Publ. Math. Debrecen
78/3-4 (2011), 583–593
DOI: 10.5486/PMD.2011.4793

Some generalizations of the Borsuk–Ulam Theorem

By DANIEL VENDRÚSCOLO, (São Carlos) PATRICIA E. DESIDERI (São Carlos)
and PEDRO L. Q. PERGHER (São Carlos)

Abstract. Let S^n be the n-dimensional sphere, $A : S^n \to S^n$ the antipodal involution and R^n the n-dimensional euclidean space. The famous Borsuk–Ulam Theorem states that, if $f : S^n \to R^n$ is any continuous map, then there exists a point $x \in S^n$ such that $f(x) = f(A(x))$. In this paper we discuss some generalizations and variants of this theorem concerning the replacement either of the domain (S^n, A) by other free involution pairs (X, T), or of the target space R^n by more general topological spaces. For example, we consider the cases where: i) (S^2, A) is replaced by a product involution $(X, T) \times (Y, S) = (X \times Y, T \times S)$, where X and Y are Hausdorff and pathwise connected topological spaces, the involution T is free and the fundamental group of X is a torsion group; ii) R^n is replaced by $M^r \times N^s$, where M^r and N^s are closed manifolds with dimensions r and s, respectively, and $r + s = n$; iii) (S^2, A) is replaced by a product involution as described in i), and R^2 is replaced by the 2-dimensional torus T^2. We remark that i) includes the case in which $(X, T) \times (Y, S) = (X, T)$, by taking $(Y, S) = (\{\text{point}\}, \text{identity})$, and in particular the popular 2-dimensional Borsuk–Ulam Theorem.

1. Introduction

Generalizations of the Borsuk–Ulam Theorem as mentioned in the abstract can be placed in the following general setting: let X, Y be topological spaces, where X is equipped with a free involution $T : X \to X$, that is, with $T(x) \neq x$ for every $x \in X$. We say that $\{(X, T), Y\}$ satisfies the Borsuk–Ulam Theorem (in

Mathematics Subject Classification: Primary: 55M20; Secondary: 55M35.
Key words and phrases: Borsuk–Ulam Theorem, involution pair, covering space, deck transformation, equivariant map, CW-complex, generalized manifold, one-point union.
Research partially supported by CNPq and FAPESP.

an abbreviated form, *satisfies BUT*) if, given any continuous map $f : X \to Y$, there exists at least one point $x \in X$ so that $f(x) = f(T(x))$. Results of this type obtained by replacing (S^n, A) by more general free involution pairs (X, T) can be found, for example, in [2], [10], [12] and [13]. In general lines, in these papers S^n is replaced by spaces X subject to certain homological conditions, and the free involutions $T : X \to X$ are arbitrary. Results referring to the replacement of R^n by other spaces can be found, for example, in [11] (Y= a differentiable manifold), [8] (Y= a compact topological manifold), [1] (Y= a generalized manifold), [4], [5], [6] and [7] (Y= a finite CW-complex). Let X be a Hausdorff and pathwise connected topological space, equipped with a free involution $T : X \to X$. Let X/T be the orbit space of X by T and $p : X \to X/T$ the quotient map. Take a point $a \in X$, and consider the homomorphism induced in the fundamental groups, $p_* : \pi_1(X, a) \to \pi_1(X/T, \overline{a})$, where $\overline{a} = p(a)$. Denote by $h_X : \pi_1(X, a) \to H_1(X)$ the Hurewicz homomorphism, where $H_1(X)$ is the one-dimensional Z-homology group of X. In Section 2, we will obtain the following algebraic criterion for $\{(X, T), R^2\}$ satisfy BUT:

Theorem 1.1. *Set $G = \pi_1(X/T, \overline{a}) - p_*(\pi_1(X, a))$. If there exists $\beta \in G$ such that $h_{X/T}(\beta)$ is a torsion element in $H_1(X/T)$, then $\{(X, T), R^2\}$ satisfies BUT.*

As a consequence, we get that involution pairs as those mentioned in the abstract (item i), together with the target space R^2, satisfy BUT, and in particular the fact that if $\pi_1(X, a)$ is a torsion group, then for any free involution $T : X \to X$, $\{(X, T), R^2\}$ satisfies BUT, which includes the popular 2-dimensional Borsuk–Ulam Theorem. Another consequence is the fact that $\{(S, T), R^2\}$ satisfies BUT, where S is any closed orientable surface with Euler characteristic congruent to $2 \mod 4$ (which includes S^2) and T is any free involution on S.

