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Mean values of L-functions and relative class numbers of cyclotomic fields

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Dedicated to Florence F.

Abstract. Using formulas for quadratic mean values of L-functions at $s = 1$, we recover previously known explicit upper bounds on relative class numbers of cyclotomic fields. We also obtain new better bounds.

1. Introduction

Various authors have given elementary proofs of upper bounds on relative class numbers h_f^- of cyclotomic fields $\mathbf{Q}(\zeta_f)$ of conductors $f \not\equiv 2 \pmod 4$. For example, we have

$$h_{2^m}^- \leq 2^m \left(2^{m-1}/32\right)^{2^{m-3}} \qquad (m \geq 2) \tag{1}$$

(see [Met3]) and

$$h_p^- \leq 2p\,(p/32)^{(p-1)/4} \qquad (p \geq 3 \text{ a prime number}) \tag{2}$$

(see [Feng]). H. Walum broached this question by studying mean values of L-functions of prime conductors. In [Lou93] (and [Lou01]) we extended H. Walum's result on mean values of L-functions and obtained new and better bounds on relative class numbers. Here, in Lemmas 3 and 6, we obtain a general result for mean values of L-functions. By using Lemma 2, we recover these bounds on

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relative class numbers and improve upon them (see (16), (17), (18), (19) and (20) below). We show that for p or m large enough, we can replace the constant 32 by any given constant less than $4\pi^2 = 39.47841\ldots$:

Proposition 1. *Fix $C < 4\pi^2 = 39.47841\ldots$. For p effectively large enough we have*

$$h_p^- \leq 2p(p/C)^{(p-1)/4}.$$

For m effectively large enough we have

$$h_{2^m}^- \leq 2^m\sqrt{2}(2^{m-1}/C)^{2^{m-3}}.$$

Using more sophisticated results, a better bound is known (see [MM]):

$$h_p^- \leq p^{31/4}\left(\frac{p}{4\pi^2}\right)^{p/4}.$$

2. The method

Let $K_f = \mathbf{Q}(\zeta_f)$ be a cyclotomic field of prime power conductor $f = p^m > 2$, $p \geq 2$ a prime, and of degree $2n = \phi(f) = p^{m-1}(p-1)$. Let K_f^+ be the maximal real subfield of K_f, of degree n. Let d_f and d_f^+ be the absolute values of the discriminants of K_f and K_f^+. Hence,

$$d_f/d_f^+ = \begin{cases} \sqrt{pd_f} = p^{(1+p^{m-1}(pm-m-1))/2} & \text{if } p \geq 3 \\ \sqrt{4d_f} = 2^{1+2^{m-2}(m-1)} & \text{if } p = 2 \end{cases}$$

(see [Was, Lemma 4.19 and Proposition 2.1]). Let

$$w_f = \begin{cases} 2f = 2p^m & \text{if } p \geq 3 \\ f = 2^m & \text{if } p = 2 \end{cases}$$

be the number of complex roots of unity in K_f. In particular,

$$w_f\sqrt{d_f/d_f^+} = \begin{cases} 2p \cdot p^{\phi(f)/4} & \text{if } f = p \geq 3 \\ \sqrt{2} \cdot 2^m \cdot (2^{m-1})^{\phi(f)/4} & \text{if } f = 2^m \geq 4. \end{cases} \tag{3}$$

Let X_f^- be the set of the $\phi(f)/2$ odd Dirichlet characters mod $f > 2$. Then,

$$h_f^- = w_f\sqrt{d_f/d_f^+}\prod_{\chi \in X_f^-}\frac{1}{2\pi}L(1,\chi)$$

(use [Was, Corollary 4.13 and page 42]). Now, we fix $f_0 \geq 1$, a product of small distinct prime numbers $q \geq 2$. We let χ_0 be the trivial character mod f_0. We assume that f run over integers coprime with f_0, and for $\chi \in X_f^-$, we let $\chi_0\chi$ be the odd character mod $f_0 f$ induced by χ. We have

$$\prod_{\chi \in X_f^-} L(1,\chi) = \Big(\prod_{q \mid f_0} \Pi(q,f)\Big)^{-1} \prod_{\chi \in X_f^-} L(1,\chi_0\chi),$$

where

$$\Pi(q,f) := \prod_{\chi \in X_f^-} \left(1 - \frac{\chi(q)}{q}\right)$$

(throughout the paper, q is a prime divisor of f_0, and p a prime divisor of f). The geometric mean being less than or equal to the arithmetic mean, we obtain:

