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Functional equation of Dhombres type in the real case

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Abstract. We consider continuous solutions $f : \mathbb{R}_+ \to \mathbb{R}_+ = (0, \infty)$ of the functional equation $f(xf(x)) = \varphi(f(x))$ where φ is a given continuous map $\mathbb{R}_+ \to \mathbb{R}_+$. If φ is an increasing homeomorphism the solutions are completely described, if not there are only partial results. In this paper we bring some necessary conditions upon a possible range R_f. In particular, if $\varphi|R_f$ has no periodic points except for fixed points then there are at most two fixed points in R_f, and all possible types of R_f and all possible types of behavior of f can be described. The paper contains techniques which essentially simplify the description of the class of all solutions.

1. Introduction

Let φ be a continuous map of the unit interval $[0, 1]$. By $\mathcal{S}(\varphi)$ we denote the set of all continuous solutions of the generalized Dhombres functional equation

$$f(xf(x)) = \varphi(f(x)), \quad x \in (0, \infty). \tag{1}$$

This equation, with $\varphi(y) = y^2$, was studied in 1975 by J. Dhombres in [3]. Equation (1) belongs to the class of *functional equations of invariant curves*, see [7] for more information. In general, it is difficult to give a solution. In 1991,

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Jarczyk [4] studied the additive version of (1) and obtained some conditions sufficient for existence of continuous solutions. A survey of recent results concerning iterative equations, including equations of the above type, can be found in [1].

If φ is an increasing homeomorphism the class $\mathcal{S}(\varphi)$ is well described. In particular, the range R_f of a non-constant solution contains no fixed points of φ, with the exception for 1: it is easy to see that if $1 \in R_f$ then it is a fixed point of φ. A characterization of monotone solutions, and of continuous solutions can be found in [5], [6] and [9]. For decreasing homeomorphisms only partial results are known (see [10]). In [8] there was for the first time considered the general case of an arbitrary continuous function φ; this paper presents some technical results which make possible a classification of the solutions. In the present paper, we consider only *regular solutions* f, i.e., solutions for which there are no points $u < v$ such that $f(u) > 1 > f(v)$. Actually, we consider special case, $R_f \subseteq (0, 1]$. But the other cases of regular solutions can be reduced to this one type, see [8]. Obviously, R_f is an interval and, by [8], if $R_f \subseteq (0, 1]$ then $R_f = \varphi(R_f)$ is an φ-invariant set.

In the next section we will give a characterization of behavior of solutions for φ without periodic points except for the fixed points, and for R_f containing a fixed point as a boundary point. Then, in Section 3, we will develop techniques for construction of more complex solutions. Using this, we will be able to give our main result, a characterization of all possible types of R_f provided all periodic points of φ are fixed points, see Theorem 3.8. This solves one of the problems from [8]. Moreover, our results will give credibility to the conjecture that there are no solutions such that R_f contains periodic points of φ of period > 2, cf. [8].

Let $f \in \mathcal{S}(\varphi)$. Then τ_f, or simply τ, defined by $\tau_f(x) = xf(x)$ is a continuous map from $(0, \infty)$ to itself. The *forward orbit* of $x_0 \in (0, \infty)$ is the sequence $\{x_n\}_{n \geq 0}$ defined by $x_n = \tau_f^n(x_0)$, $n \in \mathbb{N}$, where τ_f^n denotes the nth iterate of τ_f and $\mathbb{N}$ the set of non-negative integers. Thus,

$$\tau_f^n(x) = xf(x)\varphi(f(x))\varphi^2(f(x)) \ldots \varphi^{n-1}(f(x)),\ n \in \mathbb{N}. \tag{2}$$

Here φ^n denotes the nth iterate of φ. If φ is a homeomorphism then there is a unique *orbit* $\{x_n\}_{n \in \mathbb{Z}}$ of x_0 defined by $x_n = \tau_f^n(x_0)$, $n \in \mathbb{Z}$, and if R_f is contained in a compact subinterval of $(0, 1)$ then, for every orbit, $\lim_{n \to \infty} x_n = 0$ and $\lim_{n \to -\infty} x_n = \infty$.

However, we will be working with continuous not necessarily homeomorphic maps φ. For such maps, every point x_0 has the orbit in the sense of Whyburn and Kuratowski, consisting of all x such that there are $m, n \in \mathbb{N}$ with $\tau_f^m(x_0) = \tau_f^n(x)$.

But this notion is too general to be useful for our purposes. Therefore we define a subset of Whyburn–Kuratowski orbit, a *generalized orbit* $\{x_n\}_{n\in\mathbb{Z}}$ of x_0, briefly $GO_f(x_0)$, as follows: For $n \in \mathbb{N}$, we let $x_n = \tau_f^n(x_0)$ as before. For $u \in (0,\infty)$, let Γ_u be the map $(u,\infty) \to (0,1)$ given by $\Gamma_u(v) = u/v$. If the graphs of $f(\cdot)$ and $\Gamma_{x_0}(\cdot)$ have a point in common, denote its first coordinate by x_{-1}. By induction, if for some $n \in \mathbb{Z}\setminus\mathbb{N}$ the graphs of $f(\cdot)$ and $\Gamma_{x_n}(\cdot)$ intersect let us denote by x_{n-1} a point such that $f(x_{n-1}) = \Gamma_{x_n}(x_{n-1})$. Obviously, in general $GO_f(x_0)$ is not uniquely determined. It may even not exist. But it has some useful properties which are summarized in the following Lemma. Its proof is obvious.

