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Inversion techniques and combinatorial identities Basic hypergeometric identities

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Abstract. Based on Carlitz' q-analogue of the Gould–Hsu inverse series relations, a new insight into basic hypergeometric formulas is offered. Most q-series identities are revisited throught the embedding technique, and some new ones are demonstrated too.

0. Introduction

Recently, basic hypergeometric series (q-series) computation has attracted new interest in the mathematical world for its wide application to mathematics, physics and computer science. But the basic formulas often involved in computation are so subtle that their derivation is not easy and so in most situations they are only regarded as prescribed fact. Therefore it seems desirable to gain new insight and to revisit the most well known q-series identities so that the non-specialist may recover their connections without need of reproving each individual formula.

During the past few years, Hsu and the author [11] thought that the Gould–Hsu inverse relation [17] could be used to establish relations among binomial identities (ordinary hypergeometric identities according to Andrews' view). This has been accomplished in the author's work [9, 10] a short time ago. Now it is natural to consider the q-analogue of the work mentioned above. That is the purpose of the present paper which will show that most basic hypergeometric formulae can be reproduced in a unified manner throught the Carlitz q-analogue [8] of the Gould–Hsu inverse series relations.

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To make the paper self-contained, the basic theorem due to Carlitz can be restated as follows: Let (a_i) and (b_i) be two complex sequences such that the polynomials defined by

$$\lambda(x;n)=\prod_{k=1}^{n}(a_k+q^x b_k) \tag{0.1a}$$

differ from zero for $-x, n\in\mathbb{N}_0$ (the set of non-negative integers) with the convention $\lambda(x;0)=1$. Then there hold the inverse relations:

$$f(n)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}\lambda(-k;n)\ g(k) \tag{0.1b}$$

$$g(n)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\frac{a_{k+1}+q^{-k}b_{k+1}}{\lambda(-n;k+1)}f(k) \tag{0.1c}$$

Interchanging a and b this reciprocal pair can be reformulated in an equivalent manner:

$$f(n)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\lambda(k;n)\ g(k) \tag{0.1d}$$

$$g(n)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}\frac{a_{k+1}+q^{k}b_{k+1}}{\lambda(n;k+1)}f(k), \tag{0.1e}$$

provided that the polynomials defined by (0.1a) do not vanish for non-negative integers x and n.

For the sake of brevity, the inverse pairs (0.1b)–(0.1c) and (0.1d)–(0.1e) will be referred as C-pair and C'-pair, respectively.

Let $|q|<1$. As usual, a ${}_r\Phi_s$ basic hypergeometric series with base q is defined by (cf. e.g., [14])

$$\begin{aligned}&{}_r\Phi_s\left[\begin{matrix}a_1, & a_2, & \ldots, & a_r\\ & b_1, & \ldots, & b_s\end{matrix};Z\right]=\\ &=\sum_{k=0}^{\infty}\frac{(a_1;q)_k(a_2;q)_k\ldots(a_r;q)_k}{(b_1;q)_k(b_2;q)_k\ldots(b_s;q)_k}\frac{z^k}{(q;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{(1-r+s)}\end{aligned} \tag{0.2a}$$

whenever the series converges (e.g., if $|z|<1$) where the q-shifted factorials are defined by

$$(x,q)_\infty=\prod_{k=0}^{\infty}(1-xq^k),\quad (x;q)_n=(x;q)_\infty/(xq^n;q)_\infty. \tag{0.2b}$$

To simplify the notation we will also write $(x)_n$ in place of $(x;q)_n$ and

$$\text{(0.2c)}\qquad \left[\begin{matrix} a_1, & a_2, & \dots, & a_r \\ b_1, & b_2 & \dots, & b_s \end{matrix} ; q\right]_n = \frac{(a_1;q)_n(a_2;q)_n\dots(a_r;q)_n}{(b_1;q)_n(b_2;q)_n\dots(b_s;q)_n}$$

instead of factorial-fractions.

To carry out the exchange between C-pairs and q-series, the following transformations

$$\text{(0.3a)}\qquad (xq^k;q)_n = (x;q)_n(xq^n;q)_k/(x;q)_k.$$

$$\text{(0.3b)}\qquad (xq^{-k};q)_n = q^{-nk}(x;q)_n(qx^{-1};q)_k/(q^{1-n}x^{-1};q)_k,$$

will be used frequently without indication.

In the following sections, various q-hypergeometric identities will be demonstrated by means of the embedding technique (cf. [9-11]) and series-composition. Throughout the paper the simple yet tedious calculations will not be displayed in detail and are left to the reader.

1. q-analogue: Gauss-Vandermonde theorems and Abel-identities

For the q-shifted operator $E : Ef(x) = f(qx)$, define the q-difference operator [12]

$$\text{(1.1)}\quad (E;q)_n = (1-Eq^{n-1})\dots(1-Eq)(1-E) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}}\, E^k,$$

where $\begin{bmatrix} n \\ k \end{bmatrix} = (q)_n/(q)_k(q)_{n-k}$. The induction principle shows that

$$(E;q)_n \frac{(a)_x}{(b)_x}(b/a)^x = (b/a)_n \frac{(a)_x}{(b)_{x+n}}(b/a)^x,$$

which can be restated, when $x = 0$, as the q-analog of the Chu-Vandermonde convolution:

$$\text{(1.2a)}\qquad {}_2\Phi_1 \left[\begin{matrix} q^{-n}, & a \\ & b \end{matrix} ; q^n b/a\right] = \left[\begin{matrix} b/a \\ b \end{matrix} ; q\right]_n .$$

Reversing the summation order in the above gives that

$$\text{(1.2b)}\qquad {}_2\Phi_1 \left[\begin{matrix} q^{-n}, & a \\ & b \end{matrix} ; q\right] = a^n \left[\begin{matrix} b/a \\ b \end{matrix} ; q\right]_n .$$