Lemma 2. *If* $\gcd(f_0, f) = 1$, *then*

$$h_f^- \leq \frac{w_f \sqrt{d_f/d_f^+}}{\prod_{q \mid f_0} \Pi(q,f)} S(f_0,f)^{\phi(f)/4}, \tag{4}$$

where

$$S(f_0,f) := \frac{2}{\phi(f)} \sum_{\chi \in X_f^-} \left| \frac{1}{2\pi} L(1,\chi_0\chi) \right|^2 .$$

To use Lemma 2, we need formulae for the sums $S(f_0,f)$. If F is an n-periodic function, we let $\sum_{a \bmod^* n} F(a)$ denote a summation over any set of representatives of $(\mathbf{Z}/n\mathbf{Z})^*$. Recall from [Lou93] that if χ is an odd Dirichlet character mod $n \geq 3$ (we do not assume that χ is primitive), then

$$\frac{1}{2\pi} L(1,\chi) = \frac{1}{4n} \sum_{a \bmod^* n} \chi(a) \cot\left(\frac{\pi a}{n}\right) \tag{5}$$

and that, for $n \geq 2$, we have

$$\tilde{S}(n) := \sum_{a \bmod^* n} \cot^2\left(\frac{\pi a}{n}\right) = \frac{n^2}{3} \prod_{p \mid n} \left(1 - \frac{1}{p^2}\right) - \phi(n). \tag{6}$$

By (5), we have

$$S(f_0,f) = \frac{1}{16 f_0^2 f^2} \sum_{a \bmod^* f_0 f} \; \sum_{b \bmod^* f_0 f} \frac{2}{\phi(f)} \Bigg(\sum_{\chi \in X_f^-} \chi(a)\overline{\chi(b)}\Bigg) \cot\left(\frac{\pi a}{f_0 f}\right) \cot\left(\frac{\pi b}{f_0 f}\right).$$

Changing b into ab and using $|\chi(a)| = 1$ for $\gcd(a, f) = 1$ and

$$\sum_{\chi \in X_f^-} \overline{\chi(b)} = \begin{cases} \phi(f)/2 & \text{if } b \equiv 1 \pmod{f} \\ -\phi(f)/2 & \text{if } b \equiv -1 \pmod{f} \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$S(f_0, f) = \frac{1}{8f_0^2 f^2} \sum_{a \bmod^* f_0 f} \sum_{\substack{b \bmod^* f_0 f \\ b \equiv 1 \pmod{f}}} \cot\left(\frac{\pi a}{f_0 f}\right) \cot\left(\frac{\pi ab}{f_0 f}\right).$$

Using (6) with $n = f_0 f$, we obtain:

Lemma 3. *If* $\gcd(f_0, f) = 1$, *then*

$$S(f_0, f) = \frac{1}{24}\left\{\prod_{q \mid f_0}\left(1 - \frac{1}{q^2}\right)\right\}\left\{\prod_{p \mid f}\left(1 - \frac{1}{p^2}\right)\right\} - \frac{\phi(f_0)^2 \phi(f)}{8f_0^2 f^2} + \frac{T(f_0, f)}{8f_0^2 f^2},$$

where

$$T(f_0, f) = \sum_{a \bmod^* f_0 f} \sum_{\substack{b \bmod^* f_0 f \\ b \equiv 1 \pmod{f} \\ b \not\equiv 1 \pmod{f_0 f}}} \left(1 + \cot\left(\frac{\pi a}{f_0 f}\right) \cot\left(\frac{\pi ab}{f_0 f}\right)\right).$$