Lemma 1.1. *Let $f \in \mathcal{S}(\varphi)$ and let $\{x_n\}_{n\in\mathbb{Z}} = GO_f(x_0)$. Then $\tau_f(x_n) = x_{n+1}$ and $\varphi(f(x_n)) = f(x_{n+1})$, $n \in \mathbb{Z}$. Moreover,*

$$\text{for } J_n := [x_{n+1}, x_n],\ J_{n+1} \subseteq \tau_f(J_n) \quad \text{and} \quad R_{f|J_{n+1}} \subseteq \varphi(R_{f|J_n}), \qquad n \in \mathbb{Z}. \quad (3)$$

If the range R_f of f is contained in a compact interval $[p,q] \subset (0,1)$ then $GO_f(x_0)$ exists for any x_0, $\lim_{n\to\infty} x_n = 0$, and $\lim_{n\to-\infty} x_n = \infty$.

In Section 2 we need some results from [8]. Let us recall that a fixed point u of φ is *attracting from the right* if there is a $\delta > 0$ such that, for any $y \in [u, u+\delta)$, $\lim_{n\to\infty} \varphi^n(y) = u$; it is *repulsing from the right* if $\varphi(y) > y$ for every y in a right neighborhood of u. Similarly for the "left". A set A is *φ-invariant* if $\varphi(A) = A$.

Lemma 1.2 (See Corollary 3.3 in [8])**.** *Let $f \in \mathcal{S}(\varphi)$, and let $L = [u,v] \subseteq R_f$ be a non-degenerate φ-invariant interval end-points of which are fixed points of φ. Then for every $y \in R_f$, $y \neq 1$, there is an $n \in \mathbb{N}$ such that $\varphi^n(y) \in L$.*

Lemma 1.3 (See Proposition 3.5 in [8])**.** *Let $f \in \mathcal{S}(\varphi)$ with $R_f \subseteq (0,1)$, and let R_f contain a compact φ-invariant set $F \neq \emptyset$. Assume that $\overline{R_f} \setminus F$ contains a fixed point a of φ which is attracting from the right if $\max F > a$, and is attracting from the left if $\min F < a$. Then $\min F < a < \max F$, and there are points $u, v \in J$ such that $u < a < v$, $\varphi(u) = v$, $\varphi(v) = u$, and $\lim_{n\to\infty} \varphi^n(y) = a$, for every $y \in (u,v)$.*

Other terminology and related results concerning the equation are given later in this text, or can be found in [8], [9], or [10]. For some well-known results concerning dynamical systems we refer to [2].

2. Solutions with φ possessing no periodic points of period > 1

The following result must be known but we are not able to give a reference.

Lemma 2.1. *Let* $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi)$*,* $p = \min \mathrm{Fix}(\varphi)$*, and* $q = \max \mathrm{Fix}(\varphi)$*. If some* $u \in \mathrm{Fix}(\varphi)$ *is repulsing from the right then* $[u, q]$ *is* φ*-invariant, if* u *is repulsing from the left then* $[p, u]$ *is* φ*-invariant.*

Proof. By the symmetry, we may assume that $\varphi(y) > y$ in $(u, u+\delta]$ and $[u, q]$ is not φ-invariant. Let $L = \bigcup_{n\geq 0} \varphi^n([u, u+\delta])$. Then L must contain a $z < u$ and hence, there are $w \in (u, u+\delta]$ and $k > 0$ such that $\varphi^k(w) = z$. Since $\varphi^k(w) < w$ and $\varphi^k(y) > y$ for every $y > u$ which is sufficiently close to u, there is a $v \in (u, w)$ such that $\varphi^k(v) = v$. By the hypothesis, $v \in \mathrm{Fix}(\varphi)$. It follows that $\varphi^k([v, w]) \supset [z, v] \supset [u, v]$. Since $\varphi(y) > y$ on $(u, u+\delta]$, there is an $s > 0$ such that $\varphi^s([u, v]) \supseteq [u, w]$ and hence $\varphi^{s+j}([u, v]) \supseteq [u, w]$, for every $j \geq 0$. Consequently, $\varphi^{s+k}([u, v]) \cap \varphi^{s+k}([v, w]) \supseteq [u, w]$. In other words, $\psi := \varphi^{k+s}$ is *turbulent* and hence, it has a periodic point of period 3 (see [2], Lemma II.3), which is a contradiction. $\square$

Theorem 2.2. *Let* $f \in \mathcal{S}(\varphi)$ *with* $R_f \subseteq (0, 1]$*. If* $\mathrm{Per}(\varphi) \cap R_f = \mathrm{Fix}(\varphi) \cap R_f$ *then* R_f *contains at most one fixed point of* φ *different from* 1*.*

Proof. Assume that $R_f \cap (0, 1)$ contains more than one fixed point of φ. By Lemma 1.2, $\mathrm{Fix}(\varphi) \cap R_f$ is a nowhere dense set. Otherwise there would be a compact interval $L \subset \mathrm{Fix}(\varphi) \cap R_f$ and a point $y \in \mathrm{Fix}(\varphi) \cap R_f \setminus L$ such that $f^n(y) \in L$, for some $n \geq 0$, which is impossible.

