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Continuous solutions of an iterative-difference equation and Brillouët-Belluot's problem

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Abstract. It is an open problem proposed by N. Brillouët-Belluot to solve the equation $f^2(x) = f(x + a) - x$. Although some related results have been obtained, the problem has remained open. In this paper we prove that it has no continuous real solutions, finally answering Brillouët-Belluot's problem. Furthermore, we give existence of continuous real solutions for the general equation $f^2(x) = \lambda f(x + a) + \mu x$ on the whole $\mathbb{R}$ in some cases which neither include the equation $f^2(x) = f(x + a) - x$ nor are considered in [J. Difference Equ. Appl. **16**(11) (2010), 1237–1255].

1. Introduction

The equation

$$f^2(x) = f(x + a) - x, \quad x \in \mathbb{R}, \tag{1.1}$$

was proposed by N. Brillouët-Belluot in [3] when she investigated continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$x + f(y + f(x)) = y + f(x + f(y)). \tag{1.2}$$

This was mentioned by K. Baron again in [2] three years later. Actually, equation (1.1) is reduced from equation (1.2) by putting $y = 0$ and letting $a = f(0)$. Obviously, for $a = 0$ equation (1.1) is of the form $f^2(x) = f(x) - x$, which has no continuous real solutions by Theorem 11 in [6] or Theorem 5 in [8]. So we

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only need to consider equation (1.1) in the case $a \neq 0$, called a second order iterative-difference equation.

Obviously, equation (1.2) has no (continuous real) solutions if equation (1.1) has no (continuous real) solutions for any constant a, but the converse may not be true. Hence, it remains interesting to study the existence of continuous real solutions for equation (1.1) although nonexistence of continuous real solutions, existence of special solutions and the form of complex solutions continuous at a point for equation (1.2) were given in [5], [7] and [1] respectively. Recently, in [4] the equation

$$f^2(x) = \lambda f(x+a) + \mu x, \quad x \in J, \tag{1.3}$$

where J is an interval of $\mathbb{R}$, λ, μ and a are all real and $a\lambda \neq 0$, a generalized form of (1.1), was discussed. The authors searched all affine solutions and proved the existence of bounded continuous solutions on compact intervals under the condition

$$|\lambda| > \max\{2, 2\sqrt{2|\mu|}\} \quad \text{or} \quad 1 + 2|\mu| < |\lambda| \leq 2.$$

Then they constructed continuous solutions and piecewise continuous solutions when

$$\mu \geq 0 \quad \text{and} \quad 0 < 1 - \mu \leq \lambda/2$$

and give their maximal extensions. Finally, they proved that equation (1.3) has no continuous solution f on $\mathbb{R}$ such that either $f(x) - f(y) < -(\mu/\lambda)(x-y)$ if $\lambda > 0$ or $f(x) - f(y) > -(\mu/\lambda)(x-y)$ if $\lambda < 0$ for all $x > y$, which actually implies that equation (1.1) has no continuous solution f such that $f(x) - f(y) < x - y$ where $x > y$.

In this paper we will consider equation (1.3) again, where we always assume that $a\lambda \neq 0$. We prove that equation (1.3) has no continuous real solutions in the case that $\lambda = 1$ and $\mu \leq -1$, which implies the nonexistence of continuous real solutions for equation (1.1) and gives a negative answer to the Brillouët-Belluot's open problem. Furthermore, we give the existence of continuous real solutions for equation (1.3) on the whole $\mathbb{R}$ in some cases which neither include the form $f^2(x) = f(x+a) + \mu x$ nor are considered in [4].

Without loss of generality, we can assume that $a = 1$ in equation (1.3), i.e.,

$$f^2(x) = \lambda f(x+1) + \mu x. \tag{1.4}$$

Otherwise, we turn to consider $g(x) := \frac{1}{a} f(ax)$, which satisfies

$$g^2(x) = \frac{1}{a} f^2(ax) = \frac{1}{a}[\lambda f(ax + a) + \mu ax] = \frac{\lambda}{a} f(a(x + 1)) + \mu x = \lambda g(x + 1) + \mu x,$$

an equation of the form (1.4). Hence we only need to consider equation (1.4).