Since $T(f_0, f) = 0$ for $f_0 = 1$ and $f_0 = 2$ (the sum over b is empty), from Lemma 3, we deduce explicit formulae for $S(1, f)$ and $S(2, f)$:

Proposition 4. *We have*

$$S(1, f) = \frac{1}{24} \prod_{p \mid f}\left(1 - \frac{1}{p^2}\right) - \frac{\phi(f)}{8f^2}.$$

In particular,

$$S(1, p) = \frac{1}{24}\left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right) \quad (p \geq 3 \text{ a prime}) \tag{7}$$

and

$$S(1, 2^m) = \frac{1}{32}\left(1 - \frac{1}{2^{m-1}}\right) \quad (m \geq 1). \tag{8}$$

Proposition 5. *We have*

$$S(2,f) = \frac{1}{32}\prod_{p\mid f}\left(1-\frac{1}{p^2}\right) - \frac{\phi(f)}{32f^2}.$$

In particular,

$$S(2,p) = \frac{1}{32}\left(1-\frac{1}{p}\right) \qquad (p \geq 3 \text{ a prime}). \tag{9}$$

Now, assume that $f_0 > 2$. We will not be able to give explicit formulae for $T(f_0,f)$ (see also [Lou99]), but Lemma 6 below will enable us to compute such formulae for any given f_0. Set $\zeta_l = \exp(2\pi i/l)$. Write $a = A + kf_0 \equiv A \pmod{f_0}$ and $b = 1 + Bf \equiv 1 \pmod{f}$. We have

$$1 + \cot\left(\frac{\pi a}{f_0 f}\right)\cot\left(\frac{\pi ab}{f_0 f}\right) = 2i\cot\left(\frac{\pi AB}{f_0}\right)\left(\frac{1}{\zeta_f^k\zeta_{f_0 f}^A - 1} - \frac{1}{\zeta_f^k\zeta_{f_0 f}^{A(1+fB)} - 1}\right)$$

and

$$T(f_0,f) = 2i\sum_{A \bmod^* f_0}\ \sum_{\substack{B=1\\ \gcd(1+Bf,f_0)=1}}^{f_0-1}\cot\left(\frac{\pi AB}{f_0}\right)$$
$$\times\sum_{\substack{k=0\\ \gcd(A+kf_0,f)=1}}^{f-1}\left(\frac{1}{\zeta_f^k\zeta_{f_0 f}^A - 1} - \frac{1}{\zeta_f^k\zeta_{f_0 f}^{A(1+fB)} - 1}\right).$$

Now, if $\lambda^l \neq 1$, then

$$\sum_{k=0}^{l-1}\frac{1}{\zeta_l^k\lambda - 1} = \frac{l}{\lambda^l - 1}$$

(evaluate the logarithmic derivative of $x^l - 1$ at $x = \lambda^{-1}$, if $\lambda \neq 0$). Hence, if $\gcd(f_0,f) = 1$ and $\omega = \zeta_{f_0 f}$ or $\omega = \zeta_{f_0 f}^{1+fB}$, then

$$\sum_{A \bmod^* f_0}\cot\left(\frac{\pi AB}{f_0}\right)\sum_{\substack{k=0\\ \gcd(A+kf_0,f)=1}}^{f-1}\frac{1}{\zeta_f^k\omega^A - 1}$$
$$= \sum_{A \bmod^* f_0}\cot\left(\frac{\pi AB}{f_0}\right)\sum_{d\mid f}\mu(d)\sum_{\substack{k=0\\ d\mid A+kf_0}}^{f-1}\frac{1}{\zeta_f^k\omega^A - 1}$$
$$= \sum_{d\mid f}\mu(d)\sum_{A \bmod^* f_0}\cot\left(\frac{\pi dAB}{f_0}\right)\sum_{\substack{k=0\\ d\mid dA+kf_0}}^{f-1}\frac{1}{\zeta_f^k\omega^{dA} - 1}$$