If $R_f \cap (0, 1)$ contains exactly two fixed points $u < v$ such that $\varphi(y) > y$ for $y \in (u, v)$ then v cannot be attracting since otherwise, by Lemma 1.3 with $F := \{u\}$ and $a := v$, u would be a periodic point of φ of period 2. Hence, $v < \sup R_f$ and $\varphi(y) > y$ for $y > v$. Since R_f is φ-invariant, $\sup R_f$ is a fixed point of φ attracting from the left, contrary to Lemma 1.3. If $\varphi(y) < y$ in (u, v) the argument is similar.

If $R_f \cap (0, 1)$ contains exactly three fixed points $p < q < r$ of φ then, by Lemma 1.3, q cannot be attracting. Hence, it is repulsing from the left or from the right. In the first case, by Lemma 2.1, $[q, r]$ is φ-invariant and, by Lemma 1.2, $\varphi^n(p) \in [q, r]$, for some n, which is impossible. Similarly in the second case.

Finally, if there are more than three fixed points of φ in $R_f \cap (0, 1)$ then, since $\mathrm{Fix}(\varphi)$ is nowhere dense, there are fixed points $p < q < r < s$ in $R_f \cap (0, 1)$ such that $\varphi(y) \neq y$, for $y \in (q, r)$. If $\varphi(y) > y$ in (q, r) then, by Lemma 2.1, $[p, q]$

is φ-invariant and, by Lemma 1.2, r is eventually mapped to it. Similarly in the other case. $\square$

Lemma 2.3. *Let p, q be fixed points of φ, $0 \le p < q < 1$, and let $f \in \mathcal{S}(\varphi)$. Assume that one of the following conditions is satisfied:*

(i) $(p,q] \subseteq R_{f|(0,\alpha)}$ *for some* $\alpha > 0$, *or* $(p,q] \subseteq R_{f|(\alpha,\infty)}$ *for every* $\alpha > 0$.

(ii) $[p,q) \subseteq R_{f|(0,\alpha)}$ *for some* $\alpha > 0$, *or* $[p,q) \subseteq R_{f|(\alpha,\infty)}$ *for every* $\alpha > 0$.

Then for every $\delta > 0$ there are points $b_1 < a_2 < a_1 < b_0 < a_0$, such that $a_i = \tau_f^i(a_0)$, $b_1 = \tau_f(b_0)$,

$$f(a_i) = q \text{ and } |f(b_j) - p| < \delta, \quad i = 0,1,2,\ j = 0,1, \text{ in the case (i)}, \tag{4}$$

and

$$|f(a_i) - q| < \delta \text{ and } f(b_j) = p, \quad i = 0,1,2,\ j = 0,1, \text{ in the case (ii)}. \tag{5}$$

Proof. Fix a $k \in \mathbb{N}$ such that $q^{k+2} > p^k$, and let $\delta > 0$ satisfy $q^{k+2} > (p+\delta)^k$. Let $\eta > 0$ be such that $|x-y| \le \eta$ implies $|\varphi^j(x) - \varphi^j(y)| < \delta$, $0 \le j \le k$.

(i) There are $v < u$ such that $f(u) = q$ and $f(v) < p + \eta$. Indeed, if $R_{f|(0,\alpha)} \supseteq (p,q]$ for an $\alpha > 0$, let $u < \alpha$ such that $f(u) = q$. If $R_{f|(0,u]} \supseteq (p,q]$ we are done. Otherwise, by the hypothesis, $R_{f|[u,\alpha]} \supseteq (p,q]$ hence, by the continuity, $R_{f|[u,\alpha]} \supseteq [p,q]$. Then there is a $w > u$ such that $f(w) = p$ and, by induction, $f(wp^n) = p$ for every $n \ge 0$. Indeed, by (1), $f(wp^n) = p$ yields $f(wp^{n+1}) = f(wp^n f(wp^n)) = \varphi(f(wp^n)) = \varphi(p) = p$. Since $\lim_{n\to\infty} wp^n = 0$ we can take $v = wp^n$ for a suitable $n \in \mathbb{N}$. If $R_{f|(\alpha,\infty)} \supset (p,q]$ for *every* α the existence of u, v is obvious.

It follows that there are $u_0 > v_0$ such that $f(u_0) = q$, $f(v_0) < p + \eta$, and $f(x) < q$ for $x \in [v_0, u_0)$. Let $\{u_n\}_{n\ge 0}$ and $\{v_n\}_{n\ge 0}$ be the forward orbits of u_0 and v_0, respectively. By the choice of k and η, there is an m, $0 \le m \le k$, such that $v_{m+1} < u_{m+2} < u_{m+1} < v_m < u_m$, $f(u_m) = f(u_{m+1}) = f(u_{m+2}) = q$ and $f(v_m), f(v_{m+1}) < p + \delta$. Put $a_0 = u_m$ and $b_0 = v_m$.