This pair of identities just corresponds to the C-pair or the C'-pair if they are telescoped. These formula can be also reformulated equivalently, as q-binomial convolutions:

$$\text{(1.3a)} \qquad \sum_{k=0}^{n} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} y \\ n-k \end{bmatrix} q^{(x-k)(n-k)} = \begin{bmatrix} x+y \\ n \end{bmatrix},$$

$$\text{(1.3b)} \qquad \sum_{k=0}^{n} \begin{bmatrix} u+k \\ k \end{bmatrix} \begin{bmatrix} v+n-k \\ n-k \end{bmatrix} q^{(1+v)k} = \begin{bmatrix} u+v+n+1 \\ n \end{bmatrix}.$$

Moreover, the limiting version of (1.2a) for a and b tending to zero with $b/a = x$ reduces to Euler's identity

$$\text{(1.4)} \qquad (x,q)_n = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k.$$

Now for a non-negative integer e, we have from (1.2b)

$$\begin{bmatrix} xq^e & \\ & ;q \\ x & \end{bmatrix}_k = \begin{bmatrix} xq^k & \\ & ;q \\ x & \end{bmatrix}_e = q^{ke}{}_2\Phi_1 \begin{bmatrix} q^{-e}, & q^{-k} & \\ & & ;q \\ & x & \end{bmatrix}.$$

Consider the composition

$${}_3\Phi_2 \begin{bmatrix} q^{-n}, & xq^e, & a & \\ & & & ;q^{n-e}b/a \\ & x, & b & \end{bmatrix} =$$

$$= \sum_{k=0}^{n} \begin{bmatrix} q^{-n}, & a & \\ & & ;q \\ q, & b & \end{bmatrix}_k (q^n b/a)^k {}_2\Phi_1 \begin{bmatrix} q^{-e}, & q^{-k} & \\ & & ;q \\ & x & \end{bmatrix} =$$

$$= \sum_i (-1)^i q^{-\binom{i}{2}} \begin{bmatrix} q^{-n}, & a, & q^{-e} & \\ & & & ;q \\ q, & b, & x & \end{bmatrix}_i (q^n b/a)^i \cdot$$

$$\cdot {}_2\Phi_1 \begin{bmatrix} q^{-n+i}, & aq^i & \\ & & ;q^{n-i}b/a \\ & bq^i & \end{bmatrix},$$

where the summation indices have been exchanged. By means of the q-

Vandermonde formula, this can be simplified as a series-transformation:

$$(1.5a)\qquad {}_3\Phi_2\left[\begin{matrix} q^{-n}, & xq^e, & a \\ & x, & b \end{matrix}; q^{n-e}b/a\right] = \\ = \left[\begin{matrix} b/a \\ b \end{matrix}; q\right]_n {}_3\Phi_2\left[\begin{matrix} q^{-e}, & a, & q^{-n} \\ x, & q^{1-n} & a/b \end{matrix}; q\right],$$

the reversal of which yields

$$(1.5b)\qquad {}_3\Phi_2\left[\begin{matrix} q^{-n}, & xq^e, & a \\ & x, & b \end{matrix}; q\right] = \\ = a^n \left[\begin{matrix} b/a \\ b \end{matrix}; q\right]_n {}_3\Phi_2\left[\begin{matrix} q^{-e}, & a, & q^{-n} \\ x, & q^{1-n} & a/b \end{matrix}; q^{1+e}x/b\right].$$

This pair of transformations is useful for the evaluation of the q-series involved in the members on the left when e is quite small, and for $e = 0$ they reduce to (1.2a) and (1.2b) respectively.

By embedding in the C'-pair, we can establish the following inverse relations

(1.6a)
$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}(1-Aq^k)^n \frac{1-A-B}{1-Aq^k-B}\left(\frac{B}{1-Aq^k}; q\right)_k = B^n q^{\binom{n}{2}}$$

$$(1.6b)\qquad \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1-Aq^k}{(1-Aq^n)^{k+1}} B^k q^{\binom{k}{2}} = \\ = \frac{1-A-B}{1-Aq^n-B}\left(\frac{B}{1-Aq^n}; q\right)_n,$$

$$(1.7a)\qquad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (Cq^{-1}(1-Aq^k); q^{-1})_n \frac{1-A-B}{1-Aq^k-Bq^k} \cdot \\ \cdot \frac{(Bq(1-Aq^k)^{-1}; q)_k}{(Cq^{-1}(1-Aq^k); q^{-1})_k} C^k (1-Aq^k)^k q^{-\binom{k+1}{2}} = (BCq^{-n}; q)_n q^{\binom{n}{2}}$$

$$(1.7b)\sum_{k=0}^{n}(-1)^{k+n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1+q^{-1}AC - Cq^{-k-1}}{(Cq^{-1}(1-Aq^n); q^{-1})_{k+1}} (BCq^{-k}; q)_k q^{\binom{k}{2}} =$$

$$= \frac{1-A-B}{1-Aq^n-B}\frac{(B(1-Aq^n)^{-1};q)_n}{(Cq^{-1}(1-Aq^n);q^{-1})_n}q^{-\binom{n+1}{2}}(1-Aq^n)^nC^n.$$

Among these identities, (1.6b) and (1.7b) are special cases of (1.4) and (1.5a), respectively, while their dual versions are the q-analogues of the Abel-identities (cf. JACKSON [19]; see also [15], Sec. 3.).