2. Answer to Brillouët-Belluot's problem

Regarding Brillouët-Belluot's open problem mentioned in the beginning of the Introduction, we consider equation (1.4) in the case that $\lambda = 1$, $\mu \le -1$, i.e.,

$$f^2(x) = f(x + 1) + \mu x. \tag{2.1}$$

Then we get that equation (1.1) has no continuous real solution, which negatively answers to Brillouët-Belluot's open problem.

Theorem 1. *For $\mu \le -1$ equation (2.1) has no continuous solutions defined on $\mathbb{R}$.*

Proof of Theorem 1. By reductio ad absurdum, suppose (2.1) has a continuous solution $f : \mathbb{R} \to \mathbb{R}$. Then the graph of f intersects the line $y = x + 1$ at most at the point $(0, 1)$. Hence, one of the following cases holds:

($\mathbf{C}_{--}$) $f(x) < x + 1$ for $x \in \mathbb{R}\backslash\{0\}$,

($\mathbf{C}_{-+}$) $f(x) < x + 1$ for $x \in (-\infty, 0)$ and $f(x) > x + 1$ for $x \in (0, \infty)$,

($\mathbf{C}_{+-}$) $f(x) > x + 1$ for $x \in (-\infty, 0)$ and $f(x) < x + 1$ for $x \in (0, \infty)$,

($\mathbf{C}_{++}$) $f(x) > x + 1$ for $x \in \mathbb{R}\backslash\{0\}$.

Consider f in the case ($\mathbf{C}_{--}$), in which $f(x) \le x + 1$ for all $x \in \mathbb{R}$. Thus,

$$\lim_{x\to-\infty} f(x) = -\infty. \tag{2.2}$$

Moreover, $f(x + 1) = f^2(x) - \mu x \le f(x) + 1 - \mu x$ for all $x \in \mathbb{R}$. It follows that

$$f(x_0 + 1) < f(x_0) \tag{2.3}$$

for each fixed $x_0 < -1$. Applying (2.3) repeatedly, we get $f(x_0) < f(x_0 - 1) < \cdots < f(x_0 - n)$. Thus, by (2.2),

$$f(x_0) \le \lim_{n\to+\infty} f(x_0 - n) = \lim_{x\to-\infty} f(x) = -\infty,$$

which is a contradiction because f takes only finite values.

Consider f in the case ($\mathbf{C}_{-+}$), in which $f(0) = 1$ and there exists a point $c_1 \in (-1, 0)$ such that $f(c_1) = 0$. It implies that

$$1 = f^2(c_1) = f(c_1 + 1) + \mu c_1 > c_1 + 2 + \mu c_1 = 2 + (1 + \mu)c_1$$

because $c_1 + 1 > 0$ and $f(c_1 + 1) > (c_1 + 1) + 1$ in the case ($\mathbf{C}_{-+}$). This is a contradiction because $(1 + \mu)c_1 \geq 0$.

Consider f in the case ($\mathbf{C}_{+-}$). Then, $f(0) = 1$ and

$$f(x) > 0 \qquad \forall x \leq 0. \tag{2.4}$$

Otherwise, suppose that $f(c_2) = 0$ for $c_2 < 0$. Since $f(x) > x + 1$ for all $x < 0$, we see that $c_2 < -1$. Thus, $f(c_2 + 1) > (c_2 + 1) + 1$ and therefore

$$1 = f^2(c_2) = f(c_2 + 1) + \mu c_2 > 2 + (1 + \mu)c_2,$$

which is in contradiction to the fact that $(1 + \mu)c_2 \geq 0$. This contradiction gives a proof to (2.4). We further claim that