In Section 3, we consider the following weak version of the Borsuk–Ulam Theorem (WBUT): if G is a topological group, we say that $\{(X, T), G\}$ satisfies WBUT if, for every map $f : X \to G$, there exists $x \in X$ such that $f(x) = f(T(x))$ (mod 2-*torsion*). We will see that, for all involution pairs (X, T) considered in Section 2, $\{(X, T), T^2\}$ satisfies WBUT, where T^2 is the 2-dimensional torus, considered with its additive structure $\pmod 1$.

Given a topological space X, we define BUT(X) as the smallest natural number n so that $\{(S^n, A), X\}$ satisfies BUT. In Section 4 we make some considerations about this number, which is a topological invariant. We will see that, if X is a finite n-dimensional CW-complex, then $n \leq \text{BUT}(X) \leq 2n$, and if X is a closed n-dimensional manifold, then $\text{BUT}(X) = n$ or $n + 1$. This raises the question of finding BUT(X) for specific n-dimensional CW-complexes X (or specific closed n-dimensional manifolds X). For example, we will see that, if X is a closed n-dimensional manifold satisfying the fact that its top-dimensional non-zero Z_2-cohomology class $\alpha \in H^n(X, Z_2)$ is a cup product of lower dimensional classes, then BUT(X) $= n$.

2. A result related to the 2-dimensional Borsuk–Ulam Theorem in terms of the fundamental group

We will prove the algebraic criterion for $\{(X,T), R^2\}$ satisfy BUT given by Theorem 2.1, maintaining the notation used to state the result; we will use simple facts concerning covering spaces. If $\sigma : I = [0,1] \to X$ is a path with $\sigma(0) = a$, we denote by $[\sigma]$ the homotopy equivalence class of σ relative to the base point a; we set σ^{-1} for the inverse path $t \to \sigma(1-t)$. To prove the result, suppose by contradiction that there exists a continuous map $f : X \to R^2$ with $f(x) \neq f(T(x))$ for every $x \in X$. Then a standard and well known construction yields an equivariant map $F : X \to S^1$, that is, satisfying $F(T(x)) = -F(x)$ for every $x \in X$. Set $q : S^1 \to S^1/A$ for the quotient map. Because F is equivariant, it induces a continuous map $\overline{F} : X/T \to S^1/A$ in such a way that the diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & S^1 \\ \big\downarrow p & & \big\downarrow q \\ X/T & \xrightarrow{\overline{F}} & S^1/A \end{array}$$

is commutative. Set $F(a) = z$, that is, $q(z) = \overline{z} = \overline{F}(\overline{a})$. Take $\beta \in G$ so that $h_{X/T}(\beta)$ is a torsion element in $H_1(X/T)$, and consider the commutative diagram

$$\begin{array}{ccc} \pi_1(X/T, \overline{a}) & \xrightarrow{\overline{F}_*} & \pi_1(S^1/A, \overline{z}) \\ \big\downarrow h_{X/T} & & \big\downarrow h_{S^1/A} \\ H_1(X/T) & \xrightarrow{\overline{F}_*} & H_1(S^1/A) \end{array}$$

Then $\overline{F}_* h_{X/T}(\beta)$ is a torsion element in $H_1(S^1/A) \cong Z$, which means that $\overline{F}_* h_{X/T}(\beta) = 0$. Now choose a loop α in X/T which represents β. Then there exists a lifting $\overline{\alpha} : I \to X$ for α with $\overline{\alpha}(0) = a$. Since β does not belong to the image of p_*, one necessarily has that $\overline{\alpha}(1) = T(a)$, and since F is equivariant, $F\overline{\alpha}$ is a path in S^1 with initial point z and final point $-z$. Choose generators

$c \in \pi_1(S^1, z)$, $d \in \pi_1\big(\frac{S^1}{A}, \overline{z}\big)$, and a path μ in S^1 with initial point z and final point $-z$, so that $q\mu$ is a loop in $\frac{S^1}{A}$ representing d. Then the usual product of paths $(F\overline{\alpha}).(\mu^{-1})$ is a loop in S^1 with base point z, which means that, in $\pi_1(S^1, z)$, $[(F\overline{\alpha}).(\mu^{-1})] = rc$ for some $r \in Z$. Since q has degree two, on then has $q_*([(F\overline{\alpha}).(\mu^{-1}])) = [qF\overline{\alpha}] - [q\mu] = [qF\overline{\alpha}] - d = rq_*(c) = \pm 2rd$. Thus $[qF\overline{\alpha}] \neq 0$, and since $h_{S^1/A}$ is an isomorphism, $h_{S^1/A}([qF\overline{\alpha}]) \neq 0$. This contradicts the fact that $h_{S^1/A}([qF\overline{\alpha}]) = \overline{F}_* h_{X/T}(\beta) = 0$.