2. q–Saalschutz-type formulae

By using (1.2a)–(1.2b) after interchanging the summation indices, we can manipulate the series composition as follows (where e is a non-negative integer as in Section 1):

$${}_3\Phi_2\left[\begin{matrix} q^{-n}, & a,\ b \\ cq^e, & q^{1-n}ab/c \end{matrix} ; q^{1+e}\right] =$$

$$= \sum_{k=0}^{n}\left[\begin{matrix} q^{-n}, & b \\ q, & q^{1-n}ab/c \end{matrix} ; q\right]_k (qa/c)^k {}_2\Phi_1\left[\begin{matrix} q^{-k}, & q^e c/a \\ & q^e c \end{matrix} ; q\right] =$$

$$= \sum_i (-1)^i q^{-\binom{i}{2}} \left[\begin{matrix} q^{-n}, & b, & q^e c/a \\ q, & q^e c, & q^{1-n}ab/c \end{matrix} ; q\right]_i (qa/c)^i \cdot$$

$$\cdot\ {}_2\Phi_1\left[\begin{matrix} q^{-n+i}, & bq^i \\ q^{1-n+i} & ab/c \end{matrix} ; q^{1-i}a/c\right] =$$

$$= \left[\begin{matrix} q^{1-n}a/c \\ q^{1-n}ab/c \end{matrix} ; q\right]_n {}_3\Phi_2\left[\begin{matrix} q^{-n}, & q^e c/a, & b \\ & c/a, & q^e c \end{matrix} ; q\right].$$

By (1.5b), this transformation may be restated as

$$\text{(2.1a)} \qquad {}_3\Phi_2\left[\begin{matrix} q^{-n}, & a,\ b \\ cq^e, & q^{1-n}ab/c \end{matrix} ; q^{1+e}\right] =$$

$$= \left[\begin{matrix} c/a, & q^e c/b \\ q^e c, & c/ab \end{matrix} ; q\right]_n {}_3\Phi_2\left[\begin{matrix} q^{-e}, & b,\ q^{-n} \\ c/a, & q^{1-n-e}b/c \end{matrix} ; q/a\right],$$

whose reversal reads as

$$ (2.1b) \qquad {}_3\Phi_2\left[\begin{matrix} q^{-n}, & a, b \\ cq^e, & q^{1-n}ab/c \end{matrix}; q\right] = $$
$$ = \left[\begin{matrix} c/a, & q^e c/b \\ q^e c, & c/ab \end{matrix}; q\right]_n {}_3\Phi_2\left[\begin{matrix} q^{-e}, & b, & q^{-n} \\ c/a, & q^{1-n-e}b/c & \end{matrix}; q\right]. $$

It may be worth mentioning that the transformations (2.1a) and (2.1b) imply those demonstrated in [1] and [25] as immediate consequences, on account of the ${}_3\Phi_2$ symmetry with respect to e and n involved in the members on the right, and $e = 0$ they reduce to the famous q-Saalschutz theorem.

$$ (2.1c) \qquad {}_3\Phi_2\left[\begin{matrix} q^{-n}, & a,\ b \\ c, & q^{1-n}ab/c \end{matrix}; q\right] = \left[\begin{matrix} c/a, & c/b \\ c, & c/ab \end{matrix}; q\right]_n $$

In the remaining part of this section the terminating balanced q-series will be evaluated through the summation formulae (2.1a–c).

From the term-splitting $(qx)_k/(x)_k = (1-x)^{-1} - q^k x/(1-x)$, it is easy to see that

$$ (2.2) \qquad {}_4\Phi_3\left[\begin{matrix} a, & b, & ,c, & qx \\ & d, & e, & x \end{matrix}; q\right] = $$
$$ = \frac{1}{1-x}{}_3\Phi_2\left[\begin{matrix} a, & b, & c \\ & d, & e \end{matrix}; q\right] - \frac{x}{1-x}{}_3\Phi_2\left[\begin{matrix} a, & b, & c \\ & d, & e \end{matrix}; q^2\right]. $$

As a linear combinations of two 2–balanced summations, we can derive from (2.1a), (2.1b) and (2.1c) that

$$ (2.3a) \qquad {}_4\Phi_3\left[\begin{matrix} q^{-1}w/a, & q^2a/w, & -a^{1/2}, & q^{-n} \\ & qa/w, & w, & -q^{1-n}a^{-1/2} \end{matrix}; q\right] = $$
$$ = \left[\begin{matrix} a, & qa^{1/2}, & q^{-1}wa^{-1/2} \\ a^{1/2}, & -a^{1/2}, & w \end{matrix}; q\right]_n. $$

Telescoping this into (0.1b) yields the C-pair as follows:

$$ \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (-a^{1/2}q^{-k};q)_n \left[\begin{matrix} q^{-1}w/a, & q^2a/w, & -a^{1/2} \\ qa/w, & w, & -qa^{-1/2} \end{matrix}; q\right]_k. $$

$$\cdot q^{\binom{k+1}{2}} = \begin{bmatrix} a, & qa^{1/2}, & -q^{-1}wa^{-1/2} & \\ & a^{1/2}, & w & ;q \end{bmatrix}_n$$

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1+a^{1/2}}{(-a^{1/2}q^{-n};q)_{k+1}} \begin{bmatrix} a,\ qa^{1/2},\ -q^{-1}wa^{-1/2} & \\ a^{1/2}, \qquad w & ;q \end{bmatrix}_k =$$

$$= q^{\binom{n+1}{2}} \begin{bmatrix} q^{-1}w/a, & q^2a/w, & -a^{1/2} & \\ qa/w, & w, & -qa^{-1/2} & ;q \end{bmatrix}_n .$$

The last relation may be reformulated as (cf. (1.11) in [23])

$$\text{(2.3b)}\quad {}_4\Phi_3 \begin{bmatrix} a, & qa^{1/2}, & -q^{-1}wa^{-1/2}, & q^{-n} & \\ & a^{1/2}, & w, & -q^{1-n}a^{1/2} & ;q \end{bmatrix} = \begin{bmatrix} q^{-1}w/a, & q^2a/w, & -a^{1/2} & \\ qa/w, & w, & -a^{-1/2} & ;q \end{bmatrix}_n .$$