$$\lim_{x \to -\infty} f(x) = +\infty. \tag{2.5}$$

Otherwise, there is a sequence $\{x_n\}$ on the negative half x-axis such that $x_n \to -\infty$ but $f(x_n) \to b$ as $n \to \infty$, where $b = -\infty$ or is finite. It is impossible to have $b = -\infty$ because of (2.4). If b is finite, then from equation (2.1) we get $\lim_{n\to\infty} f(x_n + 1) = \lim_{n\to\infty} f^2(x_n) - \mu \lim_{n\to\infty} x_n = f(b) - \mu \lim_{n\to\infty} x_n = -\infty$, which contradicts (2.4), too. On the other hand, there exists a sequence $\{y_n\} \subset (0, +\infty)$ tending to $+\infty$ such that $f(y_n) \to +\infty$ as $n \to \infty$. In fact, if $\lim_{n\to\infty} y_n = +\infty$ and $\{f(y_n)\}$ tends to $-\infty$ or to a finite number b, then the sequence $\{y_n + 1\}$ is what we need because

$$\lim_{n\to\infty} f(y_n + 1) = \lim_{n\to\infty} f^2(y_n) - \mu \lim_{n\to\infty} y_n = +\infty$$

by (2.5) and the continuity of f. This implies that $f((x, +\infty)) \supset (f(x), +\infty)$ for all $x > -\infty$. Therefore, $f^2((0, +\infty)) \supset f((f(0), +\infty)) = f((1, +\infty)) \supset (f(1), +\infty)$. In particular, if $\zeta > \max\{f(1), 2\}$, then there exists $x \in (0, +\infty)$ such that $f^2(x) = \zeta$. Consequently,

$$2 < \zeta = f^2(x) = f(x + 1) + \mu x < x + 2 + \mu x = (1 + \mu)x + 2 \leq 2,$$

a contradiction.

Consider f in the case $(\mathbf{C}_{++})$, in which

$$f(x) > x+1, \quad \text{for all } x \neq 0. \tag{2.6}$$

We also claim (2.5). Otherwise, there exists a sequence $\{x_n\} \subset (-\infty, 0)$ tending to $-\infty$ such that $f(x_n) \to -\infty$ because in the case that $f(x_n) \to b$, a finite number, from equation (2.1) we have $\lim_{n\to\infty} f(x_n+1) = \lim_{n\to\infty} f^2(x_n) - \mu \lim_{n\to\infty} x_n = f(b) - \mu \lim_{n\to\infty} x_n = -\infty$, i.e., $\{x_n+1\}$ is what we need. On the other hand, from (2.6) we easily see that

$$\lim_{x\to+\infty} f(x) = +\infty. \tag{2.7}$$

By the continuity, $f(\mathbb{R}) = \mathbb{R}$ and therefore $f^2(\mathbb{R}) = \mathbb{R}$. However, in the case $(\mathbf{C}_{++})$ we have $f(x+1) \geq (x+1)+1$, implying that $f^2(x) = f(x+1) + \mu x \geq x + 2 + \mu x = (1+\mu)x + 2 \geq 2$ for all $x \leq 0$. Moreover, $f^2(x) > 1$ when $x > 0$ by $f(x) > x+1 > 1 > 0$ when $x > 0$. This implies that $f^2(x) > 1$ for all $x \in \mathbb{R}$, a contradiction which gives a proof to (2.5).

Fix $M > f(0)$ arbitrarily and let

$$\alpha := \inf\{x \in \mathbb{R} \mid f(x) = M\}.$$

By (2.5) and the continuity of f we see that $\alpha \in (-\infty, 0)$. Moreover,

$$f(\alpha) = M, \qquad f(x) > M \quad \forall x < \alpha. \tag{2.8}$$

In particular, $f(\alpha - 1) > M = f(\alpha)$. We further define

$$\beta := \sup\{x \in \mathbb{R} \mid f(x) = f(\alpha - 1)\}.$$

Since $f(0) < M < f(\alpha-1)$, by (2.7) and the continuity of f we see that $\beta \in (0, \infty)$. Moreover,

$$f(\beta) = f(\alpha - 1), \qquad f(x) > f(\beta) \quad \forall x > \beta. \tag{2.9}$$

