Publ. Math. Debrecen
78/3-4 (2011), 633–646
DOI: 10.5486/PMD.2011.4829

On the maximal operator of Walsh–Marcinkiewicz means

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Abstract. In this paper we prove that the maximal operator $\tilde{\mathcal{M}}^*f := \sup_{n\in\mathbf{P}} \frac{|\mathcal{M}_nf|}{\log^{3/2}(n+1)}$, where $\mathcal{M}_nf$ is the nth Marcinkiewicz–Fejér mean of the 2-dimensional Walsh–Fourier series, is bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$.

1. Introduction

The a.e. convergence of Walsh–Fejér means σ_nf was proved by FINE [2]. In 1975 SCHIPP [12] showed that the maximal operator σ^* is of weak type $(1,1)$ and of type (p,p) for $1 < p \leq \infty$. The boundedness fails to hold for $p = 1$. But, FUJII [3] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1. The theorem of FUJII was extended by WEISZ [17], he showed that the maximal operator σ^* is bounded from the martingale Hardy space H_p to the space L_p for $p > 1/2$. Simon gave a counterexample [13], which showes that the boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ due to GOGINAVA [6]. In the endpoint case $p = 1/2$ two positive result was showed. WEISZ [18] proved that σ^* is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. In 2008 GOGINAVA [8] proved that the maximal operator $\tilde{\sigma}^*$ defined by

$$\tilde{\sigma}^* := \sup_{n\in\mathbf{P}} \frac{|\sigma_nf|}{\log^2(n+1)}$$

Mathematics Subject Classification: 42C10.
Key words and phrases: Walsh system, Fejér means, Marcinkiewicz means, Maximal operator.

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. He also proved that for any nondecreasing function $\varphi : \mathbf{P} \to [1, \infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\log^2(n+1)}{\varphi(n)} = +\infty$$

the maximal operator $\sup_{n\in\mathbf{P}} \frac{|\sigma_n f|}{\varphi(n)}$ is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

In 1939 for the two-dimensional trigonometric Fourier partial sums $S_{j,j}(f)$ Marcinkiewicz [10] has proved for $f \in L \log L([0, 2\pi]^2)$ that the means

$$\mathcal{M}_n f = \frac{1}{n} \sum_{j=1}^{n} S_{j,j}(f)$$

converge a.e. to f as $n \to \infty$. Zhizhiashvili [19] improved this result for $f \in L([0, 2\pi]^2)$.

For the two-dimensional Walsh–Fourier series Weisz [14] proved that the maximal operator

$$\mathcal{M}^* f = \sup_{n\geq 1} \frac{1}{n} \left| \sum_{j=0}^{n-1} S_{j,j}(f) \right|$$

is bounded from the two-dimensional dyadic martingale Hardy space H_p to the space L_p for $p > 2/3$ and is of weak type $(1, 1)$. Goginava [6] proved that the assumption $p > 2/3$ is essential for the boundedness of the maximal operator $\mathcal{M}^*$ from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$. Namely, in the endpoint case $p = 2/3$ he gave a counterexample for which the boundedness does not hold. In the endpoint case $p = 2/3$, Goginava [7] proved that the maximal operator $\mathcal{M}^*$ of the Walsh–Marcinkiewicz means of double Fourier series is bounded from the Hardy space $H_{2/3}$ to the space weak-$L_{2/3}$.

In the present paper we prove that the maximal operator $\tilde{\mathcal{M}}^*$ defined by

$$\tilde{\mathcal{M}}^* := \sup_{n\in\mathbf{P}} \frac{|\mathcal{M}_n f|}{\log^{3/2}(n+1)}$$

is bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$. We also prove that for any nondecreasing function $\varphi : \mathbf{P} \to [1, \infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\log^{3/2}(n+1)}{\varphi(n)} = +\infty$$

the maximal operator $\sup_{n\in\mathbf{P}} \frac{|\mathcal{M}_n f|}{\varphi(n)}$ is not bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$. That is, we prove the analogue of the theorems of Goginava mentioned above [8].