Similarly, we can establish the following C-pairs:

$$\text{(2.4a)}\quad {}_4\Phi_3 \begin{bmatrix} a, & qa^{1/2}, & b, & q^{-n} & \\ & a^{1/2}, & qa/b, & q^{1-n}b^2 & ;q \end{bmatrix} = \begin{bmatrix} -qa^{1/2}b^{-1}, & b^{-1}, & ab^{-2} & \\ -a^{1/2}b^{-1}, & b^{-2}, & qa/b & ;q \end{bmatrix}_n$$

$$\text{(2.4b)}\quad {}_4\Phi_3 \begin{bmatrix} ab^{-2}, & b^{-1}, & -qa^{1/2}b^{-1}, & q^{-n} & \\ & qa/b, & -a^{1/2}b^{-1} & q^{1-n}b^{-2} & ;q \end{bmatrix} = \begin{bmatrix} a, & qa^{1/2}, & b & \\ b^2, & a^{1/2}, & qa/b & ;q \end{bmatrix}_n$$

where the last relation may be restated in a similar form as (2.4a)

$$ {}_4\Phi_3\left[\begin{matrix} a, & -qa^{1/2}, & b, & q^{-n} \\ & -a^{1/2}, & qa/b, & q^{1-n}b^2 \end{matrix};q\right] = \tag{2.4c}$$

$$= \left[\begin{matrix} qa^{1/2}b^{-1}, & b^{-1}, & ab^{-2} \\ a^{1/2}b^{-1}, & qab^{-1}, & b^{-2} \end{matrix};q\right]_n ,$$

and

$$ {}_4\Phi_3\left[\begin{matrix} a, & b, & qa(1+b)/(a+b), & q^{-n} \\ qa/b, & & a(1+b)/(a+b), & q^{1-n}b^2 \end{matrix};q\right] = \tag{2.5a}$$

$$= \left[\begin{matrix} b^{-1}, & ab^{-2} \\ b^{-2}, & qa/b \end{matrix};q\right]_n$$

$$ {}_3\Phi_2\left[\begin{matrix} ab^{-2}, & b^{-1}, & q^{-n} \\ & qa/b, & q^{1-n}b^{-2} \end{matrix};q\right] = \tag{2.5b}$$

$$= \frac{1+a/b}{1-a}\left(1-\frac{a(1+b)}{a+b}q^n\right)\left[\begin{matrix} a, & b \\ b^2, & qa/b \end{matrix};q\right]_n .$$

The limiting versions of these two pairs correspond to the same hypergeometric identities (cf. Bailey [7] for the notation of hypergeometric series):

$$ {}_4F_3\left[\begin{matrix} a, & 1+a/2, & b, & -n \\ & a/2, & 1+a-b, & 1+2b-n \end{matrix}\right] = \left[\begin{matrix} -b, & a-2b \\ -2b, & 1+a-b \end{matrix}\right]_n$$

$$ {}_3F_2\left[\begin{matrix} a-2b, & -b, & -n \\ & 1+a-b, & 1-2b-n \end{matrix}\right] = \left[\begin{matrix} 1+a/2 & a, & b \\ a/2, & 1+a-b, & 2b \end{matrix}\right]_n .$$

This shows that there are different basic analogues for some binomial identities.

From the separation

$$(1-aq^{2k})/(1-a) = (1-aq^k)/(1-a) + aq^k(1-q^k)/(1-a),$$

the following balanced ${}_5\Phi_4$-series can be expressed as a two 2-balanced series in the forms (2.1a) and (2.1b), and so can be evaluated as

$$ {}_5\Phi_4\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & q^{-n} \\ & a^{1/2}, & -a^{1/2}, & qa/b, & q^{2-n}b^2 \end{matrix} ; q\right] = \tag{2.6} $$
$$ = \left[\begin{matrix} -q^{1/2}a^{1/2}b^{-1}, & q^{1/2}a^{1/2}b^{-1}, & q^{-1}b^{-1}, & q^{-1}ab^{-2} \\ -q^{1/2}a^{1/2}b^{-1}, & q^{-1/2}a^{1/2}b^{-1}, & qab^{-1}, & q^{-1}b^{-2} \end{matrix} ; q\right]_n . $$

Telescoping this into the C-pair shows that this is a self-reciprocal formula, i.e., the dual version possesses the same expression as the original one under the parameter-replacement.

All the formulae (2.3)–(2.6) demonstrated above are q-analogues of the k-balanced hypergeometric summations (for $k = 1$ and 2) contained in Bailey's book [7] (p.30). They have not been stated in the present manner previously, except for (2.3a) (cf. [23]).

3. Watson-transform and Jackson-theorems

The most striking relations associated with the C-pair reads as

(3.1a)
$$ {}_3\Phi_2\left[\begin{matrix} aq^n, & qa/bc, & q^{-n} \\ & qa/b, & qa/c \end{matrix} ; q\right] = (qa/bc)^n \left[\begin{matrix} b, & c \\ qa/b, & qa/c \end{matrix} ; q\right]_n $$

(3.1b)
$$ {}_6\Phi_5\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & q^{-n} \\ & a^{1/2}, & -a^{1/2}, & qa/b, & qa/c, & q^{n+1}a \end{matrix} ; q^{1+n}a/bc\right] = $$
$$ = \left[\begin{matrix} qa, & qa/bc \\ qa/b, & qa/c \end{matrix} ; q\right]_n , $$

which follows from telescoping in the manner

$$ \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{n}}(aq^k, q)_n \left[\begin{matrix} a, & qa/bc \\ qa/b, & qa/c \end{matrix} ; q\right]_k = $$
$$ = (qa/bc)^n q^{\binom{n}{2}} \left[\begin{matrix} a, \ b, \ c \\ qa/b, \ qa/c \end{matrix} ; q\right]_n $$