Note that $f(\beta) > \beta + 1$ by (2.6), i.e., $f(\beta) - 1 > \beta$. Thus, $f(\alpha - 1) = f(\beta) < f(f(\beta) - 1)$. By (2.5) and the continuity of f we see that there exists $\alpha' < \alpha - 1$ satisfying

$$f(\alpha') = f(f(\beta) - 1). \tag{2.10}$$

Thus, $f^2(\alpha') = f^2(f(\beta)-1)$ and $f(\alpha'+1)+\mu\alpha' = f^2(\beta)+\mu(f(\beta)-1)$ by equation (2.1). It follows that $f^2(\beta) - f(\alpha'+1) = -\mu[f(\beta) - (\alpha'+1)]$, implying that

$$\frac{f^2(\beta) - f(\alpha'+1)}{f(\beta) - (\alpha'+1)} = -\mu,$$

where we note that $f(\beta) > M > 0 > \alpha > \alpha'+1$ in the denominator. This means that the slope of the line ℓ_1 connecting points $(\alpha'+1, f(\alpha'+1))$ and $(f(\beta), f^2(\beta))$ is equal to $-\mu$. Similarly, according to (2.9),

$$\frac{f(\beta+1) - f(\alpha)}{(\beta+1) - \alpha} = \frac{(f^2(\beta) - \mu\beta) - (f^2(\alpha-1) - \mu(\alpha-1))}{(\beta+1) - \alpha} = -\mu,$$

i.e., the slope of the line ℓ_2 connecting points $(\alpha, f(\alpha))$ and $(\beta+1, f(\beta+1))$ is equal to $-\mu$. However, by the choice of α' and (2.8) we have $\alpha'+1 < \alpha < 0$ and $f(\alpha'+1) > f(\alpha)$. On the other hand, $f(\beta) > \beta+1 > 0$ by (2.6) and $f^2(\beta) < f(\beta+1)$ by equation (2.1). Moreover, $f(\beta+1) > f(\beta) = f(\alpha-1) > f(\alpha)$ by (2.8) and (2.9). It follows that

$$\frac{f^2(\beta) - f(\alpha'+1)}{f(\beta) - (\alpha'+1)} < \frac{f(\beta+1) - f(\alpha)}{f(\beta) - (\alpha'+1)} < \frac{f(\beta+1) - f(\alpha)}{(\beta+1) - \alpha},$$

implying that the two lines ℓ_1 and ℓ_2 cannot be parallel. The contradiction proves the theorem in the case $(\mathbf{C}_{++})$.

The proof is completed. $\square$

3. Generalized equation

In this section we find continuous solutions for the generalized equation (1.4). Given a real constant k, let

$$\mathcal{X}(\mathbb{R}; k) := \{f : \mathbb{R} \to \mathbb{R} \mid \sup_{x\in\mathbb{R}} |f(x) - kx| < +\infty\}.$$

Clearly, functions in this class are unbounded when $k \neq 0$. This class forms a complete metric space equipped with the metric