Corollary 2.1. *Let X be a Hausdorff and pathwise connected space, and (X, T) a free involution pair. If the fundamental group of X is a torsion group (which includes the case in which $X = S^2$), then $\{(X, T), R^2\}$ satisfies BUT.*

Proof. Choose a base point $a \in X$ and write $p : X \to X/T$ for the quotient map. Set $p(a) = \overline{a}$. One has that $p : X \to X/T$ is a two-fold covering, with $\{\text{Identity}, T\}$ being the group of deck transformations of this covering. In this way, the quotient group

$$\frac{\pi_1(X/T, \overline{a})}{p_*(\pi_1(X, a))}$$

is isomorphic to Z_2, the cyclic group of 2 elements. This means that $p_*(\pi_1(X, a))$ is a subgroup of $\pi_1(X/T, \overline{a})$ of index two. The fact that $\pi_1(X, a)$ is a torsion group then implies that $\pi_1(X/T, \overline{a})$ is a torsion group. In fact, if $\beta \in \pi_1(X/T, \overline{a}) - p^*(\pi_1(X, a))$, then $\beta^2 \in p^*(\pi_1(X, a))$ and so β^2 is a torsion element. In this way, β is a torsion element, which ends the proof. □

Corollary 2.2. *Let X, Y be Hausdorff and pathwise connected spaces, and (X, T) a free involution pair like those of Theorem 2.1 (which particularly includes the case in which the fundamental group of X is a torsion group). Let (Y, S) be a involution pair, which is not necessarily free. Then $\{(X, T) \times (Y, S), R^2\}$ satisfies BUT.*

Proof. Take points $a \in X$, $c \in Y$, and write $p : X \to X/T$ and $q : X \times Y \to X \times Y/T \times S$ for the quotient maps. Set $p(a) = \overline{a}$ and $q(a, c) = \overline{(a, c)}$. Consider the maps $\theta : X \to X \times Y$, $\Phi : X \times Y \to X$, $\theta(x) = (x, c)$, $\Phi(x, y) = x$. Then θ and Φ induce maps $\overline{\theta} : X/T \to X \times Y/T \times S$, $\overline{\Phi} : X \times Y/T \times S \to X/T$ so that $\overline{\Phi} \circ \overline{\theta}$ is the identity map. Take $\beta \in G = \pi_1(X/T, \overline{a}) - p_*(\pi_1(X, a))$ with $h_{X/T}(\beta)$ being a torsion element in $H_1(X/T)$. Then $\overline{\theta}_*(h_{X/T}(\beta))$ is a torsion element in $H^1(X \times Y/T \times S)$, and since $\overline{\theta}_* \circ h_{X/T} = h_{X \times Y/T \times S} \circ \overline{\theta}_*$, it suffices to show that $\overline{\theta}_*(\beta) \in \pi_1(X \times Y/T \times S, \overline{(a, c)})$ does not belong to $q_*(\pi_1(X \times Y, (a, c))$. Otherwise, suppose $\overline{\theta}_*(\beta) = q^*(\omega)$ for some $\omega \in \pi_1(X \times Y, (a, c))$. Then $\beta = \overline{\Phi}_*(\overline{\theta}_*(\beta)) = \overline{\Phi}_*(q^*(\omega)) = p_*(\Phi_*(\omega))$, which contradicts the fact that $\beta \notin p_*(\pi_1(X, a))$. □

Remark 2.1. Concerning product involution pairs $(X \times Y, T \times S)$ with T without fixed points, we note that if $\{(X,T), Z\}$ satisfies BUT and S has a fixed point, then it is easy to prove directly that $\{(X \times Y, T \times S), Z\}$ satisfies BUT. However, if S does not have fixed points, then we have no topological way to prove that $\{(X \times Y, T \times S), Z\}$ satisfies BUT, even if also $\{(Y,S), Z\}$ satisfies BUT.

