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Some results concerning additive mappings and derivations on semiprime rings

By AJDA FOŠNER (Koper) and JOSO VUKMAN (Maribor)

Abstract. Let R be a 2-torsion free semiprime ring and let $f : R \to R$ be an additive mapping satisfying the relation $[f(x), x^2] = 0$ for all $x \in R$. We prove that in this case $[f(x), x] = 0$ holds for all $x \in R$. This result makes it possible to prove the following result. Let R be a 2-torsion free semiprime ring and let $D, G : R \to R$ be derivations. Suppose that the relation $[D^2(x) + G(x), x^2] = 0$ holds for all $x \in R$. Then D and G both map R into its center.

Throughout, R will represent an associative ring with a center $Z(R)$. A ring R is n-torsion free, where $n > 1$ is an integer, in case $nx = 0, x \in R$, implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use the commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$ for all $x, y, z \in R$. Recall that a ring R is prime if for $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies $a = 0$. An additive mapping D is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. A mapping f of a ring R into itself is called centralizing on R if $[f(x), x] \in Z(R)$ holds for all $x \in R$. In the special case when $[f(x), x] = 0$ holds for all $x \in R$ the mapping f is said to be commuting on R. An additive mapping $f : R \to R$ is called skew-commuting on R if $f(x)x + xf(x) = 0$ holds for all $x \in R$. A classical result of Posner [14] states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (Posner's second theorem). Posner's second theorem in general cannot be proved for semiprime rings as shows the following example. Take prime rings R_1, R_2, where R_1 is commutative, and set $R = R_1 \oplus R_2$. Let $D_1 : R_1 \to R_1$ be a nonzero

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derivation. A mapping $D : R \to R$, defined by $D((r_1, r_2)) = (D_1(r_1), 0)$, is then a nonzero commuting derivation. It is also easy to prove that if $D : R \to R$ is a commuting derivation on a semiprime ring R, then D maps R into $Z(R)$ (see, for example, the end of the proof of Theorem 2.1 in [17]).

We denote by A, C, and Q the central closure, the extended centroid, and the maximal right ring of quotients of a semiprime ring R, respectively. For the explanation of the central closure, the extended centroid as well as the maximal right ring of quotients of a semiprime ring we refer to [2].

In the present paper we continue the series of papers concerning arbitrary additive maps of prime and semiprime rings satisfying certain identities (see, for example [4], [5], [6], [7], [8] and the references given there).

Let us start with the following result.

Theorem 1 ([18, Theorem 4])**.** *Let R be a 2-torsion free semiprime ring. Suppose that an additive mapping $f : R \to R$ satisfies the relation*

$$\big[[f(x), x], x\big] = 0$$

for all $x \in R$. In this case f is commuting on R.

The above result was first proved by Brešar [4] in the case R is a 2-torsion free prime ring. It should be mentioned that Theorem 1 is in fact due to Brešar [9] as well.

It is our aim in this paper to prove the following result.

Theorem 2. *Let R be a 2-torsion free semiprime ring. Suppose that an additive mapping $f : R \to R$ satisfies the relation*

$$[f(x), x^2] = 0$$

for all $x \in R$. In this case f is commuting on R.

Let us point out that the above result generalizes the result proved by Brešar and Hvala in [8].

In the proof of Theorem 2 we will need the lemma and the theorem below.

Lemma 3. *Let R be a 2-torsion free prime ring and let A be its central closure. Suppose that an additive mapping $f : R \to A$ satisfies the relation*

$$[f(x), x^2] = 0$$

for all $x \in R$. In this case $[f(x), x] = 0$ holds for all $x \in R$.

Proof. The lemma was first proved by Brešar and Hvala [8] in the case when f maps into R. Fortunately, the same proof works in the case when f maps into A. $\square$

Theorem 4 ([6, Theorem 1])**.** *Let R be a 2-torsion free semiprime ring and let $f : R \to R$ be an additive mapping. If f is skew-commuting on R, then $f = 0$.*

Proof of Theorem 2. Since R is semiprime there exists a family of prime ideals $\{P_\alpha : \alpha \in I\}$ such that $\bigcap_\alpha P_\alpha = \{0\}$. Without loss of generality we may assume that the prime rings $R_\alpha = R/P_\alpha$ are 2-torsion free (see [1, page 459]). Now let us fix some $P = P_\alpha, \alpha \in I$. The theorem will be proved by showing that $[f(x), x] \in P$ for all $x \in R$.