$$d(f, g) := \sup_{x\in\mathbb{R}} |f(x) - g(x)|.$$

In fact, suppose that $\{f_n\}$ is a Cauchy sequence in the space, i.e., for any $\varepsilon > 0$ there exists an integer N_ε such that $d(f_n, f_m) < \varepsilon$ for all $m, n \geq N_\varepsilon$. In particular, for any x in $\mathbb{R}$, $|f_n(x) - f_m(x)| \leq d(f_n, f_m) < \varepsilon$ when $n, m \geq N_\varepsilon$. So $\{f_n(x)\}$ is a Cauchy sequence for every x in $\mathbb{R}$. Let $f(x) = \lim_{n\to+\infty} f_n(x)$, $x \in \mathbb{R}$. If $n, m \geq N_\varepsilon$, then $|f_n(x) - f(x)| \leq |f_m(x) - f(x)| + d(f_m, f_n) < |f_m(x) - f(x)| + \varepsilon$ for every $x \in \mathbb{R}$. Letting $m \to +\infty$ gives that $|f_n(x) - f(x)| \leq \varepsilon$ when $n \geq N_\varepsilon$, which is independent of x. Also $\sup_{x\in\mathbb{R}} |f(x) - kx| \leq \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| + \sup_{x\in\mathbb{R}} |f_n(x) - kx| < +\infty$ for $n \geq N_\varepsilon$, implying that $f \in \mathcal{X}(\mathbb{R}; k)$. Hence $d(f_n, f) \leq \varepsilon$ for $n \geq N_\varepsilon$. What has been just shown is that $d(f_n, f) \to 0$ as $n \to +\infty$. This means that $\mathcal{X}(\mathbb{R}; k)$ is complete. Given a constant L such that $L \geq |k|$, let $\mathcal{X}(\mathbb{R}; k, L) := \mathcal{X}(\mathbb{R}; k) \cap C^{0,L}(\mathbb{R})$, where

$$C^{0,L}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \mathrm{Lip}(f) \leq L\} \tag{3.1}$$

and $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f. Being a closed subset of the set $\mathcal{X}(\mathbb{R}; k)$, $\mathcal{X}(\mathbb{R}; k, L)$ is also a complete metric space.

Theorem 2. *Suppose that either* **(C1)** $|\lambda| \in (2, +\infty)$ *and* $\mu \in [-\lambda^2/4, \lambda^2/4]$ *or* **(C2)** $|\lambda| \in (1, 2]$ *and* $\mu \in (1 - |\lambda|, |\lambda| - 1)$. *Then equation* (1.4) *has a solution* $f \in \mathcal{X}(\mathbb{R}; k, L)$, *where* k *and* L *are some real constants with* $L \geq |k|$. *More concretely,*

(D1) $k = k_+$ *and* $L = L_-$ *when* $\lambda \in (-\infty, -2)$ *and* $\mu \in \big[-\frac{\lambda^2}{4}, \frac{\lambda^2}{4}\big]$,
(D2) $k = k_+$ *and* $L = L_-$ *when* $\lambda \in [-2, -1)$ *and* $\mu \in (\lambda + 1, -\lambda - 1)$,
(D3) $k = k_-$ *and* $L = L_-$ *when* $\lambda \in (1, 2]$ *and* $\mu \in (1 - \lambda, \lambda - 1)$,
(D4) $k = k_-$ *and* $L = L_-$ *when* $\lambda \in (2, +\infty)$ *and* $\mu \in \big[-\frac{\lambda^2}{4}, \frac{\lambda^2}{4}\big]$,

where $k_\pm := \frac{\lambda \pm \sqrt{\lambda^2 + 4\mu}}{2}$ *and* $L_\pm := \frac{|\lambda| \pm \sqrt{\lambda^2 - 4|\mu|}}{2}$.

Proof. The strategy of the proof is to use Banach's fixed point theorem. As assumed in the theorem, $\lambda \neq 0$. For given constants k, L with $L \geq |k|$, define a mapping $\mathcal{T} : \mathcal{X}(\mathbb{R}; k, L) \to C^0(\mathbb{R})$ by

$$\mathcal{T}f(x) := \frac{1}{\lambda}\{f^2(x-1) - \mu(x-1)\}, \qquad x \in \mathbb{R},$$

where $f \in \mathcal{X}(\mathbb{R}; k, L)$. Then the solution is a fixed point of the mapping. Obviously,

$$\sup_{x\in\mathbb{R}} |\mathcal{T}f(x) - kx| \leq \frac{1}{|\lambda|} \sup_{x\in\mathbb{R}} |f^2(x-1) - kf(x-1)| + \frac{k}{|\lambda|} \sup_{x\in\mathbb{R}} |f(x-1) - k(x-1)|$$
$$+ \sup_{x\in\mathbb{R}} \left| \left(\frac{k^2}{\lambda} - \frac{\mu}{\lambda} - k\right) x + \frac{\mu - k^2}{\lambda} \right|$$