$$= \sum_{d|f} \mu(d) \sum_{A \bmod^* f_0} \cot\left(\frac{\pi dAB}{f_0}\right) \sum_{k=0}^{f/d-1} \frac{1}{\zeta_{f/d}^k \omega^{dA} - 1}$$
$$= f \sum_{d|f} \frac{\mu(d)}{d} \sum_{A \bmod^* f_0} \cot\left(\frac{\pi dAB}{f_0}\right) \frac{1}{\omega^{fA} - 1}$$

and

$$T(f_0, f) = f \sum_{d|f} \frac{\mu(d)}{d} \sum_{A \bmod^* f_0} \sum_{\substack{0 \neq B \bmod f_0 \\ \gcd(1+Bf, f_0)=1}} \cot\left(\frac{\pi dAB}{f_0}\right) \times \left(\cot\left(\frac{\pi A}{f_0}\right) - \cot\left(\frac{\pi A(1+Bf)}{f_0}\right)\right).$$

If $ff^* \equiv d^*d \equiv A^*A \equiv 1 \pmod{f_0}$, changing B into $f^*(A^*B-1)$ and A into fA we change $(dAB, A, A(1+Bf))$ into $(d(B-A), fA, fB)$ with $B \neq A$. Finally, changing A into d^*A and B into d^*B we obtain:

Lemma 6. *Let $f_0 > 2$ be given. Assume that $\gcd(f_0, f) = 1$. We have*

$$T(f_0, f) = f \sum_{d|f} \frac{\mu(d)}{d} A(f_0, f/d),$$

where the coefficients

$$A(f_0, d) = \sum_{A \bmod^* f_0} \sum_{\substack{B \bmod^* f_0 \\ B \neq A}} \cot\left(\frac{\pi(B-A)}{f_0}\right)\left(\cot\left(\frac{\pi dA}{f_0}\right) - \cot\left(\frac{\pi dB}{f_0}\right)\right)$$

are rational numbers which depend on d mod f_0 only. Moreover,

$$A(f_0, 1) = \phi(f_0)^2 - \frac{f_0^2}{3} \prod_{q|f_0} \left(1 - \frac{1}{q^2}\right).$$

Proof. Using $\cot(y-x)(\cot x - \cot y) = (\cot x)(\cot y) + 1$ we obtain

$$A(f_0, 1) = \sum_{A \bmod^* f_0} \sum_{\substack{B \bmod^* f_0 \\ B \neq A}} \left(1 + \cot\left(\frac{\pi A}{f_0}\right) \cot\left(\frac{\pi B}{f_0}\right)\right).$$

Since $\sum_{B \bmod^* f_0} \cot\left(\frac{\pi B}{f_0}\right) = 0$ (change B into $f_0 - B$), we obtain

$$A(f_0, 1) = \phi(f_0)(\phi(f_0) - 1) - \sum_{A \bmod^* f_0} \cot^2\left(\frac{\pi A}{f_0}\right) = \phi(f_0)(\phi(f_0) - 1) - \tilde{S}(f_0)$$

and the desired result, by (6).
Finally, if a and b are rational integers, then $\cot(\pi a/f_0)\cot(\pi b/f_0)$ is in $\mathbf{Q}(\zeta_{f_0})$, and such that $\sigma_t(\cot(\pi a/f_0)\cot(\pi b/f_0)) = \cot(\pi at/f_0)\cot(\pi bt/f_0)$ whenever $\gcd(t, f_0) = 1$, where σ_t is the automorphism of $\mathbf{Q}(\zeta_{f_0})$ which sends ζ_{f_0} to $\zeta_{f_0}^t$. It follows that the $A(f_0, d)$'s are in $\mathbf{Q}(\zeta_{f_0})$ and are invariant under the actions of the Galois group of $\mathbf{Q}(\zeta_{f_0})/\mathbf{Q}$. Hence, they are rational numbers. □