Corollary 2.3 (D. L. Gonçalves, [3])**.** *Let S be a closed orientable surface with Euler characteristic congruent to $2 \mod 4$ and T a free involution on S. Then $\{(S,T), R^2\}$ satisfies BUT.*

Proof. S/T is a non-orientable closed surface with odd Euler characteristic, and in this case it is well known that there exists an element $\beta \in \pi_1(S/T)$ so that $h_S(\beta) \in H_1(S)$ is a torsion element, and that β belongs to $\pi_1(S/T) - p_*(\pi_1(S))$. □

3. A weak Borsuk–Ulam theorem for maps into the 2-dimensional torus

Let (X,T) be a free involution pair and G a topological group. Set $i : G \to G$ for the involution $i(g) = g^{-1}$ and $2G$ for the set $\{g \in G \,/\, i(g) = g\}$; evidently, the neutral element $e \in G$ belongs to $2G$. Note that the validity of BUT for $\{(X,T), G\}$ is equivalent to the fact that, for every $f : X \to G$, $F^{-1}(e)$ is nonempty, where $F : (X,T) \to (G,i)$ is the equivariant map $F(x) = f(x).(f(T(x)))^{-1}$. This motivates the following extension of the BUT property: we say that $\{(X,T), G\}$ satisfies the *weak Borsuk–Ulam Theorem* (in an abbreviated form, satisfies *WBUT*) if $F^{-1}(2G)$ is nonempty for every $f : X \to G$. If $2G = \{e\}$, BUT is equivalent to WBUT; for example, this happens with $G = R^n$, considered with its additive structure. We want to consider the case in which G is the 2-dimensional torus $T^2 = \frac{[0,1]\times[0,1]}{\sim}$, where $\sim$ identifies $(t,0)$ to $(t,1)$ and $(0,t)$ to $(1,t)$, considered with its additive structure $\pmod 1$.

Theorem 3.1. *Let (X,T) be an involution pair like those of Theorem 1.1. Then $\{(X,T), T^2\}$ satisfies WBUT.*

Proof. The argument follows the lines of the proof of Theorem 1.1, but with more technical sophistication. One has $2T^2 = \{r_1 = (0,0),\ r_2 = (0, \frac{1}{2}),\ r_3 = (\frac{1}{2}, 0),\ r_4 = (\frac{1}{2}, \frac{1}{2})\}$. Consider $K \subset T^2$, $K = \big([0,1] \times \{\frac{1}{4}\}\big) \cup \big([0,1] \times \{\frac{3}{4}\}\big) \cup \big(\{\frac{1}{4}\} \times [0,1]\big) \cup \big(\{\frac{3}{4}\} \times [0,1]\big)/\sim$. K is invariant under the map $i : T^2 \to T^2$, and (K,i) is a free involution pair. Write $q : K \to K/i$ for the quotient map. We need to describe the homomorphism $q_* : \pi_1(K,p) \to \pi_1(K/i, q(p))$, where the base point p is $p = \left(\frac{1}{4}, \frac{1}{4}\right)$. To do this, set $A = \left\{\left(t, \frac{1}{4}\right),\ t \in I\right\}$, $B = \left\{\left(t, \frac{3}{4}\right),\ t \in I\right\}$, $C = \left\{\left(\frac{1}{4}, t\right),\ t \in I\right\}$, $D = \left\{\left(\frac{3}{4}, t\right),\ t \in I\right\}$ and $E = \left(\left\{\left(t, \frac{1}{4}\right),\ \frac{1}{4} \le t \le \frac{3}{4}\right\}\right) \cup \left(\left\{\left(\frac{3}{4}, t\right),\ \frac{1}{4} \le t \le \frac{3}{4}\right\}\right) \cup \left(\left\{\left(t, \frac{3}{4}\right),\ \frac{1}{4} \le t \le \frac{3}{4}\right\}\right) \cup \left(\left\{\left(\frac{1}{4}, t\right),\ \frac{1}{4} \le t \le \frac{3}{4}\right\}\right)$. Then $\pi_1(K,p)$ is a free group in the generators a, b, c, d and e, which can be represented by loops whose images are $\frac{A}{\sim}$, $\frac{B}{\sim}$, $\frac{C}{\sim}$, $\frac{D}{\sim}$ and $\frac{E}{\sim}$, respectively. Up to isomorphism, $i_* : \pi_1(K,p) \to \pi_1(K, i(p))$ is the degree 2 isomorphism given by $i_*(a) = b$, $i_*(c) = d$ and $i_*(e) = e$; further, $i : \frac{E}{\sim} \to \frac{E}{\sim}/i$ is a two-fold covering. Therefore $\pi_1(K/i, q(p))$ is a free group in generators x, y and z, and up to isomorphism $q_* : \pi_1(K,p) \to \pi_1(K/i, q(p))$ can be described by $q_*(a) = x$, $q_*(b) = zxz^{-1}$, $q_*(c) = y$, $q_*(d) = zyz^{-1}$ and $q_*(e) = z^2$. If a word $\mathcal{J}$ in a free group has a letter u, denote by $s(u)$ the algebraic sum of the powers of u occuring in $\mathcal{J}$. Then, if $\mathcal{J} \in \pi_1(K)$ is a word in the letters a, b, c, d and e, $q_*(\mathcal{J}) \in \pi_1(K/i)$ is a word in the letters x, y and z with $s(x) = s(a) + s(b)$, $s(y) = s(c) + s(d)$ and $s(z) = 2s(e)$.