For Walsh–Kaczmarz–Marcinkiewicz means the author [9] proved, that it is of weak type (1,1) and of type (p,p) for $1 < p \le \infty$. This theorem was extended in [4].

2. Definitions and notation

Now, we give a brief introduction to the theory of dyadic analysis [11], [1]. Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2 = \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0,1\} (k \in \mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x) := G,$$
$$I_n(x) := I_n(x_0, \dots, x_{n-1}) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\},$$

$(x \in G,\ n \in \mathbf{N})$. These sets are called dyadic intervals. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G, and $I_n := I_n(0) (n \in \mathbf{N})$. Set $e_n := (0, \dots, 0, 1, 0, \dots) \in G$, the nth coordinate of which is 1 and the rest are zeros $(n \in \mathbf{N})$.

For $k \in \mathbf{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k}$$

the kth Rademacher function. If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0,1\}$ $(i \in \mathbf{N})$, i.e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is $2^{|n|} \le n < 2^{|n|+1}$.

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)\, (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbf{P}).$$

The σ-algebra generated by the dyadic 2-dimensional cube I_k^2 of measure $2^{-k} \times 2^{-k}$ will be denoted by $\mathcal{F}_k$ $(k \in \mathbf{N})$.

The space $L_p(G^2), 0 < p \leq \infty$ with norms or quasi-norms $\| \cdot \|_p$ is defined in the usual way (For details see e.g. Weisz [15]).

Denote by $f = (f_n, n \in \mathbf{N})$ the one-parameter martingale with respect to $(\mathcal{F}_n, n \in \mathbf{N})$. The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbf{N}} |f_n| .$$

For $0 < p \leq \infty$ the Hardy martingale space $H_p(G^2)$ consists of all martingales for which

$$\|f\|_{H_p} = \|f^*\|_p < \infty.$$

The Dirichlet kernels are defined by

$$D_n(x) := \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see e.g. [11])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n(0), \\ 0, & \text{if } x \notin I_n(0). \end{cases} \tag{1}$$

The Fejér kernels are defined as follows

$$K_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(x).$$

The Kroneker product $(w_{n,m} : n, m \in \mathbf{N})$ of two Walsh system is said to be the two-dimensional Walsh system. Thus,

$$w_{n,m}\left(x^1, x^2\right) = w_n\left(x^1\right) w_m\left(x^2\right).$$

If $f \in L(G^2)$, then the number $\widehat{f}\,(n,m) := \int_{G^2} f w_{n,m}$ $(n, m \in \mathbf{N})$ is said to be the (n,m)th Walsh–Fourier coefficient of f. We can extend this definition to martingales in the usual way (see Weisz [15], [16]). Denote by $S_{n,m}$ the (n,m)th rectangular partial sum of the Walsh–Fourier series of a martingale f. Namely,

$$S_{n,m}(f; x^1, x^2) := \sum_{k=0}^{n-1} \sum_{i=0}^{m-1} \widehat{f}(k,i) w_{k,i}(x^1, x^2).$$

The Marcinkiewicz–Fejér means of a martingale f are defined by

$$\mathcal{M}_n\left(f;x^1,x^2\right):=\frac{1}{n}\sum_{k=0}^{n-1}S_{k,k}(f;x^1,x^2).$$

The 2-dimensional Dirichlet kernels and Marcinkiewicz–Fejér kernels are defined by

$$D_{k,l}(x^1,x^2):=D_k(x^1)D_l(x^2),\quad K_n(x^1,x^2):=\frac{1}{n}\sum_{k=0}^{n-1}D_{k,k}(x^1,x^2).$$

For the martingale f we consider the maximal operator

$$\mathcal{M}^*f(x^1,x^2)=\sup_{n\in\mathbf{P}}|\mathcal{M}_n(f;x^1,x^2)|.$$

3. Auxiliary propositions and main results

First, we formulate our main theorems. Our theorems are the two-dimensional analogue of the theorems of Goginava [8] for Walsh–Fejér means.