3. Some explicit formulae for $S(f_0, f)$

Lemma 3 yields explicit formulae for $S(1, f)$ and $S(2, f)$, in which cases $T(1, f) = T(2, f) = 0$. We have not been able to come up with a fully explicit formula for $S(f_0, f)$ for $f_0 > 2$. If $f_0 > 2$ is given, Lemma 6 shows that

$$T(f_0, p) = pA(f_0, p) - A(f_0, 1), \tag{10}$$

where $A(f_0, p)$ depends on p mod f_0 only. (In the same way, $T(f_0, p^m) = p^m A(f_0, p^m) - p^{m-1} A(f_0, p^{m-1})$ depends only on p mod f_0 and of m mod the order of p in $(\mathbf{Z}/f_0\mathbf{Z})^*$). Therefore, for a given f_0 we can compute all the $\phi(f_0)$ possible $A(f_0, p)$ depending only on p mod f_0 and we end up with an explicit formula for $T(f_0, p)$ and $S(f_0, p)$ which will depend on p mod f_0.

For example, for $p > 5$ and $f_0 = 30$ we have $A(30, 1) = -128$ and

p mod 30	1	7	11	13
$A(30, p)$	-128	-112	160	64
$T(30, p)$	$-128(p-1)$	$-16(7p-8)$	$32(5p+4)$	$64(p+2)$
$S(30, p)$	$\frac{2}{75}(1-\frac{1}{p})$	$\frac{2}{75}(1-\frac{11}{12p})$	$\frac{2}{75}(1+\frac{1}{2p})$	$\frac{2}{75}$
f mod 30	17	19	23	29
$A(30, p)$	-64	-160	112	128
$T(30, p)$	$-64(p-2)$	$-32(5p-4)$	$16(7p+8)$	$128(p+1)$
$S(30, p)$	$\frac{2}{75}(1-\frac{2}{3p})$	$\frac{2}{75}(1-\frac{7}{6p})$	$\frac{2}{75}(1+\frac{1}{4p})$	$\frac{2}{75}(1+\frac{1}{3p})$

Table 1.

In fact, if $f = p$ is a prime, we have the following rather nice formula:

Theorem 7. *Assume that $f_0 > 2$ and set*

$$C(f_0) := \frac{1}{24}\prod_{q \mid f_0}\left(1 - \frac{1}{q^2}\right).$$

If $\gcd(f_0, f) = 1$, *set*

$$B(f_0, f) := \frac{A(f_0, f) - \phi(f_0)^2}{8f_0^2},$$

which depends on f *mod* f_0 *only. Then,*

$$S(f_0, p) = C(f_0) + \frac{B(f_0, p)}{p}.$$

In particular, if $p \equiv 1 \pmod{f_0}$, *then*

$$S(f_0, p) = C(f_0) \times \left(1 - \frac{1}{p}\right). \tag{11}$$

and if $p \equiv -1 \pmod{f_0}$, *then*

$$S(f_0, p) = C(f_0) \times \left(1 + \frac{1}{p}\right) - \frac{\phi(f_0)^2}{4f_0^2 p}.$$

Proof. For the first assertion, use Lemma 3, Lemma 6 and (10). For the other assertions, notice that $A(f_0, f) = A(f_0, 1) = A(f_0, 1)$ if $f \equiv 1 \pmod{f_0}$ and $A(f_0, f) = -A(f_0, 1) = -A(f_0, 1)$ if $f \equiv -1 \pmod{f_0}$. $\square$

Proposition 8. *If* 3 *does not divide* f, *then*

$$S(3, f) = \frac{1}{27} \prod_{p|f} \left(1 - \frac{1}{p^2}\right) - \frac{\phi(f)}{18f^2} + \frac{T(3, f)}{72f^2},$$

with

$$T(3, f) = \frac{4f}{3} \left(\frac{f}{3}\right) \prod_{p|f} (1 - \left(\frac{p}{3}\right) \frac{1}{p}).$$

