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Some applications of Bochner formula to submanifolds of a unit sphere

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Abstract. The aims of this paper are to give estimates for the eigenvalue of the Laplacian on submanifolds in a unit sphere $S^{n+p}(1)$ and to give some new characterizations of spheres in $S^{n+p}(1)$.

1. Introduction

Let M be an n-dimensional closed oriented manifold immersed in a unit sphere $S^{n+p}(1)$, where p is the codimension. One knows that minimal submanifolds and submanifolds with constant scalar curvature $n(n-1)$ in $S^{n+p}(1)$ were studied by many others, they are very interesting topics since they come from the variational problems in differential geometry.

A well-known result about minimal submanifolds in $S^{n+p}(1)$ is due to Simons [11], if M is a minimal hypersurface in $S^{n+1}(1)$ and

$$S < n,$$

then $S \equiv 0$ and M is a unit sphere $S^n(1)$. Many years later, Leung [8] has proved

Theorem 1.1 ([8]). *Let M be a closed minimal submanifold in $S^{n+p}(1)$. Let f be an eigenfunction of the Laplacian on M corresponding to a non-zero eigenvalue λ, then*

$$\int_M (\lambda + S - n)|\nabla f|^2 dv \geq 0, \tag{1.1}$$

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equality holds if and only if either M is totally geodesic and λ is a non-zero eigenvalue or $n = 2$ and $n + p = 2m$ and M is isometric to $S^2\left(\sqrt{\frac{m(m+1)}{2}}\right)$ and λ is a non-zero eigenvalue, where S denotes the squared norm of the second fundamental form of M.

On the other hand, by the study of Cheng–Yau's self-adjoint operator and some new estimates, Li [6] has obtained that if M is an n-dimensional closed hypersurface with constant scalar curvature $n(n-1)$ in $S^{n+1}(1)$ and

$$S < \frac{n^2}{2(n-2)},$$

then $S \equiv 0$ and M is a unit sphere $S^n(1)$.

By comparing the results of Simons, Li and Leung, it is nature to ask the following:

Question 1. Does there exist the similar integral inequality with (1.1) for the eigenvalue of the Laplacian on submanifolds with constant scalar curvature $n(n-1)$ in $S^{n+p}(1)$?

In this paper, we answer the above question and prove the following integral inequality:

Theorem 1.2. *Let M be a closed submanifold in $S^{n+p}(1)$ with constant scalar curvature $n(n-1)$. Let f be an eigenfunction of the Laplacian on M corresponding to a non-zero eigenvalue λ, then*

$$\int_M \left(\lambda + \frac{2(n-2)}{n}S - n\right)|\nabla f|^2 dv \geq 0, \tag{1.2}$$

equality holds if and only if M is a geodesic sphere $S^n(1)$.

Finally, we do not assume that M has constant scalar curvature $n(n-1)$ and obtain some other integral inequality which also involves the non-zero eigenvalue of the Laplacian. In fact, we prove

Theorem 1.3. *Let M be a closed submanifold in $S^{n+p}(1)$. Let f be an eigenfunction of the Laplacian on M corresponding to a non-zero eigenvalue λ, then*

$$\int_M \left(\lambda + \frac{n^2}{4(n-1)}\rho^2 - n\right)|\nabla f|^2 dv \geq 0, \tag{1.3}$$

equality holds if and only if M is a sphere, where $\rho^2 := S - nH^2$ is a non-negative function which vanishes exactly at the umbilical points of M, H is the mean curvature of M.

Remark 1. We do not assume that ρ^2 is constant in the above Theorem.

