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Perturbation theory of lower semi-Browder multivalued linear operators

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Abstract. In the present paper we introduce the notion of lower semi-Browder linear relation and we study the perturbation problem under compact operator perturbations.

1. Introduction and preliminaries

Let X be a Banach space. A multivalued linear operator on X or simply a linear relation $T : X \to X$ is a mapping from a subspace $\mathcal{D}(T) \subset X$ called the domain of T, into the collection of nonempty subsets of X such that $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$ for all nonzero α, β scalars and $x_1, x_2 \in \mathcal{D}(T)$. If T maps the points of its domain to singletons, then T is said to be a single valued linear operator or simply an operator. We denote the class of linear relations on X by $LR(X)$.

A linear relation T in X is uniquely determined by its graph, $G(T)$ which is defined by $G(T) := \{(x,y) \in X \times X : x \in \mathcal{D}(T);\ y \in Tx\}$. Let $T \in LR(X)$. The inverse of T is the linear relation T^{-1} given by $G(T^{-1}) := \{(y,x) : (x,y) \in G(T)\}$. If T^{-1} is single valued, then T is called injective, that is, T is injective if and only if its null space $\mathcal{N}(T) := \{x : (x,0) \in G(T)\} := T^{-1}(0) = \{0\}$. The range of T is the subspace $\mathcal{R}(T) := T(\mathcal{D}(T))$ and T is called surjective if $\mathcal{R}(T) = X$. When T is injective and surjective we say that T is bijective. We write $\alpha(T) := \dim \mathcal{N}(T)$; $\beta(T) := \dim X/\mathcal{R}(T)$ and the index of T, $i(T)$, is defined by $i(T) := \alpha(T) - \beta(T)$

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provided $\alpha(T)$ and $\beta(T)$ are not both infinite. The multivalued part of a linear relation T is defined by $\mathrm{mul}(T) = \{y : (0, y) \in G(T)\}$.

For $T \in LR(X)$, the linear relation T^n is defined as usual with $T^0 = I$ and $T^1 = T$. It is well known that $\{\mathcal{N}(T^k)\}$ forms an ascending sequence of subspaces. Suppose that for some k, $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1})$; we shall then write $a(T)$ for the smallest value of k for which this is true, and call the integer $a(T)$, the ascent of T. If no such integer exists, we shall say that T has infinite ascent. In a similar way, $\{\mathcal{R}(T^k)\}$ forms a descending sequence; the smallest integer for which $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$ is called the descent of T and is denoted by $d(T)$. If no such integer exists, we shall say that T has infinite descent. For a linear relation T, the root manifold $\mathcal{N}^\infty(T)$ is defined by $\mathcal{N}^\infty(T) = \cup_{n=1}^{\infty} \mathcal{N}(T^n)$. Similarly, the root manifold $\mathcal{R}^\infty(T)$, is defined by $\mathcal{R}^\infty(T) = \cap_{n=1}^{\infty} \mathcal{R}(T^n)$.

For $T, S \in LR(X)$ and $\lambda \in \mathbb{K}$, the linear relations $T + S, \lambda T, T - \lambda, TS$ are defined by $G(T + S) := \{(x, y + z) : (x, y) \in G(T), (x, z) \in G(S)\}$, $G(\lambda T) := \{(x, \lambda y) : (x, y) \in G(T)\}$, $G(T - \lambda) = \{(x, y - \lambda x) : (x, y) \in G(T)\}$ and $G(TS) := \{(x, y) : \exists\ z \in X,\ (x, z) \in G(S),\ (z, y) \in G(T)\}$ respectively. For two linear relations T and S, the notation $T \subset S$ means that $G(T) \subset G(S)$. Let M be a subspace of X such that $M \cap \mathcal{D}(T) \neq \emptyset$. Then the linear relation $T|M$ is given by $G(T|M) := \{(x, y) \in G(T) : x \in M\}$.

For a given closed subspace E of X let Q_E^X or simply Q_E denote the natural quotient map from X onto X/E. We shall denote $Q_{\overline{T(0)}}^X$ by Q_T. Clearly $Q_T T$ is single valued. For $x \in \mathcal{D}(T)$, $\|Tx\| := \|Q_T Tx\|$ and the norm of T is defined by $\|T\| := \|Q_T T\|$. We note that this quantity is not a true norm since $\|T\| = 0$ does not imply $T = 0$.