Theorem 1. *The maximal operator $\tilde{\mathcal{M}}^*$ is bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$.*

Theorem 2. *Let $\varphi:\mathbf{P}\to[1,\infty)$ be a nondecreasing function satisfying the condition*

$$\overline{\lim_{n\to\infty}}\frac{\log^{3/2}(n+1)}{\varphi(n)}=+\infty. \tag{2}$$

Then the maximal operator

$$\sup_{n\in\mathbf{P}}\frac{|\mathcal{M}_nf|}{\varphi(n)}$$

is not bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$.

To prove our Theorem 1 we need the following Lemmas of Goginava [7, Lemma 7, Lemma 9], Glukhov [5] and Weisz [16]:

Lemma 1 (Goginava [7])**.** *Let $(x^1,x^2)\in(I_{l^1}\backslash I_{l^1+1})\times(I_{m^2}\backslash I_{m^2+1})$ and $0\le l^1<N$, $0\le m^2<N$. Then*

$$\int_{I_N\times I_N}|K_n(x^1+t^1,x^2+t^2)|d\mu(t^1,t^2)$$
$$\le\frac{c}{2^{3N}}\Bigg\{2^{l^1-m^2}\sum_{r^1=l^1+1}^{m^2+1}2^{r^1}D_{2^{m^2+1}}(x^1+e_{l^1}+e_{r^1})\sum_{s=m^2+1}^{N}D_{2^s}(x^2+e_{m^2}+x^1_{m^2+1,s-1})$$

$$+ 2^{l^1+m^2} \sum_{s=l^1}^{m^2} \sum_{r^1=l^1+1}^{s} D_{2^s}(x^1 + e_{l^1} + e_{r^1}) \Bigg\}, \text{ for } n \geq 2^N,$$

with the notation $x_{i,j} := \sum_{s=i}^{j} x_s e_s$ $(x_{i,i-1} = 0)$.

Lemma 2 (Goginava [7])**.** *Let* $(x^1, x^2) \in I_N \times (I_{m^2} \setminus I_{m^2+1})$ *and* $0 \leq m^2 < N$. *Then*

$$\int_{I_N \times I_N} |K_n(x^1 + t^1, x^2 + t^2)| d\mu(t^1, t^2) \leq c \frac{2^{m^2}}{2^{2N}} \sum_{s=m^2}^{N-1} D_{2^s}(x^2 + e_{m^2}), \text{ for } n > 2^N.$$

Lemma 3 (Glukhov [5])**.** *There exists a constant* c *such that*

$$\sup_n \int_{G^2} |K_n(x^1, x^2)| d\mu(x^1, x^2) \leq c.$$

A bounded measurable function a is a p-atom, if there exists a dyadic two-dimensional cube I^2, such that

a) $\int_{I^2} a d\mu = 0$,

b) $\|a\|_\infty \leq \mu(I^2)^{-1/p}$,

c) $\operatorname{supp} a \subset I^2$.

Lemma 4 (Weisz [16])**.** *Suppose that the operator* T *is sublinear and* p-*quasilocal for any* $0 < p \leq 1$. *If* T *is bounded from* L_∞ *to* L_∞, *then*

$$\|Tf\|_p \leq c_p \|f\|_{H_p} \quad \text{for all } f \in H_p.$$

4. Proofs of the theorems

First, we prove Theorem 1.

Proof of Theorem 1. Lemma 3 yields the boundedness from the space L_∞ to the space L_∞. By Lemma 4, the proof will be complete, if we show that the maximal operator $\tilde{\mathcal{M}}^*$ is 2/3-quasilocal. That is, there exists a constant c such that

$$\int_{\overline{I^2}} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu \leq c < \infty$$

for every 2/3-atom a. where the dyadic cube I^2 is the support of the 2/3-atom a.