2. Preliminaries

In order to prove our results, we introduce some preliminaries and notations in this section. Let $S^{n+p}(1)$ be a sphere of constant sectional curvature one. Let M be an n-dimensional closed hypersurface in $S^{n+p}(1)$. For any $p \in M$, we choose a local orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ in $S^{n+p}(1)$ around p, such that $e_1, \ldots, e_n$ are tangent to M and $e_{n+1}, \cdots, e_{n+p}$ are normal to M. Take the corresponding dual coframe $\omega_1, \ldots, \omega_{n+p}$. In this paper, we make the following convention on the range of indices:

$$1 \leq A, B, C \leq n+p; \quad 1 \leq i, j, k \leq n; \quad n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

The structure equations of $S^{n+p}(1)$ are (see [7])

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} = \omega_{BA}, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \omega_A \wedge \omega_B. \tag{2.2}$$

Restricted to M, we have $\omega_\alpha = 0$, thus

$$0 = d\omega_\alpha = \sum_i \omega_{\alpha i} \wedge \omega_i, \tag{2.3}$$

from Cartan's lemma, we obtain

$$\omega_{i\alpha} = \sum_j h^\alpha_{ij} \omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.4}$$

We then get the structure equations of M as follows:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} = \omega_{ji}, \tag{2.5}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l, \tag{2.6}$$

where R_{ijkl} is the component of the curvature tensor of induced metric on M. If h^α_{ij} denotes the component of the second fundamental form of M, S denotes the squared norm of the second fundamental form, $\vec{H}$ denotes the mean curvature vector and H denotes the mean curvature of M, then we have

$$S = \sum_{i,j,\alpha} (h^\alpha_{ij})^2, \quad \vec{H} = \sum_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{n} \sum_k h^\alpha_{kk}, \quad H = |\vec{H}|. \tag{2.7}$$

M is called minimal if $\vec{H} \equiv 0$, i.e., $\sum_k h^\alpha_{kk} = 0$ for all α.
The Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2.8}$$

$$R_{ik} = (n-1)\delta_{ik} + n\sum_\alpha H^\alpha h^\alpha_{ik} - \sum_{j,\alpha} h^\alpha_{ij}h^\alpha_{jk}, \tag{2.9}$$

$$R = n(n-1) + n^2H^2 - S, \tag{2.10}$$

where R is the scalar curvature.
The Codazzi equations are

$$h^\alpha_{ijk} = h^\alpha_{ikj}, \tag{2.11}$$

where the covariant derivative of h^α_{ij} is defined by

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\omega_{ki} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}. \tag{2.12}$$

3. Proofs of theorems

Firstly, we prove the following Lemma.

Lemma 1 (Cf. Theorem B of [8])**.** *Let M be a closed submanifold in $S^{n+p}(1)$. Let f be an eigenfunction on M corresponding to a non-zero eigenvalue λ, then*

$$\lambda \int_M |\nabla f|^2 dv \geq \frac{n}{n-1}\int_M \mathrm{Ric}(\nabla f, \nabla f). \tag{3.1}$$

Proof of Lemma 1. For any smooth function $u : M \to R$, one has from Bochner formula that

$$\frac{1}{2}\triangle(|\nabla u|^2) = \mathrm{Ric}(\nabla u, \nabla u) + < \nabla u, \nabla(\triangle u) > + |\,\mathrm{Hess}\, u|^2. \tag{3.2}$$

Since M is closed, one deduces from (3.2) and $\triangle f + \lambda f = 0$ that

$$\int_M \mathrm{Ric}(\nabla f, \nabla f)dv - \lambda \int_M |\nabla f|^2 dv + \int_M |\,\mathrm{Hess}\, f|^2 dv = 0. \tag{3.3}$$

Let I denotes the identity operator on the tangent bundle TM of M. By a direct calculation, we obtain for any $t \in R$,

$$\int_M |\,\mathrm{Hess}\, f - tf\mathrm{I}|^2 dv = \int_M (|\,\mathrm{Hess}\, f|^2 - 2tf\triangle f + nf^2t^2)dv$$

$$= \int_M |\operatorname{Hess} f|^2 dv + \left(2t + \frac{n}{\lambda}t^2\right) \int_M |\nabla f|^2 dv$$

$$\geq \int_M |\operatorname{Hess} f|^2 dv + \left(2 \times \left(-\frac{\lambda}{n}\right) + \frac{n}{\lambda}\left(-\frac{\lambda}{n}\right)^2\right) \int_M |\nabla f|^2 dv$$