A linear relation $T \in LR(X)$ is said to be closed if its graph is a closed subspace, continuous if for each neighbourhood V in $\mathcal{R}(T)$, $T^{-1}(V)$ is a neighbourhood in $\mathcal{D}(T)$ equivalently $\|T\| < \infty$, open if T^{-1} is continuous equivalently $\gamma(T) > 0$ where $\gamma(T)$ is the minimum modulus of T defined by $\gamma(T) := \sup\{\lambda \geq 0 : \lambda\, \mathrm{dis}(x, \mathcal{N}(T)) \leq \|Tx\| \text{ for } x \in \mathcal{D}(T)\}$. The resolvent set of a linear relation T is the set $\rho(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is injective, open and has dense range}\}$. It is clear from the Closed Graph Theorem for linear relations that if T is closed linear relation and X is a Banach space then $\rho(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is bijective}\}$. The spectrum of T is the set $\sigma(T) := \mathbb{C} \backslash \rho(T)$.

Throughout this paper, we denote by $CR(X)$ (resp. $BR(X)$) the set of all closed (resp. bounded) linear relations on X. The set of bounded (resp. compact) operators is designed by $\mathcal{L}(X)$ (resp. $\mathcal{K}(X)$). A linear relation $T \in CR(X)$ is called lower semi-Fredholm if $\beta(T) < \infty$ and $\mathcal{R}(T)$ is closed. Let $\Phi_-(X))$ denote the set of lower semi-Fredholm relations.

Linear relations were introduced into Functional Analysis by J. von Neumann [18], motivated by the need to consider adjoints of non-densely defined linear differential operators which are considered by Coddington [8], Coddington and Dijksma [7], Dikjsma, Sabbah and De Snoo [11], among others. One main reason why linear relations are more convenient than operators is that one can define the inverse, the closure and the completion for a linear relation. Interesting works on multivalued linear operators include the treatise on partial differential relations by Gromov [16], the application of multivalued methods to solution of differential equations by Favini and Yagi [13], the development of fixed point theory for linear relations to the existence of mild solutions of quasi-linear differential inclusions of evolution and also to many problems of fuzzy theory (see, for example [1], [14], [17] and [21]) and several papers on semi-Fredholm linear relations and other classes related to them (see, for example [2], [3] and [4]).

The perturbation problem of bounded semi-Browder operators was studied by S. Grabiner [15] and V. Rakočević [19], [20]. Recently, F. Fakhfakh and M. Mnif [12] extend this analysis to closed densely defined semi-Browder linear operators. The results obtained are applied to the determination of the stability of Browder's essential defect spectrum and Browder's essential approximate point spectrum under perturbations belonging to $\mathcal{K}(X)$. The aim goal of this work is to pursue the investigation started in [15], [19], [20], [12], and to extend it to lower semi-Browder multivalued linear operators (see Definition 4.1) on Banach spaces. More precisely, let X be a Banach space and let $T \in CR(X)$ such that $\mathcal{R}(T) \subset \mathcal{D}(T)$ and $V = T + K$ where $K \in \mathcal{K}(X)$. Under the hypotheses:

(i) $\rho(T) \neq \emptyset$, $\rho(T + K) \neq \emptyset$.

(ii) $KT \subset TK$.

we show in Theorem 4.1 (see Section 4) that T is lower semi-Browder linear relation if and only if $T + K$ is lower semi-Browder linear relation. As a consequence, we obtain the invariance and the characterization of Browder's essential defect spectrum (see Definition 4.2).

A brief outline of the paper follows. Throughout Section 2, we gather some auxiliary results needed in the rest of the paper. In Section 3, we establish some new properties of semi-Fredholm linear relations. The main results of this section is Proposition 3.1. In Section 4, we present perturbation results based on the analysis given in the two previous sections. We first consider a special case of lower semi-Browder linear relation. After that, we extend this study to general lower semi-Browder linear relations and we investigate the invariance and the characterization of Browder's essential defect spectrum.

2. Algebraic properties for linear relations

This section contains some algebraic properties in the context of linear relations in Banach spaces.

We first recall the following definitions.

Definition 2.1 (see [24]). Let X be a Banach space, T and $V \in LR(X)$.

(i) We say that T and V commute if $TV \subset VT$.

(ii) We say that T and V strictly commute if $TV = VT$.

Definition 2.2. Let X be a Banach space and $T \in LR(X)$. We say that T has a trivial singular chain manifold if $\mathcal{R}_c(T) = \{0\}$ where $\mathcal{R}_c(T) := (\cup_{n=1}^{\infty} \mathcal{N}(T^n)) \cap (\cup_{n=1}^{\infty} T^n(0))$.

Our main aim is to establish some commutation properties of linear relations.

Proposition 2.1. *Let X be a Banach space, $T \in LR(X)$ and $V = T + K$ where $K \in \mathcal{L}(X)$. Assume that $KT \subset TK$. Then,*

(i) $KV \subset VK$.

(ii) $TV = VT$.

(iii) $TV^k = V^k T$ *for all* $k \in \mathbb{N}$.

(iv) $T^p V = VT^p$ *for all* $p \in \mathbb{N}$.