$$\leq \frac{1+k}{|\lambda|}\sup_{x\in\mathbb{R}}|f(x)-kx|+\sup_{x\in\mathbb{R}}\left|\left(\frac{k^2}{\lambda}-\frac{\mu}{\lambda}-k\right)x+\frac{\mu-k^2}{\lambda}\right|<+\infty$$

if

$$\frac{k^2}{\lambda}-\frac{\mu}{\lambda}-k=0. \tag{3.2}$$

Moreover, for $x_1, x_2 \in \mathbb{R}$,

$$\begin{aligned}|\mathcal{T}f(x_1)-\mathcal{T}f(x_2)| &\leq \frac{1}{|\lambda|}\{|f^2(x_1-1)-f^2(x_2-1)|+|\mu(x_1-x_2)|\}\\ &\leq \frac{L^2+|\mu|}{|\lambda|}|x_1-x_2|\leq L|x_1-x_2|\end{aligned}$$

if

$$\frac{L^2+|\mu|}{|\lambda|}\leq L. \tag{3.3}$$

It implies that $\mathcal{T}$ maps $\mathcal{X}(\mathbb{R};k,L)$ into itself if (3.2) and (3.3) are both satisfied. Furthermore, for arbitrary $f_1, f_2 \in \mathcal{X}(\mathbb{R};k,L)$,

$$\begin{aligned}&|\mathcal{T}f_1(x)-\mathcal{T}f_2(x)| \leq \frac{1}{|\lambda|}|f_1(f_1(x-1))-f_2(f_2(x-1))|\\ &\quad\leq \frac{1}{|\lambda|}\{|f_1(f_1(x-1))-f_1(f_2(x-1))|+|f_1(f_2(x-1))-f_2(f_2(x-1))|\}\\ &\quad\leq \frac{L+1}{|\lambda|}d(f_1,f_2)\quad \forall x\in\mathbb{R},\end{aligned}$$

i.e.,

$$d(\mathcal{T}f_1,\mathcal{T}f_2)\leq \frac{L+1}{|\lambda|}d(f_1,f_2).$$

It implies that $\mathcal{T}$ is a contraction if

$$|\lambda|>L+1. \tag{3.4}$$

Thus, by Banach's fixed point theorem, $\mathcal{T}$ has a unique fixed point in the class $\mathcal{X}(\mathbb{R};k,L)$, provided (3.2), (3.3) and (3.4) hold.

The conditions on λ and μ are given by relations (3.2), (3.3) and (3.4). Equation (3.2) has a real solution k if and only if

$$\lambda^2+4\mu\geq 0. \tag{3.5}$$

Similarly, inequality (3.3) has a real solution L if and only if

$$\lambda^2 - 4|\mu| \geq 0. \tag{3.6}$$

Clearly, (3.6) implies (3.5). Then under condition (3.6), we can get

$$k = k_- \quad \text{or} \quad k = k_+ \tag{3.7}$$

and

$$L_- \leq L \leq L_+ \tag{3.8}$$

from (3.2) and (3.3) respectively, where $k_\pm$ and $L_\pm$ are defined in the theorem. Note that (3.4) and the restriction $L \geq |k|$, given just before (3.1), require

$$|k| \leq L < |\lambda| - 1. \tag{3.9}$$

Thus we need conditions on λ and μ to guarantee that the intersection of (3.8) and (3.9) is nonempty. It follows from (3.8) and (3.9) that the following three inequalities all hold:

$$L_- < |\lambda| - 1, \tag{3.10}$$

$$|k| < |\lambda| - 1, \tag{3.11}$$

$$|k| \leq L_+. \tag{3.12}$$

From (3.10) we get that

either $$|\lambda| > 2 \quad \text{and} \quad |\mu| \leq \frac{\lambda^2}{4} \tag{3.10 a}$$

or $$1 < |\lambda| \leq 2 \quad \text{and} \quad |\mu| < |\lambda| - 1. \tag{3.10 b}$$

