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On the diameter and girth of ideal-based zero-divisor graphs

By SHAHABADDIN EBRAHIMI ATANI (Rasht), AHMAD YOUSEFIAN DARANI (Ardabil) and EDMUND R. PUCZYŁOWSKI (Warsaw)

Abstract. We describe the diameter and the girth of the zero-divisor graph of a commutative ring R with respect to a nonzero ideal I.

1. Introduction and preliminaries

Let R be a commutative ring with identity. The zero-divisor graph [1] of R, denoted $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero-divisors of R with two distinct vertices a and b joined by an edge if and only if $ab = 0$. This graph was studied by many authors. A particular attention was paid to its diameter and girth (cf. [2]–[5] and papers cited there).

In [6] Redmond introduced and studied the following more general concept. For a given ideal I of R the zero-divisor graph $\Gamma_I(R)$ of R with respect to I is the undirected graph with vertices $T(I) = \{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$.

Obviously $\Gamma(R) = \Gamma_0(R)$ and it is clear that $\Gamma_I(R) = \emptyset$ if and only if I is a prime ideal of R (we treat R as a prime ideal of R).

For given distinct vertices a and b in an undirected graph G, the distance between a and b, denoted $d(a,b)$, is the length of a shortest path connecting a and b, if such a path exists; otherwise, $d(a,b) = \infty$. We set $d(a,b) = 0$ if and only if $a = b$. If $d(a,b) < \infty$ for arbitrary a, b, then G is said to be connected and its diameter is defined as $\operatorname{diam}(G) = \sup\{d(a,b) \mid a, b \text{ are vertices of } G\}$. The girth

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$\mathrm{gr}(G)$ of G is defined as the length of a shortest cycle in G. If G has no cycles, we set $\mathrm{gr}(G) = \infty$.

The invariants $\mathrm{diam}(\Gamma(R))$ and $\mathrm{gr}(\Gamma(R))$ are well described (cf. [1]–[5]). In particular it is known that $\mathrm{diam}(\Gamma(R)) \leq 3$ and $\mathrm{gr}(\Gamma(R)$ is 3, 4 or ∞. In this paper we continue studies, started in [6], of $\mathrm{diam}(\Gamma_I(R))$ and $\mathrm{gr}(\Gamma_I(R))$ for $I \neq 0$.

For given $a \in R$ denote by $\bar{a}$ the image of a in R/I. In Proposition 2.1 we describe for arbitrary distinct $a, b \in T(I)$ the relationship between $d(a, b)$ and $d(\bar{a}, \bar{b})$. This allows us to find (Section 2) a precise relationship between $\mathrm{diam}(\Gamma_I(R))$ and $\mathrm{diam}(\Gamma(R/I))$, and, applying known results on $\mathrm{diam}(\Gamma(R/I))$, characterize $\mathrm{diam}(\Gamma_I(R))$. In Section 3 we get a complete, more explicit than in [6], description of $\mathrm{gr}(\Gamma_I(R))$.

Throughout the paper R is a commutative ring with identity and I is a nonzero ideal of R, which is not prime (in particular $I \neq R$).

2. Diameter

We start with describing the relationship between $d(a, b)$ and $d(\bar{a}, \bar{b})$, for arbitrary $a, b \in T(I)$. It is evident that $d(\bar{a}, \bar{b}) \leq d(a, b)$.

Proposition 2.1. *Let a, b be distinct elements of $T(I)$.*

(1) ([6], Theorem 2.5) *If $\bar{a} \neq \bar{b}$, then $d(a, b) = 1$ if and only if $d(\bar{a}, \bar{b}) = 1$;*

(2) (cf. [6], Corollary 2.6) *If $\bar{a} = \bar{b}$, then $d(a, b) = 1$ provided $a^2 \in I$ and $d(a, b) = 2$ otherwise;*

(3) *If $d(\bar{a}, \bar{b}) = 2$, then $d(a, b) = 2$;*

(4) *$d(\bar{a}, \bar{b}) = 3$ if and only if $d(a, b) = 3$.*

Proof. (1) is obvious as $ab \in I$ if and only if $\bar{a}\bar{b} = 0$.

(2) If $\bar{a} = \bar{b}$ and $a^2 \in I$, then $\bar{a}\bar{b} = \bar{a}^2 = 0$, so $ab \in I$. Consequently $d(a, b) = 1$. Suppose that $\bar{a} = \bar{b}$ and $a^2 \notin I$. Then $ab \notin I$, so $d(a, b) > 1$. Since $a \in T(I)$, there is $c \in T(I)$ such that $ac \in I$. However $\overline{ac} = \overline{bc}$, so $bc \in I$. Hence, since $a^2 \notin I$, $a \neq c \neq b$. Consequently $d(a, b) = 2$ and we are done.