We are now ready to proceed with the proof. Suppose by contradiction one has a map $f : X \to T^2$ so that the corresponding equivariant map $F : X \to T^2$ maps X into $T^2 - 2T^2$. We assert that there is an equivariant homotopy equivalence $h : (T^2 - 2T^2, i) \to (K, i)$. In fact, note that $T^2 - K$ is the disjoint union of four open disks, each one with one of the r_i's in the center. Then h can be constructed in an equivariant way by using the radial projection around r_i, for each $i = 1, 2, 3, 4$. This gives the equivariant map $g = hF : X \to K$ and the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ g\ } & K \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ X/T & \xrightarrow{\ \overline{g}\ } & K/i \end{array}$$

where $p : X \to X/T$ is the quotient map. Choose an initial base point $v \in X$. Since K is pathwise connected, up to isomorphism the corresponding base point $g(v) \in K$ can be replaced by $p = \left(\frac{1}{4}, \frac{1}{4}\right)$. Thus, without loss, in what follows $\pi_1(K)$ will always be considered with base point p, and so we will omit mention to base points. One has the commutative diagram

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\ g_*\ } & \pi_1(K) \\ {\scriptstyle p_*}\big\downarrow & & \big\downarrow{\scriptstyle q_*} \\ \pi_1(X/T) & \xrightarrow{\ \overline{g}_*\ } & \pi_1(K/i) \\ {\scriptstyle h_{X/T}}\big\downarrow & & \big\downarrow{\scriptstyle h_{K/i}} \\ H_1(X/T) & \xrightarrow{\ \overline{g}_*\ } & Z \times Z \times Z \end{array}$$

Take $\beta \in G = \pi_1(X/T) - p_*(\pi_1(X))$ as described in the hypothesis. Then, using the fact that $Z \times Z \times Z$ has not torsion elements, we get that $\overline{g}_* h_{X/T}(\beta) = 0$. On the other hand, choosing a loop α in X/T which represents β, there exists a lifting $\overline{\alpha} : I \to X$ for α with endpoints forming an orbit of T, and because g is equivariant, $g\overline{\alpha}$ is a path in K joining the points $w, i(w) \in K$, where $w = g(v)$. Now we can choose a path μ in K joining w to $i(w)$ so that $q\mu$ is a loop in K/i representing z: if $w \in E$, μ can be taken as a direct path in E from w to $i(w)$, and if $w \notin E$, μ can be taken as a path in K which runs through two consecutive edges of the square E. Then the path $(g\overline{\alpha}).(\mu^{-1})$ is a loop in K, which means that $[(g\overline{\alpha}).(\mu^{-1})] \in \pi_1(K)$ is a word $\mathcal{J}$ in the letters a, b, c, d and e. It follows that $q_*([(g\overline{\alpha}).(\mu^{-1}])) = [qg\overline{\alpha}].[q\mu^{-1}] = [qg\overline{\alpha}].z^{-1} = q_*(\mathcal{J})$. By the description of q_*, $q_*(\mathcal{J})$ is a word in the letters x, y and z such that the sum of the powers of z is even, and thus $[qg\overline{\alpha}]$ is a word in x, y, z with the sum of powers of z being odd. Then the coordinate of $h_{K/i}([qg\overline{\alpha}])$ corresponding to the $h_{K/i}(z)$-factor is nonzero, which gives that $h_{K/i}([qg\overline{\alpha}]) = \overline{g}_* h_{X/T}(\beta)$ is nonzero. □