Moreover, (3.6) and (3.11) imply that

either $$\lambda < -2 \quad \text{and} \quad -\frac{\lambda^2}{4} \leq \mu < \lambda + 1 \tag{3.11 a_-}$$

or $$1 < \lambda \leq 2,\ 1-\lambda < \mu \leq \frac{\lambda^2}{4} \quad \text{and} \quad 1-\lambda < \mu < 2\lambda^2-3\lambda+1 \tag{3.11 b_-}$$

or $$\lambda > 2 \quad \text{and} \quad -\frac{\lambda^2}{4} \leq \mu \leq \frac{\lambda^2}{4} \tag{3.11 c_-}$$

as $k = k_-$, and that

either $$\lambda < -2 \quad \text{and} \quad -\frac{\lambda^2}{4} \leq \mu \leq \frac{\lambda^2}{4} \tag{3.11 a_+}$$

or $$-2 \leq \lambda < -1,\ \lambda + 1 < \mu \leq \frac{\lambda^2}{4} \quad \text{and} \quad \lambda + 1 < \mu < 2\lambda^2 + 3\lambda + 1 \tag{3.11 b_+}$$

or $$\lambda > 2 \quad \text{and} \quad -\frac{\lambda^2}{4} \leq \mu < 1 - \lambda \tag{3.11 c_+}$$

as $k = k_+$. Similarly, from (3.6) and (3.12) we can see that

$$\text{either} \qquad \lambda < -1 \quad \text{and} \quad -\frac{\lambda^2}{4} \le \mu \le 0 \tag{3.12 a_-}$$

$$\text{or} \qquad \lambda > 1 \quad \text{and} \quad -\frac{\lambda^2}{4} \le \mu \le \frac{\lambda^2}{4} \tag{3.12 b_-}$$

as $k = k_-$, and that

$$\text{either} \qquad \lambda < -1 \quad \text{and} \quad -\frac{\lambda^2}{4} \le \mu \le \frac{\lambda^2}{4} \tag{3.12 a_+}$$

$$\text{or} \qquad \lambda > 1 \quad \text{and} \quad -\frac{\lambda^2}{4} \le \mu \le 0 \tag{3.12 b_+}$$

as $k = k_+$. Considering the intersection of those inequalities from (3.10 a) to (3.12 b_+) as $k = k_-$ and $k = k_+$, we get that

$$\begin{array}{rrcl} \text{either} & \lambda < -2 & \text{and} & -\frac{\lambda^2}{4} \le \mu < \lambda + 1 \\ \text{or} & 1 < \lambda \le 2 & \text{and} & 1 - \lambda < \mu < \lambda - 1 \\ \text{or} & \lambda > 2 & \text{and} & -\frac{\lambda^2}{4} \le \mu \le \frac{\lambda^2}{4} \end{array} \tag{3.13}$$

and

$$\begin{array}{rrcl} \text{either} & \lambda < -2 & \text{and} & -\frac{\lambda^2}{4} \le \mu \le \frac{\lambda^2}{4} \\ \text{or} & -2 \le \lambda < -1 & \text{and} & \lambda + 1 < \mu < -\lambda - 1 \\ \text{or} & \lambda > 2 & \text{and} & -\frac{\lambda^2}{4} \le \mu < 1 - \lambda \end{array} \tag{3.14}$$

respectively, which imply the conditions **(C1)** and **(C2)** in the theorem. Therefore, under **(C1)** or **(C2)**, $\mathcal{T}$ has a unique fixed point in $\mathcal{X}(\mathbb{R}; k, L)$, which gives a continuous solution of equation (1.4).