Given $x \in R$, we will write $\overline{x}$ for the coset $x + P \in R/P$. We will denote by C the extended centroid of the prime ring R/P and by A the central closure of R/P. One can consider A as a vector space over the field C. If $Z(R/P) \neq 0$, then C can be regarded as a subspace of A and there exists a subspace B of A such that $A = B \oplus C$. By π we will denote the canonical projection of A onto B. Substituting $x + p$ for x in $[f(x), x^2] = 0$ we see that $[f(p), x^2] \in P$ for all $x \in R$ and $p \in P$, that is

$$\big[\overline{f(p)}, \overline{x}^2\big] = 0$$

for all $\overline{x} \in R/P$. This relation can be written in the form

$$\big[\overline{f(p)}, \overline{x}\big]\overline{x} + \overline{x}qbig[\overline{f(p)}, \overline{x}\big] = 0$$

for all $\overline{x} \in R/P$. In other words, the mapping $\overline{x} \mapsto [\overline{f(p)}, \overline{x}]$ is skew-commuting on R/P. Applying Theorem 4 one can conclude that $[\overline{f(p)}, \overline{x}] = 0$ for all $\overline{x} \in R/P$. In other words, $\overline{f(p)}$ lies in the center of R/P. In particular, $\pi\overline{f(p)} = 0$ for all $p \in P$. Using this we see that the mapping $\overline{f} : R/P \to A$, $\overline{f}(\overline{x}) = \pi\overline{f(x)}$ is well defined. Note also that $\overline{f}$ is additive and it satisfies $[\overline{f}(\overline{x}), \overline{x}^2] = 0$ for all $x \in R$. By Lemma 3 we get $[\overline{f}(\overline{x}), \overline{x}] = 0$ for all $x \in R$. This implies that $[f(x), x] \in P$ for all $x \in R$.

In the case $Z(R/P) = 0$ the proof can be completed using the standard arguments. $\square$

Posner's first theorem [14], which states that the compositum of two nonzero derivations on a 2-torsion free prime ring cannot be a derivation, in general cannot be proved for semiprime rings (see for example [3]). However, in the case we have a semiprime ring one can easily prove the following result.

Theorem 5 ([12, Lemma 1.1.9])**.** *Let R be a 2-torsion free semiprime ring and let $D, G : R \to R$ be derivations. Suppose that $D^2(x) = G(x)$ holds for all $x \in R$. In this case $D = 0$.*

The result above motivated the following theorem.

Theorem 6. *Let R be a 2-torsion free semiprime ring and let $D, G : R \to R$ be derivations. Suppose that either*

$$\big[[D^2(x) + G(x), x], x\big] = 0 \qquad \textit{or} \qquad [D^2(x) + G(x), x^2] = 0$$

holds for all $x \in R$. Then D and G map R into $Z(R)$.

For the proof of Theorem 6 we will need Theorem 1, Theorem 2, and the lemma below.

Lemma 7 ([16, Lemma 1])**.** *Let R be a semiprime ring. Suppose the relation $axb + cxa = 0$ is fulfilled for some $a, b, c \in R$ and all $x \in R$. Then $ax(b + c) = 0$ holds for all $x \in R$.*

Lemma 7 will be used also in the proof of Theorem 8.

Proof of Theorem 6. According to Theorem 1 and Theorem 2 one can conclude that the mapping $x \mapsto D^2(x) + G(x)$ is commuting on R. Thus we have

$$[F(x), x] = 0 \tag{1}$$

for all $x \in R$, where $F(x)$ stands for $D^2(x) + G(x)$. If we linearize the above relation we obtain

$$[F(x), y] + [F(y), x] = 0$$

for all $x, y \in R$. Putting in the above relation xy for y and noting that $F(xy) = F(x)y + xF(y) + 2D(x)D(y)$ we obtain

$$\begin{aligned} 0 &= [F(x), xy] + [F(xy), x] = [F(x), xy] + [F(x)y + xF(y) + 2D(x)D(y), x] \\ &= [F(x), x]y + x[F(x), y] + [F(x), x]y + F(x)[y, x] + x[F(y), x] + 2[D(x)D(y), x] \\ &= F(x)[y, x] + 2[D(x)D(y), x]. \end{aligned}$$

We have therefore

$$F(x)[y, x] + 2[D(x)D(y), x] = 0$$

for all $x, y \in R$. The substitution yx for y in the above relation gives

$$\begin{aligned} 0 &= F(x)[y, x]x + 2[D(x)D(y)x + D(x)yD(x), x] \\ &= F(x)[y, x]x + 2[D(x)D(y), x]x + 2[D(x)yD(x), x] = 2[D(x)yD(x), x]. \end{aligned}$$