(ii) Similarly to the first part there are points $v < u$ such that $f(v) = p$ and $f(u) > q - \eta$. The remainder of the proof is similar. $\square$

Lemma 2.4. *Let p, q be fixed points of φ, $0 \le p < q < 1$, and let $p \le \varphi(y) < y$, for $p < y < q$. If $f \in \mathcal{S}(\varphi)$ such that $R_f \subseteq (0,q]$ and $q \in R_f$ then $f \equiv q$.*

Proof. By Lemma 1.3 with $F = \{q\}$ and $a = p$, $p \notin \overline{R_f}$. Hence $R_f \subseteq (p,q]$. Let $f(x_0) = q$ for some $x_0 > 0$ and let $\{x_n\}_{n\in\mathbb{N}} = \{x_0 q^n\}_{n\in\mathbb{N}}$ be the forward orbit of x_0. Put $J_n := [x_{n+1}, x_n]$ and $z_n := \min f|_{J_n}$. By (3), $(0, x_0] = \bigcup_{n\geq 0} \tau_f^n(J_0)$ since $\tau_f(x)$ is decreasing. Hence, $f((0, x_0]) = \bigcup_{n\geq 0} \varphi^n(f(J_0))$ and, by (3), $\min \varphi^n(f(J_0)) \leq \varphi^n(z_0)$. Then $\inf f((0, x_0]) = \inf_{n\geq 0} \min \varphi^n(f(J_0)) \leq \inf_{n\geq 0} \varphi^n(z_0) = p$ if $z_0 < q$ which is impossible since $p \notin \overline{R_f}$. Thus, $z_0 = q$ and $f|_{(0,x_0]} \equiv q$. □

Theorem 2.5. *Let $p, q \in \mathrm{Fix}(\varphi)$, $0 \leq p < q < 1$ such that $y \neq \varphi(y)$, for $p < y < q$. If $f \in \mathcal{S}(\varphi)$, $R_f \subseteq (0, q]$, $q \in R_f$, and $R_f \neq \{q\}$, then*

(i) $y < \varphi(y)$ *for* $y \in (p, q)$; (ii) $\lim_{t\to 0} f(t) = q$;

(iii) $R_f = (p, q]$; (iv) $\lim_{t\to\infty} f(t) = p$.

Proof. Lemma 2.4 implies (i), and Lemma 1.3 applied to $F = \{p\}$ and $a = q$ implies $p \notin R_f$. Thus, $R_f \subseteq (p, q]$. To prove (ii), let $f(x_0) = q$, for some $x_0 > 0$, and let $\{x_n\}_{n\geq 0}$ be the forward orbit of x_0. Then, by (1), we have $f(x_n) = q$ for every $n \in \mathbb{N}$. Put $J_n := [x_{n+1}, x_n]$ and $z_n := \min f|_{J_n}$, $n \in \mathbb{N}$; then $z_n \in (p, q]$. By (3), $[z_n, q] = R_{f|J_n} \subseteq \varphi^n(R_{f|J_0}) = \varphi^n([z_0, q])$. Since $f((0, x_k]) = \bigcup_{n\geq k} f(J_n) = \bigcup_{n\geq k} [z_n, q]$,

$$\begin{aligned}\inf f((0, x_k]) &= \inf_{n\geq k} z_n = \inf_{n\geq k} \min([z_n, q]) \geq \inf_{n\geq 0} \min \varphi^{n+k}([z_0, q]) \\ &= \min \varphi^k([z_0, q]) \to \{q\}, \quad \text{for } k \to \infty,\end{aligned}$$

since $\varphi(y) > y$ in (p, q), and (ii) follows.

Next we show that

$$\inf_{x>0} f(x) = p. \tag{6}$$

Indeed, assume $\inf_{x>0} f(x) = s > p$. By Lemma 1.1, there is $GO(x_0) = \{x_n\}_{n\in\mathbb{Z}}$. Put $J_n = [x_{n+1}, x_n]$ and $z_n = \min f|_{J_n}$. There is a $\delta > 0$ such that $\varphi([s, q]) = [s + \delta, q]$. Since $s = \inf_{n\in\mathbb{Z}} z_n$, there is an m such that $z_{m+1} < s + \delta$. On the other hand, $z_m \geq s$ implies

$$z_{m+1} \in R_{f|J_{m+1}} \subseteq \varphi(R_{f|J_m}) = \varphi([z_m, q]) \subseteq \varphi([s, q]) = [s + \delta, q].$$

This contradiction proves (6) and hence, (iii).

To prove (iv) we first show that

$$\text{the set } E(q) := \{x > 0; f(x) = q\} \text{ is upper bounded in } (0, \infty). \tag{7}$$

Assume the contrary. By the hypothesis, $E(q) \neq \emptyset$ hence, by (ii) and (6), $(p, q] \subseteq R_{f|(\alpha,\infty)}$, for every $\alpha > 0$. Let $0 < \delta < q - p$. By Lemma 2.3(i), there are points

$b_1 < a_2 < a_1 < b_0 < a_0$ satisfying (4). Since $\tau_f([b_0, a_0]) \supseteq [b_1, a_1]$ there is a $u \in (b_0, a_0)$ with $\tau_f(u) = a_2$. Then, by (1), $\varphi(v) = q$, for $v = f(u)$. So, $v \neq p$, and $v = q$ would imply $\tau_f(u) > a_2$. Thus, $p < v < q$ and there is the minimal $h \in (p, q)$ with $\varphi(h) = q$. Without loss of generality we may assume $h > p + \delta$. Let $u_0 \in [b_0, a_0]$ be the minimal, and $u_1 \in [a_1, b_0]$ the maximal point such that $f(u_0) = f(u_1) = h$. Obviously, $a_1 < u_1 < b_0 < u_0 < a_0$, and there is the minimal $u_2 \in [a_1, u_1]$ such that $f(u_2) = h$. Then $a_1 < u_2 \leq u_1$,

$$R_{f|[a_1,u_2]} = [h, q], \quad \text{and} \quad R_{f|(u_1,u_0)} \subset (p, h). \tag{8}$$