Next we will give a source of explicit examples of free involutions pairs (X, T) for which $\{(X, T), T^2\}$ satisfies WBUT. This will be strongly based on the paper [3] of D. L. Gonçalves. Suppose $T, P : X \to X$ are free involutions on X. We say that (X, T) and (X, P) are equivalent if there is an equivariant homeomorphism $f : (X, T) \to (X, P)$. This is an equivalence relation on the set of all free involutions on X; denote by $\mathrm{Inv}(X)$ the set of equivalence classes. $\mathrm{Inv}(X)$ may be empty; for example, if X has odd Euler characteristic, or if X is a smooth closed manifold that does not bound. The equivalence classes $(X, [T])$ are suitable objects to study the BUT (WBUT) property; in fact, if (X, T) and (X, P) are equivalent, then, for every space Y, $\{(X, T), Y\}$ satisfies BUT (WBUT) if and only if $\{(X, P), Y\}$ satisfies BUT (WBUT). In [3], D. L. Gonçalves studied the BUT property for $\{(S, T), R^2\}$, where S is a closed surface (that is, a 2-dimensional closed manifold) and S is any free involution on S. It was shown that, if S is orientable, or if S is nonorientable and the Euler characteristic of S is even, then $\mathrm{Inv}(S)$ is nonempty (in the remaining cases, it is known that $\mathrm{Inv}(S)$ is empty, because S does not bound). Further, it was shown that $\mathrm{Inv}(S)$ has $2^r - 1$ elements, where r is the number of elements of a canonical system of generators of $\pi_1(S)$. For any S for which $\mathrm{Inv}(S)$ is nonempty, the elements of $\mathrm{Inv}(S)$ which satisfy the mentioned BUT property were explicitly determined; for example, if S is orientable and the Euler characteristic of S is congruent to $2 \mod 4$ (which includes the 2-dimensional sphere), then $\{(S, T), R^2\}$ satisfies BUT for every class $[T]$ (see Corollary 2.3).

Corollary 3.2. *Let S be a closed surface for which* $\mathrm{Inv}(S)$ *is nonempty. If (S,T) is a free involution pair, then $\{(S,T),T^2\}$ satisfies WBUT if and only if $\{(S,T),R^2\}$ satisfies BUT.*

Proof. Suppose that $\{(S,T),R^2\}$ satisfies BUT. By doing a case-by-case inspection on these (S,T), and taking into account the classification theorem of surfaces, we can see that, for each such a (S,T), there exists an element $\beta \in \pi_1(S/T)$ so that $h_S(\beta) \in H_1(S)$ is a torsion element, and with β belonging to $\pi_1(S/T) - p_*(\pi_1(S))$. Now suppose that $\{(S,T),R^2\}$ does not satisfy BUT. Then there exists a continuous map $f : S \to R^2$ such that $f(x) \neq f(T(x))$ for every $x \in S$. Take a homeomorphism $g : R^2 \to B$, where B is an open ball centered in $(0,0)$ with radius equal to $\frac{1}{16}$, and consider the usual universal covering $h : R^2 \to T^2$. Then the equivariant map $F : (S,T) \to (T^2,i)$ which corresponds to the map $hgf : S \to T^2$ clearly satisfies the fact that $F^{-1}(2T^2)$ is empty. □

To increase the source of free involutions (X,T) for which $\{(X,T),T^2\}$ satisfies WBUT, one still has

Corollary 3.3. *Consider product involutions $(X \times Y, T \times S)$ like those of Corollary 2.2. Then $\{(X \times Y, T \times S), T^2$ satisfies WBUT.*

4. A topological invariant coming from the Borsuk–Ulam theorem

Let X be a topological space. Taking into account the standard equivariant inclusion $S^{n-1} \to S^n$, it is easy to see that, if $\{(S^n,A),X\}$ satisfies BUT, then $\{(S^m,A),X\}$ satisfies BUT for every $m > n$, and if $\{(S^n,A),X\}$ does not satisfy BUT, then the same is true for $\{(S^m,A),X\}$ with $m < n$. Then either $\{(S^n,A),X\}$ does not satisfy BUT for every natural number n, or there exists the smallest natural number n for which $\{(S^n,A),X\}$ satisfies BUT. In the first case, we write $\mathrm{BUT}(X) = \infty$, and in the second $\mathrm{BUT}(X) = n$. If there exists a continuous injective map $X \to Y$ (and in particular if X is a subspace of Y), then $\mathrm{BUT}(X) \leq \mathrm{BUT}(Y)$, and in particular $\mathrm{BUT}(X)$ is a topological invariant (but not a homotopic invariant). Evidently, $\mathrm{BUT}(\{point\}) = 0$ and $\mathrm{BUT}(X) > 0$ if X has at least two points; in this case, $\mathrm{BUT}(X) = 1$ if X is a discrete space. If $X = \{a,b\}$ has two points and is equipped with the trivial topology, then $\mathrm{BUT}(X) = \infty$; in fact, using induction on n we can construct, for every $n \geq 0$, a subset $P \subset S^n$ such that $P \cap A(P) = \emptyset$ and $P \cup A(P) = S^n$. Next, we consider the map $f : S^n \to X$ that sends P into a and $A(P)$ into b. We also have