(v) $KV^k \subset V^k K$ *for all* $k \in \mathbb{N}$.

Proof. (i) By [6, Theorem 2.1, 2.15, 2.14, 2.16], it is easy to see that $KV = K(T + K) = KT + K.K \subset TK + K.K = (T + K)K = VK$.

(ii) We first prove $TV \subset VT$. Indeed, from [6, Theorem 2.1, 2.16, 2.14, 2.15] we get $TV = (V - K)V \subset V.V - KV \subset V.V - VK \subset V(V - K) = VT$. Similarly, we prove $VT = (T + K)T \subset T.T + KT \subset T.T + TK \subset T(T + K) = TV$.

(iii) We first show by induction $TV^k \subset V^k T$ for all $k \in \mathbb{N}$. For $k = 1$ is clear. Assume that $TV^k \subset V^k T$ for all $k \in \mathbb{N}$. Then by [6, Theorem 2.1, 2.14], we have $TV^{k+1} = TV^k V \subset V^k TV \subset V^k VT = V^{k+1} T$. Reasoning in the same way as above, we obtain $V^{k+1} T \subset TV^{k+1}$.

(iv) This assertion can be checked in the same way as (iii).

(v) Proceeding in the same way as (iii). □

As an application, we infer

Lemma 2.1. *Let X be a Banach space, $T \in LR(X)$ and $V = T + K$ where $K \in \mathcal{L}(X)$. Assume that T is onto and $KT \subset TK$. Then for all $k \in \mathbb{N}^*$ we have*

(i) $\mathcal{R}(V^k) = \mathcal{R}(V^k T)$.

(ii) $\mathcal{R}(TV^k) = \mathcal{R}(V^kT)$.

(iii) $T(\mathcal{R}(V^k) \cap \mathcal{D}(T)) = \mathcal{R}(V^k)$.

Proof. (i) Let $y \in \mathcal{R}(V^k)$. Then $(x, y) \in G(V^k)$ for some $x \in \mathcal{D}(V^k)$. Due to the assumption T is onto, it follows that $(z, x) \in G(T)$ for some $z \in \mathcal{D}(T)$. Thus $(z, y) \in G(V^kT)$ which implies that $y \in \mathcal{R}(V^kT)$. It remains to show the converse inclusion. Let $y \in \mathcal{R}(V^kT)$ so that there exists $x \in \mathcal{D}(V^kT) := \{x \in \mathcal{D}(T) : Tx \cap \mathcal{D}(V^k) \neq \emptyset\}$ such that $y \in (V^kT)(x) = V^k(Tx) = V^kz$ where $z \in Tx \cap \mathcal{D}(V^k)$.

(ii) Let $y \in \mathcal{R}(V^kT)$ so that $(x, y) \in G(V^kT)$ for some $x \in \mathcal{D}(V^kT)$. Since $G(V^kT) = G(TV^k)$, then $y \in \mathcal{R}(TV^k)$. For the converse inclusion, we proceed in the same way as above.

(iii) Let $y \in T(\mathcal{R}(V^k) \cap \mathcal{D}(T))$. Thus there exists $z \in \mathcal{R}(V^k) \cap \mathcal{D}(T)$ such that $y \in Tz$. Using the fact $z \in \mathcal{R}(V^k)$, it follows that $(x, z) \in G(V^k)$ for some $x \in \mathcal{D}(V^k)$ and hence $(x, y) \in G(TV^k)$. Therefore by Proposition 2.1, $(x, y) \in G(V^kT)$ which shows that $y \in \mathcal{R}(V^k)$. Conversely, let $y \in \mathcal{R}(V^k) = \mathcal{R}(TV^k)$. Thus there exists $x \in \mathcal{D}(TV^k) := \{x \in \mathcal{D}(V^k) : V^kx \cap Tx \neq \emptyset\}$ such that $y \in (TV^k)(x) = T(V^kx) = Tz$ where $z \in \mathcal{R}(V^k) \cap \mathcal{D}(T)$. $\square$

The next proposition was established in [25] for linear operators. Similar statements can be made for linear relations.

Proposition 2.2. *Let X be a Banach space and $T \in CR(X)$.*

(i) *If $\mathcal{R}_c(T) = \{0\}$ and $a(T) < \infty$, then $\mathcal{N}^\infty(T) \cap \mathcal{R}^\infty(T) = \{0\}$.*

(ii) *If $\alpha(T) < \infty$ and $\mathcal{N}^\infty(T) \cap \mathcal{R}^\infty(T) = \{0\}$, then $a(T) < \infty$.*

Proof. (i) Assume that $\mathcal{R}_c(T) = \{0\}$ and $a(T) = p < \infty$. Let $x \in \mathcal{N}^\infty(T) \cap \mathcal{R}^\infty(T)$, so that there exists $y \in \mathcal{D}(T^p)$ such that $x \in T^py$ and $0 \in T^px$. Since $(x, 0) \in G(T^p)$ for some $x \in \mathcal{D}(T^p)$ and $(y, x) \in G(T^p)$ for some $y \in \mathcal{D}(T^p)$, it follows that $(y, 0) \in G(T^{2p})$ and hence $y \in \mathcal{N}(T^{2p}) = \mathcal{N}(T^p)$. Therefore by [9, Proposition 2.3, (ii)], $x \in T^p(0)$ which shows that $x \in \mathcal{R}_c(T) = \{0\}$.