Since h is minimal, $\varphi(f([b_0, u_0)) \subset (p, q)$, and since u_0 is minimal and $\tau_f([b_0, a_0]) \supseteq [b_1, a_1]$, we have $\tau_f(u_0) \leq a_2$. Finally, by the first condition in (8), $\tau_f([a_1, u_2]) \supseteq (b_1, a_2]$. Summarizing, we have $\tau_f(b_0) = b_1 < \tau_f(u_2) \leq \tau_f(u_1) < \tau_f(u_0) \leq a_2$ and hence, there is a point $r \in (b_0, u_0)$ with $\tau_f(r) = \tau_f(u_1)$. Since h is minimal, the second condition in (8), and (1) imply $q > \varphi(f(r)) = f(\tau_f(r)) = f(\tau_f(u_1)) = q$, a contradiction. This proves (7).

To finish the proof of (iv) assume on the contrary that $\limsup_{x\to\infty} f(x) = r > p$. Then there are points $v_1 < v_2 < \cdots \to \infty$ such that $\lim_{n\to\infty} f(v_n) = r$. Since $\varphi(y) > y$ in (p, q), by (1) we have $\lim_{n\to\infty} f(\tau_f(v_n)) = \lim_{n\to\infty} \varphi(f(v_n)) = \varphi(r) > r$ which is a contradiction unless $r = q$. Thus, $\limsup_{x\to\infty} f(x) = q$. Let w_0 be such that $\tau_f(w_0) > \max E(q) =: x_0$ (see (7)), and let $\{w_n\}_{n\geq 0}$ be its forward orbit. Since $w_n \to 0$, there is a $k \geq 0$ such that $x_0 \in [w_{k+1}, w_k]$. Hence, there is a $c \in [w_k, w_{k-1}]$ such that $\tau_f(c) = x_0$. Put $f(c) = d$. Then, by (7), $d < q$ and, by (1), $\varphi(d) = q$. Consequently, $\varphi(y) = q$ for $d \leq y \leq q$ and $\limsup_{x\to\infty} f(x) = q$ yields $f(x) = q$, for every $x \geq x_0$, contrary to (7). □

Lemma 2.6. *Let p, q be fixed points of φ, $0 < p < q < 1$, and let $y < \varphi(y) \leq q$, for $p < y < q$. If $f \in \mathcal{S}(\varphi)$ such that $R_f \subseteq [p, 1]$ and $p \in R_f$, then $f \equiv p$.*

Proof. By Lemma 1.3 applied to $F = \{p\}$ and $a = q$, $q \notin \overline{R_f}$. Hence, $R_f \subseteq [p, q)$. Let $f(x_0) = p$, and let $\{x_n\}_{n\in\mathbb{N}}$ be the forward orbit of x_0. Put $J_n = [x_{n+1}, x_n]$, and $z_n = \max f|_{J_n}$. Similarly as in the proof of Lemma 2.4, $f((0, x_0]) = \bigcup_{n\geq 0} f^n(J_0)$ whence, $\sup f((0, x_0]) = \sup_{n\geq 0} \max f^n(J_0) \geq \sup_{n\geq 0} \varphi^n(z_0) = q$ if $z_0 < q$ which is impossible since $q \notin \overline{R_f}$. Thus, $z_0 = p$ and $f|_{(0,x_0]} \equiv p$. □

Theorem 2.7. *Let p, q be fixed points of φ, $0 < p < q < 1$ such that $y \neq \varphi(y)$, for $p < y < q$. If $f \in \mathcal{S}(\varphi)$, $R_f \subseteq [p, 1]$, $p \in R_f$, and $R_f \neq \{p\}$, then*

(i) $y > \varphi(y)$ *for* $y \in (p, q)$; (ii) $\lim_{t\to 0} f(t) = p$;

(iii) $R_f = [p, q)$; (iv) $\lim_{t\to\infty} f(t) = q$.

Proof. Lemma 2.6 implies (i), and Lemma 1.3 applied to $F = \{q\}$ and $a = p$ implies $q \notin R_f$. To prove (ii), let $f(x_0) = p$, for some $x_0 > 0$, and let $\{x_n\}_{n\geq 0}$ be the forward orbit of x_0. By (1), $f(x_n) = p$, for every n. Put $J_n := [x_{n+1}, x_n]$ and $z_n := \max f|_{J_n}$, $n \in \mathbb{N}$; then $z_n \in [p, q)$. By (3), $[p, z_n] = R_{f|J_n} \subseteq \varphi^n(R_{f|J_0}) = \varphi^n([p, z_0])$. Since $f((0, x_k]) = \bigcup_{n\geq k} f(J_n) = \bigcup_{n\geq k}[p, z_n]$,

$$\begin{aligned}\sup f((0, x_k]) &= \sup_{n\geq k} z_n = \sup_{n\geq k} \max([p, z_k]) \leq \sup_{n\geq 0} \max \varphi^{n+k}([p, z_0]) \\ &= \max \varphi^k([p, z_0]) \to \{p\}, \quad \text{for } k \to \infty,\end{aligned}$$

since $\varphi(y) < y$ in (p, q), and (ii) follows.