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1-aq^{2k}}{(aq^n;q)_{k+1}}(qa/bc)^k q^{\binom{k}{2}} \begin{bmatrix} a,\ b,\ c \\ qa/b,\ qa/c \end{bmatrix}_k ; q = $$

$$= \begin{bmatrix} a, & qa/bc \\ qa/b, & qa/c \end{bmatrix}_n ; q .$$

Consider the series-composition (very-well-poised series)

$$ {}_8\Phi_7 \begin{bmatrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e, & q^{-n} \\ a^{1/2}, & -a^{1/2}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{n+1}a & \end{bmatrix} ; q^{2+n}a^2/bcde = $$

$$= \sum_{k=0}^{n} \begin{bmatrix} a, & qa^{1/2}, & -qa^{1/2}, & d, & e, & q^{-n} \\ q, & a^{1/2}, & -a^{1/2}, & qa/d, & qa/e, & q^{n+1}a \end{bmatrix}_k ; q \cdot$$

$$\cdot (q^{1+n}a/de)^k {}_3\Phi_2 \begin{bmatrix} aq^k, & qa/bc, & q^{-k} \\ & qa/b, & qa/c \end{bmatrix} ; q = $$

$$= \sum_{i=0}^{n} \frac{(qa)_{2i}}{(q)_i} \begin{bmatrix} & qa/bc, & d, & e, & q^{-n} \\ qa/b, & qa/c, & qa/d, & qa/e, & q^{n+1}a \end{bmatrix}_i ; q \cdot$$

$$\cdot (-1)^i q^{-\binom{i}{2}} (q^{1+n}a/de)^i \cdot$$

$$\cdot\, {}_6\Phi_5 \begin{bmatrix} aq^{2i}, & q^{1+i}a^{1/2}, & -q^{1+i}a^{1/2}, & q^i d, & d^i e, & q^{-n+i} \\ & q^i a^{1/2}, & -q^i a^{1/2}, & q^{1+i}a/d, & q^{1+i}a/e, & q^{1+n+i}a \end{bmatrix} ; q^{1+n-i}a/de $$

in which the last summation results from the interchange between summation-indices. In view of (3.1b), *Watson's q-analogue of Whipple's transform* (e.g., cf. [14]) has been established:

(3.2)

$$ {}_8\Phi_7 \begin{bmatrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e, & q^{-n} \\ & a^{1/2}, & -a^{1/2}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{n+1}a \end{bmatrix} ; q^{2+n}a^2/bcde = $$

$$= \begin{bmatrix} qa, & qa/de \\ qa/d, & qa/e \end{bmatrix}_n ; q \ {}_4\Phi_3 \begin{bmatrix} q^{-n}, & d, & e, & qa/bc \\ qa/b, & qa/c, & de/q^n a & \end{bmatrix} ; q .$$

When the ${}_4\Phi_3$ in the above reduces to a ${}_3\Phi_2$ series, it can be evaluated by the q-Saalschutz theorem. In this case we obtain a balanced summation formula: *Jackson's q-analogue of Dougall's* formula (see [14] also)

$$
\begin{aligned}
{}_8\Phi_7\left[\begin{matrix} a,\ qa^{1/2},\ -qa^{1/2}, & b, & c, & d, & e, & q^{-n} \\ a^{1/2}, \ -a^{1/2}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{n+1}a \end{matrix} ; q\right] = \\
= \left[\begin{matrix} qa, & qa/bc, & qa/bd, & qa/cd \\ qa/bcd, & qa/b, & qa/c, & qa/d \end{matrix} ; q\right]_n
\end{aligned} \tag{3.3a}
$$

provided that $q^{1+n}a^2 = bcde$.

Also, we can consider a 3-balanced very-well-poised series. In this case the ${}_4\Phi_3$ in (3.2) can be separated through (2.2) and evaluated by means of (2.1). After some trivial modification, we get another summation formula (cf. [6, 21])

$$
\begin{aligned}
{}_8\Phi_7\left[\begin{matrix} a,\ qa^{1/2},\ -qa^{1/2}, & b, & c, & d, & e, & q^{-n} \\ a^{1/2}, \ -a^{1/2}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{n+1}a \end{matrix} ; q^2\right] = \\
= \left(q^n + \frac{(1-a/bcd)(1-q^n a^2/bcd)(1-q^n)}{(1-a/bc)(1-a/bd)(1-a/cd)}\right) \cdot \\
\cdot \left[\begin{matrix} qa, & a/bc, & a/bd, & a/cd \\ a/bcd, & qa/b, & qa/c, & qa/d \end{matrix} ; q\right]_n
\end{aligned} \tag{3.3b}
$$

where $q^n a^2 = bcde$.

Jackson's theorem (3.3a) is of fundamental importance in q-series computation. Here we display some examples.

Jackson's q-analogue of the Dougall–Dixon formula

$$
\begin{aligned}
{}_6\Phi_5\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d, \\ & a^{1/2}, & -a^{1/2}, & qa/b, & qa/c, & qa/d, \end{matrix} ; qa/bcd\right] = \\
= \left[\begin{matrix} qa, & qa/bc & ,qa/bd, & qa/cd \\ qa/bcd, & qa/b, & qa/c, & qa/d \end{matrix} ; q\right]_\infty
\end{aligned} \tag{3.3c}
$$

which follows from (3.3a) or (3.3b) for n tending to infinity. This is the non-terminating version of (3.1b).