(ii) Assume that $\alpha(T) < \infty$ and $\mathcal{N}^\infty(T) \cap \mathcal{R}^\infty(T) = \{0\}$. Since $\mathcal{R}(T^n)$ is a descending sequence and $\alpha(T) < \infty$, then there exists k such that

$$\mathcal{R}(T^k) \cap \mathcal{N}(T) = \mathcal{R}^\infty(T) \cap \mathcal{N}(T) \subset \mathcal{N}^\infty(T) \cap \mathcal{R}^\infty(T) = \{0\}.$$

Hence the result follows from [22, Lemma 5.5, (i)]. $\square$

Finally, we conclude this section with the following lemma which we need in the rest of the paper.

Remark 2.1. If $T \in BR(X)$ and the linear subspace $T(0)$ is closed, then from [10, II.5.1] $T \in CR(X)$.

Lemma 2.2. *Let X be a Banach space, $T \in BR(X)$ and $V = T + K$ where $K \in \mathcal{K}(X)$. Assume that T is onto and $KT \subset TK$. Let $k \in \mathbb{N}$ and*

$$\begin{cases} \widetilde{T} : X/\mathcal{R}(V^k) \longrightarrow X/\mathcal{R}(V^k) \\ \widetilde{x} \longrightarrow \widetilde{T}(\widetilde{x}) = \{\widetilde{y} \text{ such that } y \in Tx\}. \end{cases}$$

(i) *$\widetilde{T}$ is an onto single valued linear operator.*

(ii) *If we suppose also that the linear subspace $T(0)$ is closed, then $\widetilde{T}$ is one to one single valued linear operator.*

Proof. (i) Clearly from Lemma 2.1(iii), $\widetilde{T}$ is well defined. $\widetilde{T}$ is a single valued linear operator. Indeed, from [22, Lemma 3.2] if $y \in T(0) = V(0)$, then $\widetilde{y} = \widetilde{0}$. This leads to $\widetilde{T}(\widetilde{0}) = \{\widetilde{y} \text{ such that } y \in T(0) = V(0)\} = \widetilde{0}$ so that $\mathrm{mul}(\widetilde{T}) = \{\widetilde{y} \text{ such that } (0, \widetilde{y}) \in G(\widetilde{T})\} = \{\widetilde{y} \text{ such that } \widetilde{y} \in \widetilde{T}(0)\} = \widetilde{0}$. Since T is onto, then for $\widetilde{y} \in X/\mathcal{R}(V^k)$ we have $y \in Tx$ and hence $\widetilde{y} \in \{\widetilde{z} \text{ such that } z \in Tx\} = \widetilde{T}(\widetilde{x})$.

(ii) From Remark 2.1 together with [10, III.4.2 (b)], [4, Proposition 2, (ii)] and [5, Proposition 4 (i), Lemma 1], we infer $V \in \Phi_-(X)$. Thus by [22, Lemma 5.4], $\dim(X/\mathcal{R}(V^k)) = \beta(V^k) \leq k\beta(V) < \infty$ for $k \in \mathbb{N}$. This together with (i) leads to $\widetilde{T}$ is one to one linear operator. $\square$

Remark 2.2. (i) If $T \in BR(X)$ and $V = T + K$ where $K \in \mathcal{K}(X)$ such that T is onto, $KT \subset TK$ and $T(0)$ is a closed linear subspace, then it is easy to see that $\mathcal{N}(T) \subset \mathcal{R}(V^k)$ for all $k \in \mathbb{N}$. Indeed, let $x \in \mathcal{N}(T)$ so that $\widetilde{0} \in \{\widetilde{y} \text{ such that } y \in Tx\} = \widetilde{T}(\widetilde{x})$. The use of Lemma 2.2 (ii) leads to $x \in \mathcal{R}(V^k)$.

(ii) Let $T \in CR(X)$ and $V = T + K$ such that $KT \subset TK$. It is easy to see that $V(\mathcal{R}(V^m) \cap \mathcal{D}(V)) = \mathcal{R}(V^{m+1})$ and $K(\mathcal{R}(V^m)) = \mathcal{R}(KV^m) \subset \mathcal{R}(V^m K)$ for all $m \in \mathbb{N}$.