Next we show that

$$\sup_{x>0} f(x) = q. \tag{9}$$

Indeed, assume $\sup_{x>0} f(x) = s < q$. By Lemma 1.1, there is $GO(x_0) = \{x_n\}_{n\in\mathbb{Z}}$. Put $J_n = [x_{n+1}, x_n]$ and $z_n = \max f|_{J_n}$. There is a $\delta > 0$ such that $\varphi([p, s]) = [p, s - \delta]$. Since $s = \sup_{n\in\mathbb{Z}} z_n$, there is an m such that $z_{m+1} > s - \delta$. On the other hand, since $z_m \leq s$, (3) implies

$$z_{m+1} \in R_{f|J_{m+1}} \subseteq \varphi(R_{f|J_m}) = \varphi([p, z_m]) \subseteq \varphi([p, s]) = [p, s - \delta].$$

This contradiction proves (9) and hence, (iii).

To prove (iv) we first show that

$$\text{the set } E(p) := \{x > 0; f(x) = p\} \text{ is upper bounded in } (0, \infty). \tag{10}$$

Assume the contrary. By the hypothesis, $E(p) \neq \emptyset$ hence, by (i), (ii) and (9), $[p, q) \subseteq R_{f|(\alpha,\infty)} = [p, q)$, for every $\alpha > 0$. Let $0 < \delta < q - p$. By Lemma 2.3(ii) there are points $b_1 < a_2 < a_1 < b_0 < a_0$ satisfying (5). Since $\tau_f(b_1) < b_1$, by the continuity, $\tau_f((b_1, a_1]) \supset [b_1, a_2]$. Hence, there is a $u \in (b_1, a_1)$ such that $\tau_f(u) = b_1$. Put $v = f(u)$. Then, by (1), $\varphi(v) = p$. So, $v \neq q$, and $v = p$ would imply $\tau_f(u) < \tau_f(b_0) = b_1$. Thus, $p < v < q$ and there is the maximal $h \in (p, q)$ with $\varphi(h) = p$. Without loss of generality we may assume $h < q - \delta$. Let $u_0 \in [b_0, a_0]$ and $u_1 \in [a_1, b_0]$ the minimal points such that $f(u_0) = f(u_1) = h$. Obviously, $a_1 < u_1 < b_0 < u_0 < a_0$, and there is the maximal $u_2 \in [u_1, b_0]$ such that $f(u_2) = h$. Then $a_1 < u_1 \leq u_2$,

$$R_{f|[b_0,u_0)} = [p, h), \quad \text{and} \quad R_{f|[a_1,u_1)} \subseteq (h, q). \tag{11}$$

Since $\varphi(y) < y$ in (p, q), (3) implies $\tau_f(u_0) < a_2$, and since $u_1 < u_0$, we have $\tau_f(u_0) \in \tau_f((a_1, u_1))$. Then there is an $r \in (a_1, u_1)$ with $\tau_f(r) = \tau_f(u_0)$. By the maximality of h, $p \notin \varphi(h,q)$. Hence, the second condition in (11) and (1) imply $p < \varphi(f(r)) = f(\tau_f(r)) = f(\tau_f(u_0)) = \varphi(f(u_0)) = p$, which is a contradiction. This proves (10).

To finish the proof of (iv) assume on the contrary that $\liminf_{x\to\infty} f(x) = r < q$. Then there are points $v_1 < v_2 < \cdots \to \infty$ such that $\lim_{n\to\infty} f(v_n) = r$. Since $\varphi(y) < y$ in (p,q), by (1) we have $\lim_{n\to\infty} f(\tau_f(v_n)) = \lim_{n\to\infty} \varphi(f(v_n)) = \varphi(r) < r$ which is a contradiction unless $r = p$. Thus, $\liminf_{x\to\infty} f(x) = p$. Let w_0 be such that $\tau_f(w_0) > \max E(p) =: x_0$ (see (10)), and let $\{w_n\}_{n\ge 0}$ be its forward orbit. Since $w_n \to 0$, there is a $k \ge 0$ such that $x_0 \in [w_{k+1}, w_k]$. Hence, there is a $c \in [w_k, w_{k-1}]$ such that $\tau_f(c) = x_0$. Put $f(c) = d$. Then, by (10), $d > p$ and, by (1), $\varphi(d) = p$. Consequently, $\varphi(y) = p$ for $d \ge y \ge p$ and $\liminf_{x\to\infty} f(x) = p$ yields $f(x) = p$, for every $x \ge x_0$, contrary to (9). □

Theorem 2.8. *Let $p \in (0,1)$ be a fixed point of φ such that $\varphi([p,1]) = [p,1]$ and $\varphi(y) \neq y$ for $y \in (p,1)$. Let $f \in \mathcal{S}(\varphi)$ such that $1 \in R_f$ and $f \not\equiv 1$. Then, for some $a > 0$, one of the following is true.*

(i) *$\varphi(y) < y$ for $y \in (p,1)$, $f(t) = 1$ for $t \ge a$, $R_f \subseteq [p,1]$, and $\lim_{t\to 0} f(t) = p$;*

(ii) *$\varphi(y) > y$ for $y \in (p,q)$, $f(t) = 1$ for $t \le a$, $R_{f|(a,\infty)} = (p,1)$ and $\lim_{t\to\infty} f(t) = p$.*

Moreover, in case (i), *there can be $p \in R_f$.*

Proof. By Proposition 2.2 in [8], if there are points $u < v < w$ such that $f(u) = f(w) = 1$ and $f(v) < 1$ then φ has a periodic point of period 3. Hence, in our case, $A = \{x > 0; f(x) = 1\}$ is a connected set.