Let a be an arbitrary $2/3$-atom with support I^2, and $\mu(I^2) = 2^{-2N}$. Without loss of generality, we may assume that $I^2 := I_N \times I_N$. It is evident that $\tilde{\mathcal{M}}_n(a) = 0$ if $n \le 2^N$ (with the notation $\tilde{\mathcal{M}}_n(f) := \frac{|\mathcal{M}_n f|}{\log^{3/2}(n+1)}$). Therefore, we set $n > 2^N$.

By $\|a\|_\infty \le 2^{3N}$ we have that

$$\begin{aligned}\frac{|\mathcal{M}_n(a;x^1,x^2)|}{\log^{3/2}(n+1)} &\le \frac{1}{\log^{3/2}(n+1)} \int_{I_N\times I_N} |a(t^1,t^2)||K_n(x^1+t^1,x^2+t^2)|d\mu(t^1,t^2) \\ &\le \frac{c2^{3N}}{\log^{3/2}(n+1)} \int_{I_N\times I_N} |K_n(x^1+t^1,x^2+t^2)|d\mu(t^1,t^2)\end{aligned}$$

and

$$|\tilde{\mathcal{M}}^* a| \le \frac{c2^{3N}}{N^{3/2}} \sup_{n>2^N} \int_{I_N\times I_N} |K_n(x^1+t^1,x^2+t^2)|d\mu(t^1,t^2). \tag{3}$$

We write that

$$\begin{aligned}\int_{\overline{I_N\times I_N}} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu &= \int_{I_N\times\overline{I_N}} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu + \int_{\overline{I_N}\times I_N} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu \\ &\quad + \int_{\overline{I_N}\times\overline{I_N}} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu =: L_1 + L_2 + L_3.\end{aligned}$$

First, we discuss L_1 by the help of Lemma 2 and inequality (3) (the discussion of L_2 goes analogously). We introduce the notation $J_t := I_t \backslash I_{t+1}$ $(t \in \mathbf{N})$.

$$\begin{aligned}L_1 &= \sum_{m^2=0}^{N-1} \int_{I_N\times J_{m^2}} |\tilde{\mathcal{M}}^* a(x^1,x^2)|^{2/3} d\mu(x^1,x^2) \\ &\le \frac{c}{N} \sum_{m^2=0}^{N-1} \int_{I_N\times J_{m^2}} \left| 2^{3N} \sup_{n>2^N} \int_{I_N\times I_N} |K_n(x^1+t^1,x^2+t^2)|d\mu(t^1,t^2)\right|^{2/3} d\mu(x^1,x^2) \\ &\le \frac{c}{N} \sum_{m^2=0}^{N-1} \int_{I_N\times J_{m^2}} \left| 2^{m^2+N} \sum_{s=m^2}^{N-1} D_{2^s}(x^2+e_{m^2})\right|^{2/3} d\mu(x^1,x^2).\end{aligned}$$

We decompose J_{m^2} as the following disjoint union:

$$J_{m^2} = \bigcup_{q^2=m^2+1}^{N} I_N^{m^2,q^2},$$

where

$$I_N^{m^2,q^2} := \begin{cases} I_{q^2+1}(0,\dots,0,x_{m^2}=1,0,\dots,0,x_{q^2}=1), & \text{for } m^2<q^2<N, \\ I_N(0,\dots,0,x_{m^2}=1,0,\dots,0), & \text{for } q^2=N. \end{cases}$$

From (1), we get

$$
\begin{aligned}
L_1 &\le \frac{c2^{-N/3}}{N}\sum_{m^2=0}^{N-1}\sum_{q^2=m^2+1}^{N}\int_{I_N^{m^2,q^2}}\left|2^{m^2}\sum_{s=m^2}^{q^2}2^s\right|^{2/3}d\mu(x^2)\\
&\le \frac{c2^{-N/3}}{N}\sum_{m^2=0}^{N-1}\sum_{q^2=m^2+1}^{N}2^{2m^2/3+2q^2/3}2^{-q^2}\le c.
\end{aligned}
$$

Now, we discuss L_3.