3. Semi-Fredholm linear relations

In this section we are concerned with the study of some properties of semi-Fredholm linear relations in Banach spaces.

In order to show these properties, we need a bit of preparation.

Lemma 3.1. *Let X be a Banach space and $T \in CR(X)$ such that $\rho(T) \neq \emptyset$. Then for every polynomial P, the linear relation $P(T)$ is closed.*

Proof. Let $\eta \in \rho(T)$ and $\lambda \in \mathbb{C}$. Take

$$\begin{aligned} P(\lambda) &= a_0 + a_1\lambda + \cdots + a_n\lambda^n = a_0 + a_1((\lambda-\eta)+\eta) + \cdots + a_n((\lambda-\eta)+\eta)^n \\ &= b_1(\lambda-\eta) + b_2(\lambda-\eta)^2 + \cdots + b_n(\lambda-\eta)^n + c \\ &= (\lambda-\eta)(b_1 + b_2(\lambda-\eta) + \cdots + b_n(\lambda-\eta)^{n-1}) + c. \end{aligned}$$

We define the linear relation

$$\begin{aligned} P(T) &:= (T-\eta)(b_1 + b_2(T-\eta) + \cdots + b_n(T-\eta)^{n-1}) + cI \\ &:= (T-\eta)Q(T) + cI \end{aligned}$$

where $Q(T)$ is a polynomial of degree $n-1$. Now we prove the result by induction. For $n = 1$ is clear. Assume that the result holds for any polynomial of degree $n-1$. Clearly, $T-\eta$ is closed and bijective linear relation. Therefore by the induction hypothesis and [10, II.5.17, III.4.2 (b)], $(T-\eta)Q(T)$ is closed. Since cI is a single valued continuous, then it follows from [5, Lemma 1] that $P(T) := (T-\eta)Q(T) + cI$ is closed. □

Lemma 3.2. *Let X, Y be two Banach spaces and $T \in CR(X,Y)$. Suppose that there exists a closed subspace Y_0 of Y such that $\mathcal{R}(T) \oplus Y_0$ is closed. Then the subspace $\mathcal{R}(T)$ is also closed.*

Proof. Let Π_1 and Π_2 the following operators:

$$\begin{cases} \Pi_1 : \mathcal{D}(T) \times Y_0 \longrightarrow \mathcal{D}(T) \\ (x, y_0) \longrightarrow x, \end{cases} \quad \text{and} \quad \begin{cases} \Pi_2 : \mathcal{D}(T) \times Y_0 \longrightarrow Y_0 \subset Y \\ (x, y_0) \longrightarrow y_0. \end{cases}$$

$$\text{Take} \quad \begin{cases} T_0 : \mathcal{D}(T_0) = \mathcal{D}(T) \times Y_0 \longrightarrow Y \\ (x, y_0) \longrightarrow T_0(x, y_0) = T \circ \Pi_1(x, y_0) + \Pi_2(x, y_0) = Tx + y_0. \end{cases}$$

T_0 is a closed linear relation. Indeed, let $((x_n, y_n), z_n) \in G(T_0) = \{((x,y),z)$ such that $(x,y) \in \mathcal{D}(T_0)$ and $z \in T_0(x,y)\}$ such that $(x_n, y_n) \to (x,y)$ and $z_n \to z$. Then, $x_n \to x$, $y_n \to y$ and $z_n = z_{n_1} + y_n$ where $z_{n_1} \in Tx_n$. Hence $y \in Y_0$ and $z_{n_1} \to z - y$. Since T is a closed linear relation, $x \in \mathcal{D}(T)$ and $z - y \in Tx$. Consequently $(x,y) \in \mathcal{D}(T_0) = \mathcal{D}(T) \times Y_0$ and $z \in Tx + y = T_0(x,y)$. On the other hand, it is easy to see that $\mathcal{N}(T_0) = \mathcal{N}(T) \times \{0\}$ and $\mathcal{R}(T_0) = \mathcal{R}(T) \oplus Y_0$. Hence from hypothesis and [10, II.3.2 (b), III.4.2 (b)], there exists $\lambda \geq 0$ such that $\|T_0(x,y_0)\| \geq \lambda d((x,y_0), \mathcal{N}(T_0))$ $\forall (x,y_0) \in \mathcal{D}(T_0)$. Therefore $\|Tx\| = \|T_0(x,0)\| \geq \lambda d((x,0), \mathcal{N}(T) \times \{0\}) = \lambda d(x, \mathcal{N}(T))$ $\forall x \in \mathcal{D}(T)$ which implies by the use of [10, II.3.2 (b), III.4.2 (b)] that $\mathcal{R}(T)$ is closed. □

A direct consequence of Lemma 3.2, we obtain

Corollary 3.1. *Let X, Y be two Banach spaces and $T \in CR(X,Y)$. If $\beta(T) = \dim(Y/\mathcal{R}(T)) < \infty$, then $\mathcal{R}(T)$ is closed.*

Now we are in the position to give the first fundamental result which generalizes Lemma 3.2 in [12] for multivalued linear operators.