We have therefore

$$[D(x)yD(x), x] = 0$$

for all $x, y \in R$. This can be written in the form

$$D(x)yD(x)x - xD(x)yD(x) = 0$$

for all $x, y \in R$. Applying Lemma 7 one can conclude that

$$D(x)y[D(x), x] = 0 \tag{2}$$

for all $x, y \in R$. Putting first xy for y in the above relation, then multiplying the relation (2) from the left side by x, and then subtracting the relations so obtained one from another we obtain $[D(x), x]y[D(x), x] = 0$ for all $x, y \in R$. It follows that

$$[D(x), x] = 0$$

for all $x \in R$ by the semiprimeness of R. In other words, D is commuting on R, whence it follows that D maps R into $Z(R)$. We have therefore $[D(x), y] = 0$ for all $x, y \in R$. Putting $D(x)$ for x we obtain $[D^2(x), y] = 0$ which reduces the relation (1) to $[G(x), x] = 0$, $x \in R$. Thus, G maps R into $Z(R)$ as well. The proof is completed. □

Let us point out that the first part of Theorem 6 generalizes Theorem 4 in [15].

We proceed with the following result which generalizes Theorem 5.

Theorem 8. *Let R be a 2-torsion free semiprime ring and let $D, G : R \to R$ be derivations. Suppose that*

$$(D^2(x) + G(x))x^2 + x^2(D^2(x) + G(x)) = 0$$

holds for all $x \in R$. In this case $D = G = 0$.

In the proof of Theorem 8 and also in the proof of Theorem 9 we shall use the fact that any semiprime ring R and its maximal right ring of quotients Q satisfy the same differential identities which is very useful since Q contains the identity element (see Theorem 3 in [13]). For the explanation of differential identities we refer to [10].

Proof of Theorem 8. By [13, Theorem 3] we have

$$F(x)x^2 + x^2F(x) = 0 \tag{3}$$

for all $x \in Q$. Here, $F(x)$ stands for $D^2(x) + G(x)$. Note that $F(1) = 0$. Thus, by (3),

$$F(x)(x+1)^2 + (x+1)^2 F(x) = 0 \tag{4}$$

holds for all $x \in Q$. It follows from (3) and (4) that

$$F(x)x + xF(x) + F(x) = 0 \tag{5}$$

for all $x \in Q$. Replacing x by $x+1$ in (5), we see that $F(x)x+xF(x)+3F(x) = 0$ for all $\in x \in Q$, implying that $2F(x) = 0$ holds for all $x \in Q$. Since R is 2-torsion free, $F = 0$. Thus, by Theorem 5, $D = 0$ and $G = 0$, as asserted. □

Theorem 9. *Let R be a 2-torsion free semiprime ring and let $D, G : R \to R$ be derivations. Suppose that the mapping*

$$x \mapsto (D^2(x) + G(x))x + x(D^2(x) + G(x))$$

is skew-commuting on R. In this case $D = G = 0$.

Proof. The assumption of the theorem can be written in the form

$$F(x)x^2 + 2xF(x)x + x^2 F(x) = 0 \tag{6}$$

for all $x \in R$. Again, $F(x)$ stands for $D^2(x) + G(x)$. By [13, Theorem 3] the above identity holds for all $x \in Q$. Note that $F(1) = 0$. By (6), we obtain

$$F(x)(x+1)^2 + 2(x+1)F(x)(x+1) + (x+1)^2 F(x) = 0 \tag{7}$$

for all $x \in R$. By (6) and (7), it follows that $F(x)x + xF(x) + F(x) = 0$ for all $x \in R$. Here we have used the assumption that R is 2-torsion free. Thus,

$$-F(x) = F(x)x + xF(x) = -(F(x)x + xF(x))x - x(F(x)x + xF(x)) = 0$$

for all $x \in R$. By Theorem 5, $D = 0$ and $G = 0$, as asserted. □

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AJDA FOŠNER
FACULTY OF MANAGEMENT
UNIVERSITY OF PRIMORSKA
CANKARJEVA 5
SI-6104 KOPER
SLOVENIA

E-mail: ajda.fosner@fm-kp.si

JOSO VUKMAN
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
FACULTY OF NATURAL SCIENCES AND MATHEMATICS
UNIVERSITY OF MARIBOR
KOROŠKA CESTA 160
SI-2000 MARIBOR
SLOVENIA

E-mail: joso.vukman@uni-mb.si

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