(i) Assume $a_0 := \min A > 0$. Let $y \in R_{f|(0,a_0)}$. Since $y < 1$ there is a maximal $x_0 < a_0$ such that $f(x_0) = y$. Then $R_{f|(x_0,a_0)} = (f(x_0), 1)$. Let $\{x_n\}_{n\ge 0}$ be the forward orbit of x_0. Since $\tau_f([x_0,a_0]) \supseteq [x_1,a_0]$, there is a $u \in (x_0,a_0)$ with $\tau_f(u) = x_0$. Since $f(u) \in R_{f|(x_0,a_0)}$, $f(u) > f(x_0)$ and, by (1), $\varphi(f(u)) = f(x_0)$. This implies $\varphi(y) < y$ in $(p,1)$. For $J_n = [x_{n+1}, x_n]$ and $z_n = \max f(J_n)$, $n \ge 0$, we get, similarly as in the proof of Theorem 2.7, that $\lim_{t\to 0} f(t) = p$.

To finish the proof assume $a_1 := \max A < \infty$. Let $x_0 > a_1/p$, and let $\{x_n\}_{n\ge 0}$ be the forward orbit of x_0. Then $f(x_0) \in [p,1)$ hence, $x_1 = x_0 f(x_0) \in (a_1, x_0)$. Let $J_n = [x_{n+1}, x_n]$, $n \in \mathbb{N}$. Since $\varphi(y) < y$ in $(p,1)$, by (3) we get $R_{f|(0,x_0]} \subseteq \bigcup_{n\ge 0} \varphi^n(R_{f|J_0}) \subseteq f(J_0) \subseteq [p,1)$ – a contradiction. Thus, for $a := a_0$ we have the situation described in (i).

(ii) So assume $A = (0,a]$. Then $\varphi(y) > y$ for $y \in (p,1)$. Indeed, let $x_0 > a$, and let $\{x_n\}_{n\ge 0}$ be its forward orbit. Since $f(x_0) < 1$, $\varphi(y) < y$ would imply

$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \varphi^n(f(x_0)) = p$ and, by (2), $\lim_{n\to\infty} x_n = 0$, a contradiction. Similarly, $p \notin R_f$ since, by (1), $f(u) = p$ would imply $f(up^n) = p < 1$ for $n \ge 0$ and $\lim_{n\to\infty} up^n = 0$, which is impossible. To finish the argument assume on the contrary that $\limsup_{t\to\infty} f(t) = r > p$. Then, similarly as in the proof of Theorem 2.5, we get $\limsup_{t\to\infty} f(t) = 1$ which is impossible since $f(t) < 1$ for $t > a$ and $\varphi(y) > y$, for $p < y < 1$. Thus, $r = p$. The fact that $R_{f|(a,\infty)} = (p, 1)$ is obvious.

It remains to show an example satisfying (i), with $p \in R_f$. Let $f(x) = \frac{1}{2}$ for $x \le \frac{3}{4}$, $f(x) = 2x - 1$ for $\frac{3}{4} < x \le 1$, and $f(x) = 1$ for $x > 1$. Let $\varphi(y) = \frac{1}{2}$ for $\frac{1}{2} \le y \le \frac{1}{2}(\sqrt{7} - 1)$, and $\varphi(y) = y^2 + y - 1$ for $\frac{1}{2}(\sqrt{7} - 1) < y \le 1$. It is easy to verify that, for $p = \frac{1}{2}$, $\varphi(y) < y$ in $(p, 1)$, $f \in \mathcal{S}(\varphi)$ and $R_f = [p, 1]$. $\square$

Remark 2.9. The example given in the last part of the proof of Theorem 2.8 gives a positive answer to the problem in [8], whether the range of a solution of (1) can contain two fixed points of φ. Note that similar behavior is impossible if φ is a homeomorphism of $[p, 1]$ with $\varphi(y) \ne y$ in $(p, 1)$. For details and examples, see [5]. Theorem 2.2 shows that the results given in Theorems 2.5, 2.7 and 2.8 are the best possible ones.

3. Technical tools for solutions with more complex φ

Let ψ be a continuous map $[p, q] \to [p, q]$, where $0 \le p < q < 1$. In this section, points p, q need not be periodic unless explicitly stated. Let $J_0 = [a_1, a_0]$, and let $g_0 : J_0 \to [p, q]$ be continuous. Then g_0 satisfies the *initial condition (with respect to ψ) on J_0* if $a_1 = \tau_{g_0}(a_0)$ $(= a_0 g_0(a_0))$ and $g_0(a_1) = \psi(g_0(a_0))$. A function $g_0 : J_0 \to [p, q]$ is *forward ψ-extendable* if it satisfies the initial conditions on J_0, and if there is a continuous function $g : (0, a_0] \to [p, q]$ such that $g|_{J_0} = g_0$ and

$$g(xg(x)) = \psi(g(x)), \quad x \in (0, a_0]. \tag{12}$$

Note that g_0 cannot be forward ψ-extendable if there are points $u \ne v$ in J_0 such that $\tau_{g_0}(u) = \tau_{g_0}(v)$ and $\psi(g_0(u)) \ne \psi(g_0(v))$. The following lemma gives a sufficient condition for ψ-extendability.