Proposition 3.1. *Let X be a Banach space and $T \in CR(X)$ such that $\rho(T) \neq \emptyset$. Let $p \in \mathbb{N}$. If $T \in \Phi_-(X)$, then $T^p \in \Phi_-(X)$.*

Proof. From [22, Lemma 5.4], we infer $\beta(T^p) \leq p\beta(T) < \infty$ for all $p \in \mathbb{N}$. This together with Corollary 3.1 and Lemma 3.1 leads to $T^p \in \Phi_-(X)$ for all $p \in \mathbb{N}$. □

4. Perturbation of lower semi-Browder linear relations

Our aim is to set up a theorem of perturbation of lower semi-Browder linear relations under compact operator perturbations.

Definition 4.1. Let X be a Banach space and $T \in CR(X)$. We say that T is lower semi-Browder linear relation, denoted $T \in \mathcal{B}_-CR(X)$, if $T \in \Phi_-(X)$, $d(T) < \infty$ and $i(T) \geq 0$.

We begin our investigation with a proposition which treats a special case of lower semi-Browder linear relation (T is onto).

Proposition 4.1. *Let X be a Banach space, $T \in BR(X)$ and $V = T + K$ where $K \in \mathcal{K}(X)$. Assume that $T(0)$ is closed and $KT \subset TK$. If T is onto, then $d(V) < \infty$.*

Proof. Since T is an onto closed linear relation, then by [10, III.4.2 (b), II.3.2 (b)] there is a positive number γ for which $\|Tx\| \geq \gamma d(x, \mathcal{N}(T)) \forall\ x \in \mathcal{D}(T) = X$. Suppose that $x \in \mathcal{D}(T) = \mathcal{D}(V) = X$ and $z \in \mathcal{R}(V^k)$. Therefore by Lemma 2.1 (iii), there is $y \in \mathcal{R}(V^k) \cap \mathcal{D}(T)$ such that $z \in Ty$ ($Ty = z + T(0) = z + \overline{T(0)}$). Thus we have

$$\begin{aligned} \|T(x-y)\| &:= \|Q_T T(x-y)\| \\ &= \|Q_V Tx - Q_V z\| \geq \gamma d(x-y, \mathcal{N}(T)) \geq \gamma d(x, \mathcal{R}(V^k)), \end{aligned} \tag{4.1}$$

since $\mathcal{N}(T) \subset \mathcal{R}(V^k)$ for all $k \in \mathbb{N}$ (see Remark 2.2 (i)). Since this holds for all $z \in \mathcal{R}(V^k)$, we obtain

$$d(Q_V Tx, Q_V \mathcal{R}(V^k)) \geq \gamma d(x, \mathcal{R}(V^k)) \ \forall\, x \in \mathcal{D}(T) = X. \tag{4.2}$$

Suppose that V had infinite descent. Then there would be a bounded sequence $\{x_n\}$ with $x_n \in \mathcal{R}(V^n)$ and $d(x_n, \mathcal{R}(V^{n+1})) \geq 1$. Assume $m > n > 0$, then

$$\begin{aligned}\|Q_V K x_n - Q_V K x_m\| &= \|Q_V K x_n - Q_V K x_m + Q_V T x_n - Q_V T x_n\| \\ &= \|Q_V V x_n - Q_V T x_n - Q_V K x_m\|.\end{aligned}$$

The use of Remark 2.2 (ii) and equation (4.2) leads to

$$\|Q_V K x_n - Q_V K x_m\| \geq d(Q_V T x_n, Q_V \mathcal{R}(V^{n+1})) \geq \gamma d(x_n, \mathcal{R}(V^{n+1})) \geq \gamma \tag{4.3}$$

which contradicts the compactness of the linear operator $Q_V K$. □

Now as a corollary, we get

Corollary 4.1. *Let X be a Banach space, $T \in CR(X)$ and $V = T + K$ where $K \in \mathcal{K}(X)$ such that $KT \subset TK$. Let Y be a closed subspace $\subset \mathcal{D}(T)$ satisfying $T(Y) \subset Y$ and $V(Y) \subset Y$. We denote by $\overline{T}$ and $\overline{V}$, respectively, the restriction of T and V to Y.*

If $\overline{T}$ is onto, then $\overline{V}$ has finite descent.