(3) Suppose that $d(\bar{a}, \bar{b}) = 2$. Then $ab \notin I$, so $d(a, b) > 1$. Since $d(\bar{a}, \bar{b}) = 2$ there is $c \in T(I)$ such that $\bar{a} \neq \bar{c} \neq \bar{b}$ and $ac \in I$, $bc \in I$. Obviously c is distinct from a and b, so $d(a, b) = 2$.

(4) The "if" part follows from the inequality $d(\bar{a}, \bar{b}) \leq d(a, b)$ and (1)–(3). Conversely suppose that $d(\bar{a}, \bar{b}) = 3$. Then there is a path $\bar{a}, \bar{a}_1, \bar{a}_2, \bar{b}$ in $\Gamma(R/I)$

of length 3. Obviously a, a_1, a_2, b is a path in $\Gamma_I(R)$, so $d(a,b) \leq 3$. Since $3 = d(\bar{a},\bar{b}) \leq d(a,b)$, we get $d(a,b) = 3$. The proof is complete. $\square$

Proposition 2.1 together with the fact that $\mathrm{diam}(\Gamma(R/I)) \leq 3$ give a precise description of $d(a,b)$ in terms of $d(\bar{a},\bar{b})$. It in particular implies ([6], Theorem 2.4) that $\mathrm{diam}(\Gamma_I(R)) \leq 3$. This allows us to find a precise relation between $\mathrm{diam}(\Gamma_I(R))$ and $\mathrm{diam}(\Gamma(R/I))$.

Corollary 2.1. (1) $\mathrm{diam}(\Gamma_I(R)) = 1$ *if and only if* $\mathrm{diam}(\Gamma(R/I)) = 1$ *or* 0*;*

(2) $\mathrm{diam}(\Gamma_I(R)) = 2$ *if and only if* $\mathrm{diam}(\Gamma(R/I)) = 2$*;*

(3) $\mathrm{diam}(\Gamma_I(R))) = 3$ *if and only if* $\mathrm{diam}(\Gamma(R/I)) = 3$.

Proof. (1) is an immediate consequence of Proposition 2.1 (1) and (2) and the fact that $\mathrm{diam}(\Gamma(R/I)) = 0$ if and only if $T(I) = r+I$ for an element $r \in R\setminus I$ such that $r^2 \in I$.

(2) is an immediate consequence of Proposition 2.1.

We know that $\mathrm{diam}(\Gamma(R/I)) \leq 3$, $\mathrm{diam}(\Gamma_I(R)) \leq 3$ and, by (1) and (2), $\mathrm{diam}(\Gamma_I(R)) \leq 2$ if and only if $\mathrm{diam}(\Gamma(R/I)) \leq 2$. These and Proposition 2.1 (4) imply (3). $\square$

Applying [3, Theorem 2.6] one obtains a description of $\mathrm{diam}(\Gamma(R/I))$, which together with Corollary 2.1 give the following description of $\mathrm{diam}(\Gamma_I(R))$ (since $\mathrm{diam}\,\Gamma_I(R) \leq 3$ it is enough to describe cases in which the diameter is equal 1 or 2).

Theorem 2.1. (1) $\mathrm{diam}(\Gamma_I(R)) = 1$ *if and only if one of the following conditions holds*

(i) $R/I \simeq Z_2 \times Z_2$*;*

(ii) $T(I) = \sqrt{I} \setminus I$, *where* $\sqrt{I}$ *is the radical of* I, *i.e.*, $\sqrt{I} = \{r \in R \mid r^n \in I$ *for a positive integer* $n\}$ *and* $(\sqrt{I})^2 \subseteq I$.

(2) $\mathrm{diam}(\Gamma_I(R)) = 2$ *if and only if either*

(i) $I = P_1 \cap P_2$ *for some distinct prime ideals* P_1, P_2 *of* R *and* $R/I \not\simeq Z_2 \times Z_2$

or

(ii) *the former conditions do not hold and each pair of distinct zero divisors of* R/I *has a nonzero annihilator.*

3. Girth

In this section we describe $\mathrm{gr}(\Gamma_I(R))$ (recall that $I \neq 0$ and it is not a prime ideal).