Taking d equal to $a^{1/2}$, $-a^{1/2}$ and infinity in (3.3c), we obtain the following q-Dixon formulae:

(3.4a)
$$ {}_4\Phi_3\left[\begin{matrix} a, & -qa^{1/2}, & b, & c \\ & -a^{1/2}, & qa/b, & qa/c \end{matrix}; qa^{1/2}/bc\right] = $$
$$ = \left[\begin{matrix} qa, & qa/bc, & qa^{1/2}/b, & qa^{1/2}/c \\ qa/b, & qa/c, & qa^{1/2}, & qa^{1/2}/bc \end{matrix}; q\right]_\infty $$

(3.4b)
$$ {}_4\Phi_3\left[\begin{matrix} a, & qa^{1/2}, & b, & c \\ & a^{1/2}, & qa/b, & qa/c \end{matrix}; -qa^{1/2}/bc\right] = $$
$$ = \left[\begin{matrix} qa, & qa/bc, & -qa^{1/2}/b, & -qa^{1/2}/c \\ qa/b, & qa/c, & -qa^{1/2}, & -qa^{1/2}/bc \end{matrix}; q\right]_\infty $$

(3.4c)
$$ {}_5\Phi_5\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c \\ 0 & a^{1/2}, & -a^{1/2}, & qa/b, & qa/c \end{matrix}; -qa/bc\right] = $$
$$ = \left[\begin{matrix} qa, & qa/bc \\ qa/b, & qa/c \end{matrix}; q\right]_\infty . $$

For $c = -a^{1/2}$, (3.4a) reduces to the q-analogue of Kummer's formula:

(3.5)
$$ {}_2\Phi_1\left[\begin{matrix} a, & b \\ qa/b & \end{matrix}; -q/b\right] = \left[\begin{matrix} qa, & -qa^{1/2}/b, & qa^{1/2}/b, & -q \\ qa/b, & -qa^{1/2}, & qa^{1/2}, & -q/b \end{matrix}; q\right]_\infty . $$

4. q-analogues of the Watson and Whipple summations

For the specialized Saalschutz summation

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n/2}, & q^{(1-n)/2}, & q^{1/2}c^2/b \\ & q^{3/2-n}/b, & q^{1/2}c^2 \end{matrix}; q\right] = $$
$$ = \left[\begin{matrix} bq^{-1/2} \\ c^2 \end{matrix}; q^{1/2}\right]_n \left[\begin{matrix} c^2 \\ bq^{-1/2} \end{matrix}; q\right]_n . $$

Changing the base from q to q^2 and then embedding in (0.1b) yields the

C-pair:

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ 2k \end{bmatrix} (bq^{-1-2k};q^2)_n \left[\begin{matrix} q, & qc^2/b \\ qc^2, & q^3/b \end{matrix} ;q^2\right]_k q^{\binom{2k+1}{2}} =$$

$$= \left[\begin{matrix} q^{-1}b \\ c^2 \end{matrix} ;q\right]_n (c^2;q^2)_n$$

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-bq^{k-1}}{(bq^{-1-n};q^2)_{k+1}} \left[\begin{matrix} bq^{-1} \\ c^2 \end{matrix} ;q\right]_k (c^2;q^2)_k =$$

$$= \begin{cases} 0, & (n\,\text{odd}) \\ q^{\binom{n+1}{2}} \left[\begin{matrix} q, & qc^2/b \\ qc^2, & q^3/b \end{matrix} ;q^2\right]_m & (n=2m). \end{cases}$$

The last one may be reformulated as the q-analogue of the Watson formula

$${}_4\Phi_3 \left[\begin{matrix} q^{-n},\ b,\ c,\ -c \\ c^2,\ (q^{1-n}b)^{1/2},\ -(q^{1-n}b)^{1/2} \end{matrix} ;q\right] =$$

$$= \begin{cases} 0, & (n\,\text{odd}) \\ \left[\begin{matrix} q, & qc^2/b \\ qc^2, & q/b \end{matrix} ;q^2\right]_m & (n=2m), \end{cases}$$

which may also be expressed in a symmetrical version (cf. Andrews [4])

$${}_4\Phi_3 \left[\begin{matrix} a,\ b,\ c,\ -c \\ c^2,\ (qab)^2,\ -(qab)^{1/2} \end{matrix} ;q\right] =$$

$$= \left[\begin{matrix} qa, & qb, & qc^{-2}, & qabc^{-2} \\ q, & qab, & qac^{-2}, & qbc^{-2} \end{matrix} ;q^2\right]_\infty$$

where a or b is equal to q^{-n}, n being a non-negative integer.

Again, the special Saalschutz formula

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, & q^{1/4}ae, & q^{1/4}e/a \\ & q^{(1-n)/2}e, & q^{(2-n)/2}e \end{matrix} ;q\right] = \left[\begin{matrix} q^{1/4}a, & q^{1/4}/a \\ q^{1/2}e, & e^{-1} \end{matrix} ;q^{1/2}\right]_n $$

could be used to establish the following C-pair:

$$ \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (e^{-1}q^{-k};q^{1/2})_n \frac{(q^{1/4}ae, q^{1/4}e/a;q)_k}{(eq^{1/2};q^{1/2})_{2k}} q^{\binom{k+1}{2}} = $$

$$ = \left[\begin{matrix} q^{1/4}a, \ q^{1/4}/a \\ eq^{1/2} \end{matrix} ;q\right]_n $$

$$ \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-e^{-1}q^{-k/2}}{(e^{-1}q^{-n};q^{1/2})_{k+1}} \left[\begin{matrix} q^{1/4}a, \ q^{1/4}/a \\ eq^{1/2} \end{matrix} ;q^{1/2}\right]_k = $$

$$ = q^{\binom{n+1}{2}} \left[\begin{matrix} q^{1/4}ae, & q^{1/4}e/a \\ eq^{1/2}, & eq \end{matrix} ;q\right]_n . $$

Replacing the base q by q^2, we may restate the last relation as the q-analogue of the Whipple formula (cf. Jain [20]).

