomorphic image of RG and RG is n-clean, it follows by Proposition 2.1 that $R(G/t(G))$ is n-clean. We may therefore assume that G is torsion-free. If G has rank greater than 1, then G has a torsion-free quotient G' of rank 1. But since RG' is also n-clean, we can assume that G is of rank 1. Thus, G is isomorphic to a subgroup of $(\mathbb{Q}, +)$. Since R is commutative, then R/M is a field where M is a maximal ideal of R. Furthermore, $(R/M)G$ is n-clean because $(R/M)G \cong RG/MG$ is a homomorphic image of RG (by Proposition 2.1). Hence, we can assume that R is a field. Since G is torsion-free, there exists a $g \in G$ such that $g^{-1} \neq g$. Now since $g + \cdots + g^n + g^{-1} + \cdots + g^{-n}$ is n-clean in RG, there exists a finitely generated subgroup G_1 of G such that $g \in G_1$ and $g + \cdots + g^n + g^{-1} + \cdots + g^{-n}$ is n-clean in RG_1. From above, G_1 is isomorphic to a finitely generated subgroup of $(\mathbb{Q}, +)$. Since every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic, so is G_1, and we can write $G_1 = \langle h \rangle$. Thus, $g = h^k$, $g^{-1} = h^{-k}$ for some $k \in \mathbb{N}$. Note that there is a natural isomorphism $R\langle h \rangle \cong R[x, x^{-1}]$ with

$h^k+\cdots+h^{nk}+h^{-k}+\cdots+h^{-nk} \leftrightarrow x^k+\cdots+x^{nk}+x^{-k}+\cdots+x^{-nk}$. This implies that $x^k+\cdots+x^{nk}+x^{-k}+\cdots+x^{-nk}$ is n-clean in $R[x,x^{-1}]$ which is impossible because $Id(R[x,x^{-1}]) \subseteq R$ and $U(R[x,x^{-1}]) \subseteq \{ax^i \mid 0 \neq a \in R, i \in \mathbb{Z}\}$. Hence, G must be locally finite. $\square$

In [2], it was shown that if R is a clean ring and G is a finite group such that $|G|$ is a unit in R, then RG is not necessarily clean. Here we prove the following:

Theorem 3.2. *Let R be a commutative clean ring and let G be a locally finite p-group with $p \in J(R)$. Then RG is an n-clean ring for any positive integer n.*

Proof. Let n be a positive integer and suppose that $x \in RG$ is not n-clean. By Corollary 2.5, there exists a proper ideal I of R such that R/I is local and $x+IG \in RG/IG$ is not n-clean. If $p+I$ is a unit in R/I, then $pr-1 \in I$ for some $r \in R$. But $p \in J(R)$ implies that $pr-1$ is a unit in R. Hence $I=R$; a contradiction. Thus $p+I$ is not a unit in R/I and therefore, $p+I \in J\left(R/I\right)$. By Proposition 2.6, $RG/IG \cong (R/I)\,G$ is local. It follows by Proposition 2.2 that RG/IG is n-clean; a contradiction. Hence, $x \in RG$ must be n-clean. $\square$

Remark. Let $\mathbb{Z}_{(7)} = \{\frac{m}{n} \in \mathbb{Q} \mid 7 \text{ does not divide } n\}$ and let C_3 be the cyclic group of order 3. The example in [2] that the group ring $\mathbb{Z}_{(7)}C_3$ is not clean shows that the condition $p \in J(R)$ in Theorem 3.2 is not superfluous.

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