$\mathrm{BUT}(S^\infty) = \infty$, where $S^\infty = lim_n(S^n)$ with the weak topology. The Borsuk–Ulam Theorem implies that $\mathrm{BUT}(R^n) \le n$, and in fact $\mathrm{BUT}(R^n) = n$. This is a very special particular case of the following

Theorem 4.1. *Suppose X is a finite n-dimensional CW-complex. Then*

i) $n \le \mathrm{BUT}(X) \le 2n$;

ii) *if X is a not closed topological manifold (which includes R^n), then* $\mathrm{BUT}(X) = n$;

iii) *if X is a closed topological manifold, then* $\mathrm{BUT}(X) = n$ *or* $n+1$. *In this case, if X satisfies the fact that its top-dimensional nonzero Z_2-cohomology class $\alpha \in H^n(X, Z_2)$ is a cup product of lower dimensional classes, then* $\mathrm{BUT}(X) = n$.

Proof. In fact, the stated results follow immediately from known results of the literature. Inside the interior of an n-cell of X we can take a copy homeomorphic of S^{n-1}, which means that $\mathrm{BUT}(X) \ge n$. On the other hand, in [5], D. L. Gonçalves, P. Pergher and J. Jaworowski proved that, if $m \ge 2n$, then $\{(S^m, A), X\}$ satisfies BUT. This gives i). In [11], P. E. Conner and E. E. Floyd proved that if X is a differentiable manifold and $m > n$, then $\{(S^m, A), X\}$ satisfies BUT; in this case, they also proved that, if $m = n$ and $f : S^m \to X$ is a continuous map satisfying the fact that its induced homomorphism in Z_2-cohomology $f^* : H^n(X) \to H^n(S^m)$ is trivial, then there exists $x \in S^m$ with $f(x) = f(A(x))$. In [8], Munkholm showed the same result without the differentiability hypothesis, but with X compact. Recently, in [1], this Conner-Floyd result was proved for X a generalized manifold not necessarily compact (see the remark below). Since $H^n(X, Z_2) = 0$ if X is a not closed manifold, ii) and the first statement of iii) are established. The second statement of iii) follows from the fact that, for any map $f : S^n \to X$, $f^* : H^n(X) \to H^n(S^n)$ is a ring homomorphism and $H^j(S^n) = 0$ if $0 < j < n$. □

Remark 4.1. Using iii), we get that $\mathrm{BUT}(X) = dim(X)$ if $X = M^r \times N^s$, where M^r and N^s are closed manifolds with dimensions $r, s > 0$. The same is valid for real, complex and quaternionic projective spaces, Dold manifolds and projective space bundles associated to real, complex or quaternionic vector bundles over closed manifolds. One has $\mathrm{BUT}(S^n) = n + 1$, and in fact it is the only example we know of a closed manifold with this property.

Remark 4.2. A generalized manifold of dimension n is a topological space X which is an ENR and, for every $x \in X$, $H_*(X, X - \{x\}; Z)$ is isomorphic to $H_*(R^n, R^n - \{0\}; Z)$. Recently, such manifolds have been extensively studied.

In [1], C. Biasi, E. L. Santos and D. de Mattos showed that the Conner–Floyd theorem mentioned above remains valid if we replace manifolds by generalized manifolds. In this way, $\mathrm{BUT}(X) \leq n+1$.

Remark 4.3. If X is a compact metric space of topological dimension n, then X can be imbedded in R^{2n+1}, which gives that $\mathrm{BUT}(X) \leq 2n+1$ (for the definition of topological dimension, see for example [9]). The same argument shows that, if $\mathrm{BUT}(X) = \infty$, then X cannot be imbedded in a euclidean space.

Remark 4.4. Take any pathwise connected topological space X, and consider $X^* = (X \times X) - \{(x,y) \in X \times X \,/\, x = y\}$; note that on X^* one has the free involution $T_X(x,y) = (y,x)$. If $H_n(X^*/T_X, Z_2) = 0$, then $\mathrm{BUT}(X) \leq n$ (see [12; Theorem 3]).