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, & -q^{-n}, & q^{1/2}a, & q^{1/2}/a \\ & -q, & e, & q^{1-2n}e^{-1} \end{matrix} ;q\right] = \left[\begin{matrix} q^{1/2}ae, & q^{1/2}e/a \\ e, & eq \end{matrix} ;q^2\right]_n . $$

5. More evaluations

For the Hagen-Rothe identities (cf. [16]), one special case is

$$ \sum_{k=0}^{n} \frac{x}{x+2k}\binom{x+2k}{k}\binom{y-2k}{n-k} = \binom{x+y}{n}, \tag{5.1} $$

whose hypergeometric reformulation has been rediscovered several times (cf. [5]). Here we will discuss its q-analogue and the related C-pairs.

Consider the Saalschutz formula

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, & -q^{-n}, & b/c \\ & -b, & q^{1-2n}/c \end{matrix} ;q\right] = \left[\begin{matrix} c^2 \\ b^2 \end{matrix} ;q^2\right]_n \left[\begin{matrix} b \\ c \end{matrix} ;q\right]_{2n} . $$

After having changed the base from q to $q^{1/2}$ and telescoped into (0.1c), it will generate the C'-pair:

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-c^{-1}q^{-k/2}}{(c^{-1}q^{-n};q^{1/2})_{k+1}} \frac{1-c}{1-cq^{k/2}} \left[\begin{matrix} -q^{1/2},\ b/c \\ -b \end{matrix} ; q^{1/2}\right]_k =$$

$$= q^{\binom{n+1}{2}} \left[\begin{matrix} c^2 \\ b^2 \end{matrix} ; q\right] \left[\begin{matrix} b \\ cq^{1/2} \end{matrix} ; q^{1/2}\right]_{2n}$$

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (c^{-1}q^{-k};q^{1/2})_n \left[\begin{matrix} c^2 \\ b^2 \end{matrix} ; q\right]_k \left[\begin{matrix} b \\ cq^{1/2} \end{matrix} ; q^{1/2}\right]_{2k} q^{\binom{k+1}{2}} =$$

$$= \frac{1-c}{1-cq^{n/2}} \left[\begin{matrix} -q^{1/2},\ b/c \\ -b \end{matrix} ; q^{1/2}\right]_n .$$

The last equality can be translated into (cf. (4.3) in [5] and (4.22) in [15])

(5.2a)
$$ {}_4\Phi_3 \left[\begin{matrix} q^{-n}, & b, & bq^{1/2}, & c^2 \\ & cq^{(1-n)/2}, & cq^{(2-n)/2}, & b^2 \end{matrix} ; q\right] =$$

$$= \frac{1-c}{1-cq^{n/2}} \left[\begin{matrix} -q^{1/2}, & b/c \\ -b, & c^{-1} \end{matrix} ; q^{1/2}\right]_n .$$

Replace b and c by $aq^{1/2}$ and $cq^{n/2}$ respectively, in the above. Then the following C'-pair can be established through telescoping.

(5.2b)
$$ {}_4\Phi_3 \left[\begin{matrix} q^{-n}, & aq^{1/2}, & aq, & c^2q^n \\ & cq^{1/2}, & cq, & a^2q \end{matrix} ; q\right] =$$

$$= (aq^{1/2})^n \frac{1-c}{1-cq^n} \left[\begin{matrix} -q^{1/2}, & c/a \\ -q^{1/2}a, & c \end{matrix} ; q^{1/2}\right]_n ,$$

(5.2c)
$$ {}_6\Phi_5 \left[\begin{matrix} -c,\ q(-c)^{1/2},\ -q(-c)^{1/2},\ c/a,\ q^{-n},\ -q^{-n} \\ (-c)^{1/2},\ -(-c)^{1/2},\ -qa,\ -cq^{n+1},\ cq^{n+1} \end{matrix} ; aq^{2n+1}\right] =$$

$$= \begin{bmatrix} qa, & q^2a, & q^2c^2 & \\ & & & ;q^2 \\ qc, & q^2c, & q^2a^2 & \end{bmatrix}_n .$$

Again, from (2.1) and (2.2) we can compute the balanced summation (5.3a)

$$ {}_4\Phi_3 \begin{bmatrix} q^{-n}, & -q^{-n}, & cq, & a/c & \\ & q^{1-2n}/c, & c, & -aq & ;q \end{bmatrix} = \begin{bmatrix} a, & qa, & q^2c^2 & \\ & & & ;q^2 \\ c, & qc, & q^2a^2 & \end{bmatrix}_n ,$$

which may be used to yield the C-pair after the base change from q to $q^{1/2}$:

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-c^{-1}q^{-k/2}}{(c^{-1}q^{-n};q^{1/2})_{k+1}} \begin{bmatrix} -q^{1/2}, a/c & \\ & ;q^{1/2} \\ -aq^{1/2} & \end{bmatrix}_k =$$

$$= q^{\binom{n+1}{2}} \begin{bmatrix} a, & aq^{1/2}, & c^2q & \\ & & & ;q \\ cq, & cq^{1/2}, & a^2q & \end{bmatrix}_n$$

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (c^{-1}q^{-k};q^{1/2})_n \begin{bmatrix} a, & aq^{1/2}, & c^2q & \\ & & & ;q \\ cq, & cq^{1/2}, & a^2q & \end{bmatrix}_k q^{\binom{k+1}{2}} =$$

$$= \begin{bmatrix} -q^{1/2}, & a/c & \\ & & ;q^{1/2} \\ -aq^{1/2} & & \end{bmatrix}_n .$$

The last relation may be rewritten in the form (cf. [2,23,26])

$$ {}_4\Phi_3 \begin{bmatrix} q^{-n}, & a, & aq^{1/2}, & c^2q & \\ & cq^{(1-n)/2}, & cq^{(2-n)/2}, & a^2q & ;q \end{bmatrix} = \tag{5.3b}$$

$$= \begin{bmatrix} -q^{1/2}, & a/c & \\ & & ;q^{1/2} \\ -aq^{1/2}, & c^{-1} & \end{bmatrix}_n .$$

After following the replacement of c by $cq^{(n-1)/2}$ this summation produces a pair of q-series identities by means of the embedding process.