$$
\begin{aligned}
L_3 &= \sum_{l^1=0}^{N-1}\sum_{m^2=0}^{N-1}\int_{J_{l^1}\times J_{m^2}}|\tilde{\mathfrak{M}}^*a|^{2/3}d\mu\\
&= \sum_{l^1=0}^{N-1}\sum_{m^2=0}^{l^1-1}\int_{J_{l^1}\times J_{m^2}}|\tilde{\mathfrak{M}}^*a|^{2/3}d\mu\\
&\quad+\sum_{l^1=0}^{N-1}\sum_{m^2=l^1}^{N-1}\int_{J_{l^1}\times J_{m^2}}|\tilde{\mathfrak{M}}^*a|^{2/3}d\mu = L_{3,1}+L_{3,2}.
\end{aligned}
$$

We discuss $L_{3,2}$ (the discussion of $L_{3,1}$ goes analogously). By the inequality (3) we have that

$$
\begin{aligned}
L_{3,2} &\le \frac{c}{N}\sum_{l^1=0}^{N-1}\sum_{m^2=l^1}^{N-1}\int_{J_{l^1}\times J_{m^2}}\\
&\quad\times\left|2^{3N}\sup_{n>2^N}\int_{I_N\times I_N}|K_n(x^1+t^1,x^2+t^2)|d\mu(t^1,t^2)\right|^{2/3}d\mu(x^1,x^2)\\
&=: \frac{c}{N}\sum_{l^1=0}^{N-1}\sum_{m^2=l^1}^{N-1}L_{3,2}^{l^1,m^2}.
\end{aligned}
$$

To discuss $L_{3,2}^{l^1,m^2}$, we write the set J_{l^1} in the form of following disjoint union:

$$
J_{l^1} = \bigcup_{k=l^1+1}^{m^2+1} I_{m^2+1}^{l^1,k}.
$$

That is,

$$
L_{3,2}^{l^1,m^2} = \sum_{k=l^1+1}^{m^2+1}\int_{I_{m^2+1}^{l^1,k}\times J_{m^2}}|\qquad|^{2/3}d\mu(x^1,x^2)
$$

By Lemma 1, $\sum_{r^1=l^1+1}^{m^2+1} D_{2^{m^2+1}}(.+e_{l^1}+e_{r^1})=0$ and $\sum_{r^1=l^1+1}^{m^2+1} D_{2^{m^2+1}}(.+e_{l^1}+e_{r^1})\neq 0$ determine two cases. Thus, we write $I_{m^2+1}^{l^1,k}=(I_{m^2+1}^{l^1,k}\cap\overline{\cup_{r^1=l^1+1}^{m^2+1} I_{m^2+1}(e_{l^1}+e_{r^1})})\bigcup(I_{m^2+1}^{l^1,k}\cap(\cup_{r^1=l^1+1}^{m^2+1} I_{m^2+1}(e_{l^1}+e_{r^1})))$. Thus, this divide the expression $L_{3,2}^{l^1,m^2}$ into two parts $L_{3,2,1}^{l^1,m^2}$ and $L_{3,2,2}^{l^1,m^2}$. In $L_{3,2,1}^{l^1,m^2}$ we integrate on the set $I_{m^2+1}^{l^1,k}\cap\overline{\cup_{r^1=l^1+1}^{m^2+1} I_{m^2+1}(e_{l^1}+e_{r^1})}$, while in $L_{3,2,2}^{l^1,m^2}$ we integrate on the set $I_{m^2+1}^{l^1,k}\cap(\cup_{r^1=l^1+1}^{m^2+1} I_{m^2+1}(e_{l^1}+e_{r^1}))=I_{m^2+1}(e_{l^1}+e_k)$. Using Lemma 1, we immediately have

$$L_{3,2,1}^{l^1,m^2}\leq c2^{-m^2}\sum_{k=l^1+1}^{m^2+1}\int_{I_{m^2+1}^{l^1,k}}\left|2^{l^1+m^2}\sum_{s=l^1}^{m^2} D_{2^s}(x^1+e_{l^1}+e_k)\right|^{2/3}d\mu(x^1).$$