Remark 4.5. The theorem established in this section raises the question of improving the estimative $n \leq \mathrm{BUT}(X) \leq 2n$ for special families of n-dimensional CW-complexes X; for example, see the improvement for manifolds. In this setting, the next natural case after manifolds is the one-point union of two closed n-dimensional manifolds, $M^n \vee V^n$. The first unsolved case is $\mathrm{BUT}(M^2 \vee V^2)$, where at least one of these closed surfaces is non-orientable. In this case, $\mathrm{BUT}(X)$ may be 3 or 4.

Remark 4.6. We point the following additional questions:

i) to estimate $\mathrm{BUT}(X \times Y)$ in terms of $\mathrm{BUT}(X)$ and $\mathrm{BUT}(Y)$ (certainly, $\max\{\mathrm{BUT}(X), \mathrm{BUT}(Y)\} \leq \mathrm{BUT}(X \times Y)$);

ii) to find a space X with $\mathrm{BUT}(X)$ finite and such that X cannot be imbedded in a euclidean space, or to show that, if $\mathrm{BUT}(X)$ is finite, then X can be imbedded in some euclidean space.

Acknowledgement We are very grateful to the referee for suggestions that helped to clarify considerably the original version.

References

[1] C. Biasi, D. de Mattos and E. L. dos Santos, A Borsuk–Ulam Theorem for maps from a sphere to a generalized manifold, *Geom. Dedicata* **107** (2004), 101–110.

[2] C. T. Yang, On the theorems of Borsuk–Ulam, Kakutani–Yamabe–Yujobô and Dyson I, *Ann. of Math.* **60**, no. 2 (1954), 262–282.

[3] D. L. Gonçalves, The Borsuk–Ulam theorem for surfaces, *Quaest. Math.* **29**, no. 1 (2006), 117–123.

[4] D. L. Gonçalves, J. Jaworowski, P. L. Q. Pergher and A. Yu. Volovikov, Coincidences for maps of spaces with finite group actions, *Topology Appl.* **145**, no. 1–3 (2004), 61–68.

[5] D. L. Gonçalves, J. Jaworowski and P. L. Q. Pergher, G-coincidences for maps of homotopy spheres into CW-complexes, *Proc. Amer. Math. Soc.* **130**, no. 10 (2002), 3111–3115.

[6] D. L. Gonçalves, J. Jaworowski and P. L. Q. Pergher, Measuring the size of the coincident set, *Topology Appl.* **125**, no. 3 (2002), 465–470.

[7] D. L. Gonçalves and P. L. Q. Pergher, Z_p-coincidences for maps of spheres into CW–complexes, *Kobe J. Math.* **15**, no. 2 (1998), 191–195.

[8] H. J. Munkholm, A Borsuk–Ulam theorem for maps from a sphere to a compact topological manifold, *Illinois J. Math.* **13** (1969), 116–124.

[9] J. Nagata, A Survey of Dimension Theory, General Topology and its relations to modern Analysis and Algebra, II, *Proc. Second Prague Topological Sympos.*, 1967, 259–270.

[10] J. W. Walker, A homology version of the Borsuk–Ulam theorem, *Amer. Math. Monthly* **90** (1983), 466–468.

[11] P. E. Conner and E. E. Floyd, Differentiable Periodic Maps, *Springer-Verlag*, *Berlin*, 1964.

[12] P. L. Q. Pergher, D. de Mattos and E. L. dos Santos, The Borsuk–Ulam theorem for general spaces, *Arch. Math. (Basel)* **81**, no. 1 (2003), 96–102.

[13] T. E. Barros and C. Biasi, A note on the theorems of Lusternik-Schnirelmann and Borsuk–Ulam, *Colloq. Math.* **111**, no. 1 (2008), 35–42.

DANIEL VENDRÚSCOLO
DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DE SÃO CARLOS
CAIXA POSTAL 676
SÃO CARLOS, SP 13565-905
BRAZIL

E-mail: daniel@dm.ufscar.br

PATRICIA E. DESIDERI
DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DE SÃO CARLOS
CAIXA POSTAL 676
SÃO CARLOS, SP 13565-905
BRAZIL

E-mail: patricia.desideri@gmail.com

PEDRO L. Q. PERGHER
DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DE SÃO CARLOS
CAIXA POSTAL 676
SÃO CARLOS, SP 13565-905
BRAZIL

E-mail: pergher@dm.ufscar.br

(Received December 10, 2009; revised July 28, 2010)