Theorem 3.1. (1) *Suppose that R/I is a reduced ring, i.e., it contains no nonzero nilpotent elements. Then $gr(\Gamma_I(R)) = 4$ if and only if $I = P_1 \cap P_2$ for distinct prime ideals P_1, P_2 of R. In this case $\Gamma_I(R)$ is a complete bipartite graph. Otherwise $gr(\Gamma_I(R)) = 3$.*

(2) *Suppose that the ring R/I is not reduced. Then $gr(\Gamma_I(R)) = 3$ or $gr(\Gamma_I(R)) = \infty$. The latter holds if and only if (up to isomorphism)*

(i) $R = Z_2 \oplus Z_4$ *and* $I = Z_2$;

(ii) $R = Z_2 \oplus (Z_2[x]/(x^2))$ *and* $I = Z_2$;

(iii) $R = Z_8$ *and* $I = 4R$;

(iv) $R = Z_4[x]/(x^2 - 2, 2x)$ *and* $I = (2, x^2)/(x^2 - 2, 2x)$;

(v) $R = Z_4[x]/(2x, x^2)$ *and* $I = (2, x^2)/(2x, x^2)$ *or* $I = (x)/(2x, x^2)$ *or* $I = (x - 2, 2x)/(2x, x^2)$;

(vi) $R = Z_2[x]/(x^3)$ *and* $I = (x^2)/(x^3)$;

(vii) $R = Z_2[x, y]/(x^2, y^2, xy)$ *and* $I = (x, y^2, xy)/(x^2, y^2, xy)$.

Proof. (1) This part can be deduced from results obtained in [5]. We give a short direct proof. Note first that since $\bar{R} = R/I$ is a reduced ring, for any ideals K, L of $\bar{R}$, $K \cap L = 0$ if and only if $KL = 0$. It is evident that if $\bar{R}$ contains a direct sum of more than two nonzero ideals, then $gr(\Gamma_I(R)) = 3$. Hence, since I is not a prime ideal, $gr(\Gamma_I(R)) \neq 3$ if and only if $\bar{R}$ contains a direct sum of two but not more nonzero ideals. Thus there are nonzero $a, b \in \bar{R}$ such that $ab = 0$ and $\bar{R}a, \bar{R}b$ are domains. Let $A = \{x \in \bar{R} \mid ax = 0\}$ and $B = \{x \in \bar{R} \mid xb = 0\}$. If for some $x, y \in \bar{R}$, $xya = 0$, then $xaya = 0$. Since $\bar{R}a$ is a domain, $xa = 0$ or $ya = 0$. This shows that A is a prime ideal. Similarly B is a prime ideal. Now the sum $Ra + Rb$ is direct and since $\bar{R}$ does not contain direct sums of more than two nonzero ideals, for every $0 \neq x \in \bar{R}$, $xa \neq 0$ or $xb \neq 0$. This shows that $A \cap B = 0$. Let P_1 and P_2 be the preimages of A and B in R, respectively. Then P_1, P_2 are distinct prime ideals of R such that $P_1 \cap P_2 = I$. Clearly $T(I) = (P_1 \setminus I) \cup (P_2 \setminus I)$ and $x, y \in T(I)$ are adjacent if and only if one of them belongs to $P_1 \setminus I$ and the other to $P_2 \setminus I$. Since $I \neq 0$, none of $P_1 \setminus I$ and $P_2 \setminus I$ is singleton. It is clear that $\Gamma_I(R)$ contains no triangle but if x_1, x_2 are distinct elements of $P_1 \setminus I$ and y_1, y_2 are distinct elements of $P_2 \setminus I$, then x_1, y_1, x_2, y_2 form a cycle. Hence $gr(\Gamma_I(R)) = 4$. It is clear that $\Gamma_I(R)$ is a complete bipartite graph.

(2) Applying Zorn's Lemma we can find an ideal J of R containing I and maximal with respect to $J^2 \subseteq I$. If $J \setminus I$ contains more than 2 elements, then $gr(\Gamma_I(R)) = 3$. In particular, this holds when I contains more than 2 elements. Let A/I be the annihilator of J/I in R/I. Obviously $J \subseteq A$. If $A \neq J$, then for arbitrary $a \in A \setminus J$, $j \in J \setminus I$ and $0 \neq i \in I$, the elements j, a, $i + j$ form a triangle in $\Gamma_I(R)$. Hence in this case $gr(\Gamma_I(R)) = 3$.