Proof. From [5, Lemma 1], $V = T + K \in CR(X)$. Since $T \in CR(X)$ and Y is a closed subspace, $\overline{T} = T|Y$ is also closed. Similarly, $\overline{V} = V|Y \in CR(X)$. So that by [4, Lemma 5], the operators $Q_{\overline{T}}\overline{T}$ and $Q_{\overline{V}}\overline{V}$ are closed and the linear subspace $\overline{T}(0)$ is also closed. Hence applying the closed graph theorem, we get $Q_{\overline{T}}\overline{T}$ and $Q_{\overline{V}}\overline{V}$ are bounded operators which implies that the relations $\overline{T}$ and $\overline{V} \in BR(Y)$. According to the hypothesis $KT \subset TK$, it follows that $\overline{K}\,\overline{T} \subset \overline{T}\,\overline{K}$. Indeed, let $(x, y) \in G(\overline{K}\,\overline{T})$ so that there exists $z \in \mathcal{D}(\overline{K}) = Y$ such that $(x, z) \in G(\overline{T})$ and $(z, y) \in G(\overline{K})$. Since $Y \subset \mathcal{D}(T)$, then $(x, y) \in G(KT) \subset G(TK)$ and hence $G(\overline{K}\,\overline{T}) \subset G(KT) \subset G(TK) \subset G(\overline{T}\,\overline{K})$. It remains to show that $\overline{V} = \overline{T} + \overline{K}$ with $\overline{K} \in \mathcal{K}(Y)$. To do this, we consider $x \in Y$, then $\overline{V}x = (\overline{T + K})x = (T + K)x = Tx + Kx = \overline{T}x + \overline{K}x$. On the other hand, let $(x_n)_{n\in\mathbb{N}}$ a bounded sequence of $Y \subset X$, then $(Kx_n)_{n\in\mathbb{N}}$ admits a subsequence which converge in X. Moreover, $(\overline{K}x_n)_{n\in\mathbb{N}} = (Kx_n)_{n\in\mathbb{N}} \in Y$ and Y is a closed subspace of X. Therefore, $(\overline{K}x_n)_{n\in\mathbb{N}}$ admits a subsequence which converge in Y. This implies, by the use of Proposition 4.1 that $d(\overline{V}) < \infty$. □

We now extend our study to the general case.

Theorem 4.1. *Let X be a Banach space, $T \in CR(X)$ such that $\mathcal{R}(T) \subset \mathcal{D}(T)$ and $V = T + K$ where $K \in \mathcal{K}(X)$. Assume that:*

(i) $\rho(T) \neq \emptyset$, $\rho(T + K) \neq \emptyset$.

(ii) $KT \subset TK$.

Then $\qquad T \in \mathcal{B}\_CR(X)$ *if and only if* $T + K \in \mathcal{B}\_CR(X)$.

Proof. Let $T \in \Phi_-(X)$ such that $d(T) = p < \infty$ and $i(T) \geq 0$. From [10, III.4.2 (b)] together with [4, Proposition 2, (ii)] and [5, Proposition 4 (i), Lemma 1], we deduce that $V = T + K \in \Phi_-(X)$. The subspace $\mathcal{R}(T^p)$ is closed and satisfies $T(\mathcal{R}(T^p)) \subset \mathcal{R}(T^p)$ and $V(\mathcal{R}(T^p)) \subset \mathcal{R}(T^p)$. Indeed, let $z \in T(\mathcal{R}(T^p))$ so that there exists $y \in \mathcal{R}(T^p) \subset \mathcal{D}(T)$ such that $z \in Ty$. The fact that $y \in \mathcal{R}(T^p)$ proves that $(x, y) \in G(T^p)$ for some $x \in \mathcal{D}(T^p)$ and hence $(x, z) \in G(T^{p+1})$. So $z \in \mathcal{R}(T^{p+1}) = \mathcal{R}(T^p)$. Similarly, by the use of Proposition 2.1 (iv), we show that $V(\mathcal{R}(T^p)) \subset \mathcal{R}(T^p)$. The closure of the subspace $\mathcal{R}(T^p)$ follows, immediately, from Proposition 3.1. Since the restriction of T to $\mathcal{R}(T^p)$ is onto, therefore by Corollary 4.1, $V|\mathcal{R}(T^p)$ has finite descent. So that there is an integer k for which

$$\mathcal{R}(V^m T^p) = \mathcal{R}((V|\mathcal{R}(T^p))^k) = \mathcal{R}(V^k|\mathcal{R}(T^p) \cap \mathcal{D}(V^k)) = \mathcal{R}(V^k T^p)\ \forall\, m \geq k.$$

Let $m_1 \geq k$. So that

$$\beta(V^{m_1}) = \dim(X/\mathcal{R}(V^{m_1})) \leq \dim(X/\mathcal{R}(V^{m_1}T^p)) = \dim(X/\mathcal{R}(V^k T^p))\ . \tag{4.4}$$