$$= \int_M |\operatorname{Hess} f|^2 dv - \frac{\lambda}{n} \int_M |\nabla f|^2 dv, \tag{3.4}$$

it follows that

$$\int_M |\operatorname{Hess} f|^2 dv = \int_M |\operatorname{Hess} f - tf\mathrm{I}|^2 dv + \frac{\lambda}{n} \int_M |\nabla f|^2 dv \geq \frac{\lambda}{n} \int_M |\nabla f|^2 dv. \tag{3.5}$$

Equality holds if and only if

$$\operatorname{Hess} f = -\frac{\lambda}{n} f\mathrm{I}. \tag{3.6}$$

Thus, we conclude from (3.3) and (3.5) that

$$\lambda \int_M |\nabla f|^2 dv \geq \frac{n}{n-1} \int_M \operatorname{Ric}(\nabla f, \nabla f).$$

□

In order to prove Theorem 1.2 and Theorem 1.3, we need the following Lemma which can be found in [9].

Lemma 2. *Let M^n be a submanifold of $S^{n+p}(1)$. Let* Ric *denotes the function that assign to each point of M the minimum Ricci curvature. Then*

$$\operatorname{Ric} \geq \frac{n-1}{n}\left\{n + 2nH^2 - S - \frac{n-2}{\sqrt{n-1}}\sqrt{nH^2}\sqrt{S - nH^2}\right\}. \tag{3.7}$$

Proof of Theorem 1.2. Since M has constant scalar curvature $n(n-1)$, one has from the Gauss equation (2.10) that

$$n^2H^2 = S.$$

According to Lemma 2 and the above equation, we obtain

$$\operatorname{Ric} \geq \frac{n-1}{n}\left\{n + 2nH^2 - S - \frac{(n-2)\sqrt{n-1}}{n-1}\sqrt{nH^2}\sqrt{S - nH^2}\right\}$$

$$= (n-1)\{1 - (2n-4)H^2\} = (n-1)\left\{1 - \frac{2n-4}{n^2}S\right\}. \tag{3.8}$$

Substituting (3.8) into (3.1), we infer

$$\int_M \left(\lambda + \frac{2(n-2)}{n}S - n\right)|\nabla f|^2 dv \geq 0.$$

On the other hand, we obtain that equality holds if and only if M is a sphere by using of Obata's result [10]. Combining with $S = n^2H^2$, one concludes that $M = S^n(1)$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Using a well-known inequality, we have, for an arbitrary real number $a > 0$,

$$2|H||\rho| \leq aH^2 + \frac{1}{a}\rho^2. \tag{3.9}$$

By using of Lemma 2, we infer

$$\begin{aligned}\mathrm{Ric} &\geq \frac{n-1}{n}\left\{n + nH^2 - \rho^2 - \frac{n(n-2)}{\sqrt{n(n-1)}|H|\rho}\right\}\\ &\geq \frac{n-1}{n}\left\{n + \left(n - \frac{n(n-2)}{2\sqrt{n(n-1)}}a\right)H^2 + \left(-1 - \frac{n(n-2)}{2\sqrt{n(n-1)}}\frac{1}{a}\right)\rho^2\right\}.\end{aligned} \tag{3.10}$$

Choosing $a = \frac{2\sqrt{n(n-1)}}{n-2}$, one obtains from (3.10)

$$\mathrm{Ric} \geq \frac{n-1}{n}\left\{n - \frac{n^2}{4(n-1)}\rho^2\right\}. \tag{3.11}$$

Hence, we have from the above arguments and Lemma 1

$$\int_M \left(\lambda + \frac{n^2}{4(n-1)}\rho^2 - n\right)|\nabla f|^2 dv \geq 0.$$

Moreover, we then have that equality in the above formula implies that M is a sphere according to Obata's result [10]. This finishes the proof of Theorem 1.3. □

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