Now, we decompose the set $I_{m^2+1}^{l^1,k}$ in the following form of disjoint union:

$$I_{m^2+1}^{l^1,k}=\bigcup_{r=k+1}^{m^2+1} I_{m^2+1}^{l^1,k,r},$$

where $I_{m^2+1}^{l^1,k,r}:=I_{r+1}(0,\dots,0,\,x_{l^1}^1=1,0,\dots,0,\,x_k^1=1,0,\dots,x_r^1=1)$ for $k<r\leq m^2$ and $I_{m^2+1}^{l^1,k,r}:=I_{m^2+1}(e_{l^1}+e_k)$ for $r=m^2+1$. This yields

$$\begin{aligned}L_{3,2,1}^{l^1,m^2}&\leq c2^{-m^2}\sum_{k=l^1+1}^{m^2+1}\sum_{r=k+1}^{m^2+1}\int_{I_{m^2+1}^{l^1,k,r}}\left|2^{l^1+m^2}\sum_{s=l^1}^{r}2^s\right|^{2/3}d\mu(x^1)\\&\leq c2^{-m^2}\sum_{k=l^1+1}^{m^2+1}\sum_{r=k+1}^{m^2+1}2^{2(l^1+m^2+r)/3}2^{-r}\leq c2^{l^1/3-m^2/3}.\end{aligned}\tag{4}$$

Now, we turn our attention to $L_{3,2,2}^{l^1,m^2}$.

$$\begin{aligned}L_{3,2,2}^{l^1,m^2}&\leq c\sum_{k=l^1+1}^{m^2+1}\int_{I_{m^2+1}(e_{l^1}+e_k)\times J_{m^2}}(\qquad)^{2/3}d\mu(x^1,x^2)\\&\leq c\sum_{k=l^1+1}^{m^2+1}\sum_{\substack{x_i^1=0\\ i\in\{m^2+1,\dots,N-1\}\\ x_j^1=0\text{ otherwise}}}^{1}\int_{I_N(e_{l^1}+e_k+x^1)\times J_{m^2}}(\qquad)^{2/3}d\mu(x^1,x^2)\end{aligned}$$

For fixed $x_i^1,\ m^2+1\leq i<N$ we decompose the set J_{m^2} in the form of following disjoint union:

$$J_{m^2}=\bigcup_{q^2=m^2+1}^{N} I_N^{m^2,q^2}(x^1_{m^2+1,q^2-1}),$$

where $I_N^{m^2,q^2}(x^1_{m^2+1,q^2-1}) := I_{q^2+1}(0,\dots,0,x^2_{m^2}=1,x^1_{m^2+1},\dots,x^1_{q^2-1},1-x^1_{q^2})$, for $m^2<q^2<N$ and $I_N^{m^2,q^2}(x^1_{m^2+1,q^2-1}) := I_N(0,\dots,0,x^2_{m^2}=1,x^1_{m^2+1},\dots, x^1_{N-1})$, for $q^2=N$. That is, by Lemma 1

$$\begin{aligned} L^{l^1,m^2}_{3,2,2} &\le c\sum_{k=l^1+1}^{m^2+1}\sum_{\substack{x^1_i=0\\ i\in\{m^2+1,\dots,N-1\}\\ x^1_j=0 \text{ otherwise}}}^{1}\sum_{q^2=m^2+1}^{N}\int_{I_N(e_{l^1}+e_k+x^1)\times I_N^{m^2,q^2}(x^1_{m^2+1,q^2-1})} \\ &\quad\times\left(2^{l^1+k}\sum_{s=m^2+1}^{q^2}2^s+2^{l^1+m^2}\sum_{s=l^1}^{m^2}2^s\right)^{2/3}d\mu(x^1,x^2)\\ &\le c\sum_{k=l^1+1}^{m^2+1}\sum_{q^2=m^2+1}^{N}2^{2(l^1+m^2+q^2)/3}2^{-m^2-q^2}\\ &\le c(m^2+1-l^1)2^{2l^1/3-2m^2/3}. \end{aligned}$$