The use of [23, Lemma 5.1] together with Proposition 3.1 makes us to conclude

$$\beta(V^{m_1}) \leq \beta(V^{m_1}T^p) = \beta(V^k T^p) \leq \beta(V^k) + \beta(T^p) < \infty. \tag{4.5}$$

So by [22, Lemma 6.2], we obtain $d(V) < \infty$. It remains to show that $i(T+K) \geq 0$. Let $t_0 \in [0, 1]$. A similar reasoning as above gives $T + t_0K \in \Phi_-(X)$. Therefore, there is an $\alpha > 0$ such that $\forall\, t \in [0, 1]$ satisfying $|t - t_0| < \alpha$ we have by [10, V.15.7], $i(T + t_0K) = i(T + t_0K + (t - t_0)K) = i(T + tK)$. From the Heine–Borel theorem, there is a finite number of sets which cover $[0, 1]$. Since each of theses sets overlaps with at least one other and the index is constant on each one, we see that $i(T + K) = i(T) \geq 0$. Conversely, let $V = T + K \in \Phi_-(X)$ such that $d(T + K) < \infty$ and $i(T + K) \geq 0$. The use of Proposition 2.1 (i) leads to $-K(T + K) = -KV \subset -VK = -(T + K)K$. On the other hand, let $y \in \mathcal{R}(T + K)$ so that there exists $x \in \mathcal{D}(T)$ such that $y \in (T + K)x = Tx + Kx$. Since $\mathcal{D}(T) = \mathcal{D}(KT) \subset \mathcal{D}(TK) := \{x \in \mathcal{D}(K) : Kx \cap \mathcal{D}(T) \neq \emptyset\}$, then it follows that $Kx \in \mathcal{D}(T)$ and hence $y \in \mathcal{D}(T) = \mathcal{D}(T) \cap \mathcal{D}(K) = \mathcal{D}(T+K)$. Now, applying the reasoning above we get $T + K - K = T \in \mathcal{B}\_CR(X)$. $\square$

As an application, we infer the invariance and the characterization of Browder's essential defect spectrum under compact operator perturbations.

Definition 4.2. (i) The Browder essential defect spectrum of $T \in CR(X)$ is the set $\sigma_{db}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}\_CR(X)\}$.

(ii) The defect spectrum of $T \in CR(X)$ is the set $\sigma_d(T) := \{\lambda \in \mathbb{C} : \lambda - T$ is not onto$\}$.

(As it is usual, we write $\lambda - T := \lambda I_X - T,\ \lambda \in \mathbb{C}$.)

Theorem 4.2. *Let X be a Banach space, $T \in CR(X)$ such that $\mathcal{R}(T) \subset \mathcal{D}(T)$ and $V = T + K$ where $K \in \mathcal{K}(X)$. Assume that:*

(i) $\rho(T) \neq \emptyset$, $\rho(T + K) \neq \emptyset$.

(ii) $KT \subset TK$.

Then
$$\sigma_{db}(T + K) = \sigma_{db}(T).$$

Proof. We first claim that $\sigma_{db}(T+K) \subset \sigma_{db}(T)$. Indeed, if $\lambda \notin \sigma_{db}(T)$ then $T-\lambda \in \mathcal{B}\_CR(X)$. From [10, VI.5.4], we infer $\rho(T-\lambda) \neq \emptyset$ and $\rho(T+K-\lambda) \neq \emptyset$. On the other hand, by [6, Theorem 2.1, 2.14, 2.15, 2.16] we have $K(T - \lambda) = KT-\lambda K \subset TK-\lambda K = (T-\lambda)K$. This implies by Theorem 4.1 that $T+K-\lambda \in \mathcal{B}\_CR(X)$. So $\lambda \notin \sigma_{db}(T + K)$. Similarly, we show $\sigma_{db}(T) \subset \sigma_{db}(T + K)$. $\square$

Theorem 4.3. *Let X be a Banach space and $T \in CR(X)$ such that $\mathcal{R}(T) \subset \mathcal{D}(T)$ and $\rho(T) \neq \emptyset$. Let $G = \{K \in \mathcal{K}(X)$ such that $KT \subset TK$ and $\rho(T+K) \neq \emptyset\}$. Then*
$$\sigma_{db}(T) \subset \bigcap_{K\in G} \sigma_d(T + K).$$

Proof. If $\lambda \notin \{\sigma_d(T+K)$ with $K \in G\}$ thus $\exists K_0 \in G$ such that $T+K_0-\lambda$ is onto. Hence $T + K_0 - \lambda \in \mathcal{B}\_CR(X)$. Therefore, by Theorem 4.2 $\lambda \notin \sigma_{db}(T \boxplus K_0) = \sigma_{db}(T)$.

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