In particular,

$$S(3, p) = \frac{1}{27} \left(1 - \frac{3 - (\frac{p}{3})}{2p}\right) \quad (p \neq 3)$$

and

$$S(3, 2^m) = \frac{1}{36} \left(1 - \frac{1 - (-1)^m}{2^m}\right). \tag{12}$$

If $\gcd(f, 6) = 1$, *then*

$$S(6, f) = \frac{1}{36} \prod_{p|f} \left(1 - \frac{1}{p^2}\right) - \frac{\phi(f)}{72f^2} + \frac{T(6, f)}{288f^2},$$

with

$$T(6,f) = -4f\left(\frac{f}{3}\right)\prod_{p|f}(1-\left(\frac{p}{3}\right)\frac{1}{p}).$$

In particular,

$$S(6,p) = \frac{1}{36}\left(1-\frac{1+(\frac{p}{3})}{2p}\right) \quad (p>3). \tag{13}$$

PROOF. Assume that $f_0 = 3$ or $f_0 = 6$. Then, $f \equiv \pm 1 \pmod{f_0}$ and $\phi(f_0) = 2$. In Lemma 6, A must be equal to $+1$ or -1 mod f_0 and B which can take only one value mod f_0 must be equal to $-A$ mod f_0. Hence, we obtain

$$A(f_0,d) = 4\cot\left(\frac{-2\pi}{f_0}\right)\cot\left(\frac{\pi d}{f_0}\right) = \begin{cases} \frac{4}{3}\left(\frac{d}{3}\right) & \text{if } f_0 = 3 \\ -4\left(\frac{d}{3}\right) & \text{if } f_0 = 6. \end{cases}$$

The desired result follows. □

Lemma 9 (E.g., see [Lou93, Lemme (c)]). *Let l be the order of q mod f. Then,*

$$\Pi(q,f) = \begin{cases} (1+q^{-l/2})^{\phi(f)/l} & \text{if } l \text{is even and } q^{l/2} \equiv -1 \pmod{f} \\ (1-q^{-l})^{\phi(f)/2l} & \text{otherwise,} \end{cases}$$

Moreover, if $f = p^k$ with $p \geq 3$ and l is even, then $q^{l/2} \equiv -1 \pmod{f}$. Finally, $e^{-1/2l} \leq \Pi(q,f) \leq e^{1/l}$, hence $\Pi(q,f) = 1 + O\big(\frac{\log q}{\log f}\big)$.

PROOF. To prove the lower bound on $\Pi(q,f)$, notice that $q^l \geq f+1$ and $\phi(f)\log(1-1/(f+1)) \geq (f-1)\log(1-1/(f+1)) \geq -1$ for $f > 0$. To prove the upper bound, notice that in the first case we have $q^{f/2} \geq f-1$ and $(1+1/(f-1))^{\phi(f)} \leq (1+1/(f-1))^{f-1} \leq \exp(1)$ for $f \geq 2$. □

Lemma 10. *We have:*

m	2	3	≥ 4
$\Pi(3,2^m)$	$1+3^{-1}$	$1-3^{-2^{m-2}}$	$1-3^{-2^{m-2}}$
$\Pi(5,2^m)$	$1-5^{-1}$	$1-5^{-2^{m-2}}$	$1-5^{-2^{m-2}}$

and $\Pi(2,3^m) = 1+2^{3^{m-1}}$ *for* $m \geq 1$.