Consequently $gr(\Gamma_I(R)) = 3$ unless $\operatorname{card} I = 2$, $A = J$ and $\operatorname{card}(J/I) = 2$. Suppose that these conditions hold. Take $0 \neq a \in J/I$ and define the map $f : R/I \to J/I$ by $f(x) = ax$. Obviously f is an epimorphism of abelian groups and $\operatorname{Ker} f = A/I = J/I$. Consequently $\operatorname{card}(R) = 8$. It is clear that $\Gamma_I(R)$ has precisely two vertices and they are adjacent, so $gr(\Gamma_I(R)) = \infty$. It is also clear that I is isomorphic (as a ring) to Z_2 or $I^2 = 0$.

If $I \simeq Z_2$, then $R = I \oplus I'$ for an ideal I' of R with $\operatorname{card} I' = 4$. Hence in this case, up to isomorphism, $R = Z_2 \oplus Z_4$ or $R = Z_2 \oplus (Z_2[x]/(x^2))$ and $I = Z_2$ These give (i) and (ii), respectively.

Suppose now that $I^2 = 0$. The order of the identity element 1 of R in the additive group of R is 8, 4 or 2. If it is 8, then obviously $R \simeq Z_8$ and $I = 4R$, so we get (iii).

Suppose that the order of 1 is 4. Then the subring of R generated by 1 can be identified with Z_4. Moreover, since $\operatorname{card} J = 4$, $J \cap Z_4 = \{0, 2\}$. Every $j \in J$ is nilpotent, so $1 - j$ is an invertible element of R and consequently $2 \notin 2J$. This implies that $2J = 0$ as otherwise we would have $J = \{0, 2\} + 2J$ and $0 \neq 2J = 2(\{0, 2\} + 2J) = 0$, a contradiction. Pick $a \in J \setminus I$. Then $a^2 = 2$ or $a^2 = 0$. In the former case $a^3 = 0$ and $I = 2Z_4$. If the latter holds, then I can be any nontrivial subgroup of the additive group of $\{0, 2, a, a + 2\}$. Let f be the Z_4-algebra epimorphism of $Z_4[x]$ onto R such that $f(x) = a$. In the former case $\operatorname{Ker} f = (x^2 - 2, 2x, x^3) = (x^2 - 2, 2x)$ and $f^{-1}(I) = 2Z_4 + \operatorname{Ker} f = (2, x^2)$. This gives (iv). If the latter holds, then $\operatorname{Ker} f = (2x, x^2)$ and $f^{-1}(I)$ is equal to $2Z_4 + (2x, x^2) = (2, x^2)$ or (x) or $(x - 2, 2x, x^2) = (x - 2, 2x)$. These give (v).

Finally suppose that the order of 1 is 2. Then $2R = 0$ and R is a Z_2-algebra. Let $a \in J \setminus I$ and $I = \{0, i\}$. Then $a^2 = i$ or $a^2 = 0$. Since $1 - a$ is an invertible element of R, $ai \neq i$. Hence $ai = 0$. If $a^2 = i$, then $a^3 = 0$ and the homomorphism $f : Z_2[x] \to R$ of Z_2-algebras such that $f(x) = a$ is an epimorphism with $\operatorname{Ker} f = (x^3)$ and $f^{-1}(I) = (x^2)$. Thus (up to isomorphism) $R = Z_2[x]/(x^3)$ and $I = (x^2)/(x^3)$, so we get (vi). If $a^2 = 0$, then the homomorphism $f : Z_2[x, y] \to R$ of Z_2-algebras such that $f(x) = i$ and $f(y) = a$ is an epimorphism with $\operatorname{Ker} f = (x^2, y^2, xy)$ and $f^{-1}(I) = (x, y^2, xy) = (x, y^2)$. Hence (up to isomorphism) $R = Z_2[x, y]/(x^2, y^2, xy)$ and $I = (x, y^2)/(x^2, y^2, xy)$, so we get (vii).

These show that if $gr(\Gamma_I(R)) \neq 3$, then one of conditions (i)–(vii) holds. It is not hard to see that if any of conditions (i)–(vii) is satisfied, then $gr(\Gamma_I(R)) = \infty$. $\square$

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SHAHABADDIN EBRAHIMI ATANI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GUILAN
P.O. BOX 1914, RASHT
IRAN

E-mail: ebrahimi@guilan.ac.ir
URL: http://staff.guilan.ac.ir/ebrahimi/?lg=1

AHMAD YOUSEFIAN DARANI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MOHAGHEGH ARDABILI
P.O. BOX 179, ARDABIL
IRAN

E-mail: yousefian@uma.ac.ir
URL: http://www.yousefiandarani.com

EDMUND R. PUCZYŁOWSKI
INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW
BANACHA 2
02-097 WARSAW
POLAND

E-mail: edmundp@mimuw.edu.pl
URL: http://www.mimuw.edu.pl/badania/publikacje/

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