This and inequality (4) yield that

$$\begin{aligned} L_{3,2} &\le \frac{c}{N}\sum_{l^1=0}^{N-1}\sum_{m^2=l^1}^{N-1}2^{l^1/3-m^2/3}+\frac{c}{N}\sum_{l^1=0}^{N-1}\sum_{m^2=l^1}^{N-1}(m^2+1-l^1)2^{2l^1/3-2m^2/3}\\ &\le\frac{cN}{N}\le c. \end{aligned}$$

This completes the proof of Theorem 1. □

Next, we prove Theorem 2.

Proof of Theorem 2. Let

$$f_A(x^1,x^2) := (D_{2^{A+1}}(x^1)-D_{2^A}(x^1))(D_{2^{A+1}}(x^2)-D_{2^A}(x^2)).$$

A simple calculation yields

$$\widehat{f}_A(i,k)=\begin{cases}1, & \text{if } i,k=2^A,\dots,2^{A+1}-1,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\begin{aligned} &S_{i,j}(f;x^1,x^2)\\ &=\begin{cases}(D_i(x^1)-D_{2^A}(x^1))(D_j(x^2)-D_{2^A}(x^2)), & \text{if } i,j=2^A+1,\dots,2^{A+1}-1,\\ f_A(x^1,x^2), & \text{if } i,j\ge 2^{A+1},\\ 0, & \text{otherwise.}\end{cases} \end{aligned}$$

We can write the nth Dirichlet kernel in the following form:

$$D_n(x) \quad = \quad D_{2^{|n|}}(x) + r_{|n|}(x) D_{n-2^{|n|}}(x)$$

Thus, we have for a nondecreasing function φ that

$$\begin{aligned}
\tilde{\mathcal{M}}^* f_A(x^1,x^2) &= \sup_{n\in\mathbf{P}} \frac{|\mathcal{M}_n(f_A;x^1,x^2)|}{\varphi(n)} \geq \max_{t:1\leq 2^t\leq 2^A} \frac{|\mathcal{M}_{2^A+2^t}(f_A;x^1,x^2)|}{\varphi(2^A+2^t)} \\
&\geq \max_{t:1\leq 2^t\leq 2^A} \frac{1}{(2^A+2^t)\varphi(2^A+2^t)} \left| \sum_{k=0}^{2^A+2^t-1} S_{k,k}(f_A;x^1,x^2) \right| \\
&\geq \max_{t:1\leq 2^t\leq 2^A} \frac{1}{2^{A+1}\varphi(2^{A+1})} \left| \sum_{k=2^A+1}^{2^A+2^t-1} (D_k(x^1)-D_{2^A}(x^1))(D_k(x^2)-D_{2^A}(x^2)) \right| \\
&= \max_{t:1\leq 2^t\leq 2^A} \frac{1}{2^{A+1}\varphi(2^{A+1})} \left| \sum_{k=2^A+1}^{2^A+2^t-1} r_A(x^1) D_{k-2^A}(x^1) r_A(x^2) D_{k-2^A}(x^2) \right| \\
&= \max_{t:1\leq 2^t\leq 2^A} \frac{1}{2^{A+1}\varphi(2^{A+1})} \left| \sum_{l=1}^{2^t-1} D_l(x^1) D_l(x^2) \right| \\
&= \frac{1}{2^{A+1}\varphi(2^{A+1})} \max_{t:1\leq 2^t\leq 2^A} 2^t |K_{2^t}(x^1,x^2)|.
\end{aligned}$$