PROOF. Using $3^{2^{k-3}} \equiv 1+2^{k-1} \pmod{2^k}$ for $k \geq 4$, and $5^{2^{k-3}} \equiv 1+2^{k-1} \pmod{2^k}$ for $k \geq 3$, we obtain that the order l of 3 mod 2^m is equal to 2^{m-2} and $3^{l/2} \not\equiv -1 \pmod{2^m}$ for $m \geq 3$, and that the order l of 5 mod 2^m is equal to 2^{m-2} and $5^{l/2} \not\equiv -1 \pmod{2^m}$ for $m \geq 3$. Using $2^{3^{k-2}} \equiv -1+3^{k-1} \pmod{3^k}$ for $k \geq 3$, we obtain that the order l of 2 mod 3^m is equal to $2\cdot 3^{m-1}$ and $2^{l/2} \equiv -1 \pmod{2^m}$ for $m \geq 1$. □

4. Proof of Proposition 1

Clearly, $A(f_0,f)=O(f_0^4)$, and $T(f_0,f)=O(f_0^4 f\sum_{d|f}\frac{1}{d})=O(f_0^4 f\log f)$. Therefore,

$$S(f_0,f)=\frac{1}{24}\Bigg\{\prod_{q|f_0}\left(1-\frac{1}{q^2}\right)\Bigg\}\Bigg\{\prod_{p|f}\left(1-\frac{1}{p^2}\right)\Bigg\}+O\left(\frac{f_0^2\log f}{f}\right)$$

can be made less than $1/4\pi^2$ by putting enough prime factors in f_0. By Lemma 9, the desired result follows.

5. Upper bounds on relative class numbers

We are now in a position to obtain explicit upper bounds on relative class numbers of cyclotomic fields. To simplify, we restrict ourselves to cyclotomic fields of prime conductors $p\geq 3$ or of 2-power conductors $f=2^m\geq 4$.

5.1. The case $f_0=1$. Using (3), (4) and (7), which yields $S(1,p)\leq 1/24$, we obtain

$$h_p^-\leq 2p\left(\frac{p}{24}\right)^{(p-1)/4}\qquad (p\geq 3\text{ a prime})\tag{14}$$

(see also [Lep], [Met1] and [Met2]). Using (3), (4) and (8), which yields $S(1,2^m)\leq 1/32$, we obtain $h_{2^m}^-\leq 2^m\sqrt{2}(2^{m-1}/32)^{2^{m-3}}$, a bound slightly weaker than (1).

5.2. The case $f_0=2$. Using (3), (4) and (9), and $\Pi(2,p)\geq(1-2^{-l})^{(p-1)/6}\geq(1-1/p)^{(p-1)/4}$, we obtain:

$$h_p^-\leq\frac{2p}{\Pi(2,p)}\left(\frac{p}{32}\left(1-\frac{1}{p}\right)\right)^{(p-1)/4},\tag{15}$$

which implies (2), a better bound than (14) (see also [Feng], and the recent worse bound in [Jak]).

5.3. The cases $f_0=3$. Using (3), (4), (12) and Lemma 10, we obtain

$$h_{2^m}^-\leq\frac{2^m\sqrt{2}}{1-3^{-2^{m-2}}}\left(\frac{2^{m-1}}{36}\right)^{2^{m-3}}\qquad (m\geq 2),\tag{16}$$

which is a better bound than all the previously known ones quoted in [Met3].

5.4. The cases $f_0 = 6$. Using (3), (4) and (13), we obtain the following improvement on (2):

$$h_p^- \leq \frac{2p}{\Pi(2,p)\Pi(3,p)} \left(\frac{p}{36}\right)^{(p-1)/4} \quad (p \geq 5 \text{ a prime}). \tag{17}$$

5.5. The cases $f_0 = 15$.

Proposition 11. *We have*

$$T(15, 2^m) = 2^{m+3} \times \begin{cases} 7 & \text{if } m \equiv 0 \pmod 4, \\ -8 & \text{if } m \equiv 1 \pmod 4, \\ -4 & \text{if } m \equiv 2 \pmod 4, \\ -10 & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Hence,

$$S(15, 2^m) = \frac{2}{75}\left(1 - \frac{2}{3 \cdot 2^m} + \frac{T(15, 2^m)}{48 \cdot 2^{2m}}\right) \leq \frac{2}{75}\left(1 + \frac{1}{2^{m+1}}\right).$$