(5.3c)

$$
{}_4\Phi_3\left[\begin{matrix} q^{-n}, & a, & aq^{1/2}, & c^2q^n \\ & c, & cq^{1/2}, & a^2q \end{matrix} ;q\right] = a^n \left[\begin{matrix} -q^{1/2}, & a/c \\ -q^{1/2}a, & c \end{matrix} ;q^{1/2}\right]_n ,
$$

$$
{}_8\Phi_7\left[\begin{matrix} -c, qc^{1/2}, -qc^{1/2}, q(-c)^{1/2}, -q(-c)^{1/2}, c/a, q^{-n}, -q^{-n} \\ -c^{1/2}, c^{1/2}, (-c)^{1/2}, -(-c)^{1/2}, -qa, -cq^{n+1}, cq^{n+1} \end{matrix} ;aq^{2n}\right] =
$$

(5.3d)
$$
= \left[\begin{matrix} a, & qa, & q^2c^2 \\ c, & qc, & q^2a^2 \end{matrix} ;q^2\right]_n
$$

Among the formulae demonstrated above, (5.2b) and (5.3c) are inverses of each other. And they are the exact q-analogues of (5.1) if the latter is reformulated as a hypergeometric summation. By specifying the parameters in the Watson transformation (3.2) so that the ${}_4\Phi_3$-series involved there may be evaluated through (5.2b) and (5.3c), we can establish the following q-series identities, which have not appeared in the literature explicitly.

(5.4a)

$$
{}_8\Phi_7\left[\begin{matrix} u, qu^{1/2}, -qu^{1/2}, v^{1/2}, q^{1/2}v^{1/2}, u/v, q^{1+n}u^2/v, q^{-n} \\ u^{1/2}, -u^{1/2}, qu/v^{1/2}, q^{1/2}u/v^{1/2}, qv, q^{-n}v/u, q^{1+n}u \end{matrix} ;q^{1/2}v/u\right] =
$$

$$
= \left[\begin{matrix} -q^{1/2}, & q^{1/2}v^{1/2} \\ -q^{1/2}u/v, & q^{1/2}u/v^{1/2} \end{matrix} ;q^{1/2}\right]_n \left[\begin{matrix} qu, & qu^2/v^2 \\ qv, & qu/v \end{matrix} ;q\right]_n .
$$

(5.4b)

$$
{}_8\Phi_7\left[\begin{matrix} u, qu^{1/2}, -qu^{1/2}, v^{1/2}, q^{1/2}v^{1/2}, qu/v, q^{n}u^2/v, q^{-n} \\ u^{1/2}, -u^{1/2}, qu/v^{1/2}, q^{1/2}u/v^{1/2}, v, q^{1-n}v/u, q^{1+n}u \end{matrix} ;q^{1/2}v/u\right] =
$$

$$
= \frac{1-u/v^{1/2}}{1-q^nu/v^{1/2}}q^{-n/2} \left[\begin{matrix} -q^{1/2}, & v^{1/2} \\ -q^{1/2}u/v, & u/v^{1/2} \end{matrix} ;q^{1/2}\right]_n \left[\begin{matrix} qu, & qu^2/v^2 \\ v, & u/v \end{matrix} ;q\right]_n .
$$

(5.4c)

$$ {}_8\Phi_7\left[\begin{matrix} u, qu^{1/2}, -qu^{1/2}, v^{1/2}, q^{1/2}v^{1/2}, qu/v, q^{1+n}u^2/v, q^{-n} \\ u^{1/2}, -u^{1/2}, qu/v^{1/2}, q^{1/2}u/v^{1/2}, v, q^{-n}v/u, q^{1+n}u \end{matrix} ; q^{-1/2}v/u\right] = $$

$$ = q^{-n/2}\left[\begin{matrix} -q^{1/2}, & v^{1/2} \\ -qu/v, & q^{1/2}u/v^{1/2} \end{matrix} ; q^{1/2}\right]_n \left[\begin{matrix} qu, & q^2u^2/v^2 \\ v, & qu/v \end{matrix} ; q\right]_n . $$

(5.4d)

$$ {}_8\Phi_7\left[\begin{matrix} u, qu^{1/2}, -qu^{1/2}, v^{1/2}, q^{1/2}v^{1/2}, u/v, q^{n}u^2/v, q^{-n} \\ u^{1/2}, -u^{1/2}, qu/v^{1/2}, q^{1/2}u/v^{1/2}, qv, q^{1-n}v/u, q^{1+n}u \end{matrix} ; q^{3/2}v/u\right] = $$

$$ = \frac{1-u/v^{1/2}}{1-q^n u/v^{1/2}}\left[\begin{matrix} -q^{1/2}, & q^{1/2}v^{1/2} \\ -u/v, & u/v^{1/2} \end{matrix} ; q^{1/2}\right]_n \left[\begin{matrix} qu, & u^2/v^2 \\ qv, & u/v \end{matrix} ; q\right]_n . $$

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