Furthermore, under **(C1)** or **(C2)** we can give appropriate choices of L. We can see that $L_- \ge |k|$ if and only if either

$$\begin{array}{rrcl} \text{either} & \lambda < -2 & \text{and} & \mu = -\frac{\lambda^2}{4} \\ \text{or} & 1 < \lambda \le 2 & \text{and} & 1 - \lambda < \mu < \lambda - 1 \\ \text{or} & \lambda > 2 & \text{and} & -\frac{\lambda^2}{4} \le \mu \le \frac{\lambda^2}{4} \end{array}$$

as $k = k_-$ or

$$\begin{array}{llll} \text{either} & \lambda < -2 & \text{and} & -\dfrac{\lambda^2}{4} \le \mu \le \dfrac{\lambda^2}{4} \\ \text{or} & -2 \le \lambda < -1 & \text{and} & \lambda + 1 < \mu < -\lambda - 1 \\ \text{or} & \lambda > 2 & \text{and} & \mu = -\dfrac{\lambda^2}{4} \end{array}$$

as $k = k_+$. Thus the best choices of k and L are the following:

(D1) $k = k_+$ and $L = L_-$ when $\lambda \in (-\infty, -2)$ and $\mu \in \big[-\frac{\lambda^2}{4}, \frac{\lambda^2}{4}\big]$,

(D2) $k = k_+$ and $L = L_-$ when $\lambda \in [-2, -1)$ and $\mu \in (\lambda + 1, -\lambda - 1)$,

(D3) $k = k_-$ and $L = L_-$ when $\lambda \in (1, 2]$ and $\mu \in (1 - \lambda, \lambda - 1)$,

(D4) $k = k_-$ and $L = L_-$ when $\lambda \in (2, +\infty)$ and $\mu \in [-\frac{\lambda^2}{4}, \frac{\lambda^2}{4}]$.

The proof is completed. $\square$

Remark that for $\lambda \in (-\infty, -2)$ and $\mu \in [-\frac{\lambda^2}{4}, \lambda + 1)$, a subcase of the case **(D1)**, we can actually find another solution. In fact, from (3.13) we see that in the subcase we can find a solution with $k = k_-$, a different constant from k_+, to which the case **(D1)** is corresponding from (3.14). Similarly, for $\lambda \in (2, +\infty)$ and $\mu \in [-\frac{\lambda^2}{4}, 1 - \lambda)$, a subcase of the case **(D4)**, we can also find another solution.

There are some differences between Theorem 2 of [4] and our Theorem 2. When $\lambda = 5$ and $\mu = 4$, we cannot apply Theorem 2 of [4] to give the existence of bounded continuous solutions of equation (1.4) because $|\lambda| \le \max\{2, 2\sqrt{2|\mu|}\}$. However, our Theorem 2 implies equation (1.4) has a solution in $\mathcal{X}(\mathbb{R}; \frac{5-\sqrt{41}}{2}, 1)$ because λ, μ satisfy the condition **(D4)**. More generally, when $|\lambda| > 2$ and $\frac{\lambda^2}{8} \le |\mu| \le \frac{\lambda^2}{4}$, Theorem 2 of [4] is not available but our Theorem 2 is applicable. Another difference is that Theorem 2 of [4] gives bounded continuous solutions on a compact interval but the solutions obtained in our Theorem 2 defined on the whole $\mathbb{R}$ are unbounded when $k \ne 0$.

Theorem 2 does not answer to the following cases:

(E1) $|\lambda| \in (0, 1]$,

(E2) $|\lambda| \in (1, 2]$ and $|\mu| \in [|\lambda| - 1, +\infty)$,

(E3) $|\lambda| \in (2, +\infty)$ and $|\mu| \in (\frac{\lambda^2}{4}, +\infty)$.

When $\lambda = 1$ and $\mu \le -1$, a special case of **(E1)**, our Theorem 1 gives the nonexistence of continuous real solutions. Equation (1.1) is exactly the equation (1.3) with $\lambda = 1$ and $\mu = -1$, which lies in the case **(E1)**. Therefore, although Brillouët-Belluot's open problem is answered by Theorem 1, the existence or nonexistence of continuous solutions of the generalized equation (1.3) remains unknown almostly in cases **(E1)**, **(E2)** and **(E3)**.

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