Since, we have

$$f_A^*(x^1,x^2) = \sup_{n\in\mathbf{N}} |S_{2^n,2^n}(f_A;x^1,x^2)| = |f_A(x^1,x^2)|$$

and

$$\|f_A\|_{H_{2/3}} = \|f_A^*\|_{2/3} = c2^{-A}.$$

We obtain

$$\frac{\|\tilde{\mathcal{M}}^* f_A\|_{2/3}}{\|f_A\|_{H_{2/3}}} \geq \frac{c}{2^A\varphi(2^{A+1})2^{-A}} \left(\int_{G^2} \max_{t:1\leq 2^t\leq 2^A} (2^t|K_{2^t}(x^1,x^2)|)^{2/3} d\mu(x^1,x^2) \right)^{3/2}.$$

To investigate the integral $\int_{G^2} \max_{t:1\leq 2^t\leq 2^A} (2^t|K_{2^t}(x^1,x^2)|)^{2/3} d\mu(x^1,x^2)$, we decompose the set G as the following disjoint union

$$G = I_A \cup \bigcup_{s=0}^{A-1} (I_s \backslash I_{s+1}).$$

It is easy to show that, for $(x^1,x^2)\in I_s\times I_s$

$$K_{2^s}(x^1,x^2)=\frac{(2^s-1)(2^{s+1}-1)}{6}.$$

Therefore,

$$\begin{aligned}
&\int_{G\times G}\max_{t:1\le 2^t\le 2^A}(2^t|K_{2^t}(x^1,x^2)|)^{2/3}d\mu(x^1,x^2)\\
&\quad\ge\sum_{s=1}^{A-1}\int_{(I_s\setminus I_{s+1})\times(I_s\setminus I_{s+1})}\max_{t:1\le 2^t\le 2^A}(2^t|K_{2^t}(x^1,x^2)|)^{2/3}d\mu(x^1,x^2)\\
&\quad\ge\sum_{s=1}^{A-1}\int_{(I_s\setminus I_{s+1})\times(I_s\setminus I_{s+1})}(2^s|K^w_{2^s}(x^1,x^2)|)^{2/3}d\mu(x^1,x^2)\\
&\quad=\sum_{s=1}^{A-1}\int_{(I_s\setminus I_{s+1})\times(I_s\setminus I_{s+1})}\left(2^s\frac{(2^s-1)(2^{s+1}-1)}{6}\right)^{2/3}d\mu(x^1,x^2)\\
&\quad\ge c\sum_{s=1}^{A-1}\int_{(I_s\setminus I_{s+1})\times(I_s\setminus I_{s+1})}\left(2^{3s}\right)^{2/3}d\mu(x^1,x^2)\ge c(A-2).
\end{aligned}$$

That is,

$$\frac{\|\tilde{\mathfrak{M}}^*f_A\|_{2/3}}{\|f_A\|_{H_{2/3}}}\ge\frac{c(A+1)^{3/2}}{\varphi(2^{A+1})}$$

for A big enough.

Now, let $\{n_k:k\in\mathbf{P}\}$ be an increasing sequence of positive integers such that

$$\lim_{k\to\infty}\frac{\log^{3/2}n_k}{\varphi(n_k)}=+\infty.$$

There exists a positive integer m_k' such that $2^{m_k'}\le n_k<2\cdot 2^{m_k'}$. φ is a nondecreasing function, then we have

$$\overline{\lim_{k\to\infty}}\frac{(m_k')^{3/2}}{\varphi(2^{m_k'})}\ge c\lim_{k\to\infty}\frac{\log^{3/2}n_k}{\varphi(n_k)}=+\infty.$$

Let $\{m_k:k\in\mathbf{P}\}\subset\{m_k':k\in\mathbf{P}\}$ be such that

$$\lim_{k\to\infty}\frac{(m_k)^{3/2}}{\varphi(2^{m_k})}=+\infty.$$

This yields

$$\frac{\|\tilde{\mathfrak{M}}^*f_{m_k-1}\|_{2/3}}{\|f_{m_k-1}\|_{H_{2/3}}}\ge\frac{c(m_k)^{3/2}}{\varphi(2^{m_k})}.$$

$k\to\infty$ completes the proof of this theorem. □

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(Received February 3, 2010; revised January 27, 2011)