Using (4), and Lemma 10, we obtain a better bound than (16):

$$h_{2^m}^- \leq \frac{2^m\sqrt{2}}{(1 - 3^{-2^{m-2}})(1 - 5^{-2^{m-2}})} \left(\frac{2^{m+1}+1}{150}\right)^{2^{m-3}} \quad (m \geq 2). \tag{18}$$

5.6. The case $f_0 = 30$. According to Table 1, $S(30, p) \leq \frac{2}{75}\big(1 + \frac{1}{2p}\big)$ and we obtain a better bound than (17):

$$h_p^- \leq \frac{2p}{\Pi(2,p)\Pi(3,p)\Pi(5,p)} \left(\frac{2p+1}{75}\right)^{(p-1)/4} \quad (p \geq 7 \text{ a prime}). \tag{19}$$

5.7. The case $p \equiv 1 \pmod{f_0}$. Using (3), (4) and (11), we obtain

$$h_p^- \leq \frac{2p}{\prod_{q \mid f_0} \Pi(q,p)} \left(\frac{p}{24}\left(\prod_{q \mid f_0}\left(1 - \frac{1}{q^2}\right)\right)\left(1 - \frac{1}{p}\right)\right)^{(p-1)/4}.$$

By Lemma 9, we deduce that if $p \geq p_0(f_0)$ is large enough, then

$$h_p^- \leq 2p\left(\frac{p}{24}\prod_{q \mid f_0}\left(1 - \frac{1}{q^2}\right)\right)^{(p-1)/4} \tag{20}$$

(more explicitly, by Lemma 9 we have $\Pi(q,p) \geq \exp\big(-\frac{\log q}{2\log p}\big)$, which yields $\prod_{q \mid f_0} \Pi(q,p) \geq \exp\big(-\frac{\log f_0}{2\log p}\big)$, and using $(1 - 1/p)^{(p-1)/4} \leq \exp(-\frac{1}{8})$ for $p \geq 3$ we see that is suffices to have $p \geq p_0(f_0) := f_0^4$).

References

[Feng] K. Feng, On the first factor of the class number of a cyclotomic field, *Proc. Amer. Math. Soc.* **84** (1982), 479–482.

[Jak] S. Jakubec, On some new estimates for $h^-(\mathbf{Q}(\zeta_p))$, *Acta Arith.* **137** (2009), 43–50.

[Lep] T. Lepistö, On the growth of the first factor of the class number of the prime cyclotomic field, *Ann. Acad. Sci. Fenn.* Ser. AI, N **577** (1974), 21.

[Lou93] S. Louboutin, Quelques formules exactes pour des moyennes de fonctions L de Dirichlet, *Canad. Math. Bull.* **36** (1993), 190–196, Addendum: *Canad. Math. Bull.* **37** (1994), p. 89.

[Lou99] S. Louboutin, On the mean value of $|L(1,\chi)|^2$ for odd primitive Dirichlet characters, *Proc. Japan Acad. Ser. A Math. Sci.* **75** (1999), 143–145.

[Lou01] S. Louboutin, The mean value of $|L(k,\chi)|^2$ at positive rational integers $k \geq 1$, *Colloq. Math.* **90** (2001), 69–76.

[Met1] T. Metsänkylä, On the growth of the first factor of the cyclotomic class number, *Ann. Univ. Turku.* Ser. A, N **155** (1972).

[Met2] T. Metsänkylä, Class numbers and μ-invariants of cyclotomic fields, *Proc. Amer. Math. Soc.* **43** (1974), 299–300.

[Met3] T. Metsänkylä, Letter to the editor, *J. Number Theory* **64** (1997), 162–163.

[MM] J. M. Masley and H. L. Montgomery, Cyclotomic fields with unique factorization, *J. Reine Angew. Math.* **286/287** (1976), 248–256.

[W] H. Walum, An exact formula for an average of L-series, *Illinois J. Math.* **26** (1982), 1–3.

[Was] L. C. Washington, Introduction to cyclotomic fields (Second Edition), Graduate Texts in Mathematics, **83**, *Springer-Verlag, New York*, 1997.

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