Lemma 3.1. *Assume $\psi : [p, q] \to [p, q]$, $0 \le p < q < 1$, and g_0 satisfies the initial condition with respect to ψ on $J_0 = [a_1, a_0]$.*

(i) *Function g_0 is forward ψ-extendable if there is a decreasing sequence $\{a_n\}_{n\ge0}$ in $(0, a_0]$ with $\lim_{n\to\infty} a_n = 0$, and a sequence $\{g_n\}_{n\ge0}$ of functions*

$g_n : J_n \to [p,q]$ *where* $J_n = [a_{n+1}, a_n]$ *such that, for every* $n \in \mathbb{N}$,

$$a_n g_n(a_n) = a_{n+1},\ \tau_{g_n}(J_n) = J_{n+1},\ \text{and } g_{n+1}(\tau_{g_n}(x)) = \psi(g_n(x)), \quad \text{for } x \in J_n.$$

(ii) *If* g_0 *is forward* ψ*-extendable and* $\{g_n\}_{n\geq 0}$ *are functions given in* (i) *then* $g := \bigcup_{n\geq 0} g_n$ *is a solution of* (12).

(iii) *Function* g_0 *is forward* ψ*-extendable if*

$$\tau_{g_0}^n(x) := xg_0(x)\psi(g_0(x))\psi^2(g_0(x))\dots\psi^{n-1}(g_0(x)) \text{ is increasing on } J_0, \quad \text{for every } n \in \mathbb{N}. \tag{13}$$

Proof. Parts (i) and (ii) are obvious. To prove (iii) note that formula (13) is a particular case of (2). If τ_{g_0} is increasing on $[x_1, x_0]$ then, by (1), $\tau_{g_0}(J_0) = J_1$ and $g_1(x) = \varphi(\tau_{g_0}(x))$ is a continuous function on J_1. By induction we get that if $\tau_{g_0}^n$ is increasing on J_0 for every n, then the intervals J_n are non-overlapping, $\tau_{g_0}^n = \tau_{g_k}^{n-k}$, whenever $0 \leq k < n$, and g_n is a continuous map $J_n \to [p,q]$. □

Proposition 3.2. *Let* $\psi : [p,q] \to [p,q]$, $0 \leq p < q < 1$, *be continuous,* g_0 *on* $J_0 = [a_1, a_0]$ *forward* ψ*-extendable, with* $g_0(a_0) = q$, *and let* $g = \bigcup_{n\geq 0} g_n$. *Then, for every* $b_0 > a_0$, *there are a continuous* $\varphi : [p,1] \to [p,1]$ *and an* $f \in \mathcal{S}(\varphi)$ *such that*

$$\varphi|_{[p,q]} = \psi, f|_{(0,a_0]} = g,\ f \text{ linearly increases on } [a_0, b_0],\ \text{and } f|_{[b_0,\infty)} \equiv 1. \tag{14}$$

Proof. Let f be as in (14). Thus, for $x \in [a_0, b_0]$ we have $f(x) = (x - a_0)(1 - g(a_0))(b_0 - a_0)^{-1} + g(a_0)$, and since τ_f is increasing on $[a_0, b_0]$ there is exactly one $x_0 \in (a_0, b_0)$ such that $\tau_f(x_0) = a_0$. Denote by f^{-1} the inverse function to $f|_{[a_0,b_0]}$, and define φ as follows:

$$\varphi(y) = f(yf^{-1}(y)) \quad \text{for } f(x_0) \leq y \leq 1, \qquad \text{and}$$
$$\varphi(y) = g(yf^{-1}(y)) \quad \text{for } g(a_0) \leq y \leq f(x_0).$$

Then $f \in \mathcal{S}(\varphi)$. □

Remark 3.3. A special case of the construction described in Proposition 3.2 is the function given in the last part of the proof of Theorem 2.8. Keeping the notation from Proposition 3.2, we have $q = \frac{1}{2}$, $a_0 = \frac{3}{4}$, $b_0 = 1$, $x_0 = \frac{1}{2}(\sqrt{7} - 1)$, and $g(x) = \frac{1}{2}$ for $x \in [a_0, x_0]$.

Example 3.4. We provide another example of a forward extendable function $f_0 : [1,2] \to \left[\frac{1}{\sqrt{6}}, \frac{1}{2}\right]$. Let

$$f_0(x) = \frac{1}{2\sqrt{x}}, \quad x \in \left[1, \frac{3}{2}\right], \qquad f_0(x) = \frac{1}{2\sqrt{3-x}}, \quad x \in \left[\frac{3}{2}, 2\right],$$
$$\text{and} \qquad \varphi(y) = \sqrt{\frac{y}{2}}, \quad y \in [0,1].$$

Then, by (2),

$$\varphi^k(f_0(x)) = \frac{1}{2}x^{-1/2^{k+1}} \text{ and } \tau_{f_0}^{n+1}(x) = \left(\frac{1}{2}\right)^{n+1} x^{1/2^{n+1}}, \quad x \in \left[1, \frac{3}{2}\right], \ k, n \in \mathbb{N},$$

whence, $\tau_{f_0^{n+1}}(x)$ is increasing on $[1, \frac{3}{2}]$. Similarly,

$$\varphi^k(f_0(x)) = \frac{1}{2}(3-x)^{-1/2^{k+1}} \quad \text{and} \quad \tau_{f_0}^{n+1}(x) = \left(\frac{1}{2}\right)^{n+1} (3-x)^{1/2^{n+1}},$$
$$x \in \left[\frac{3}{2}, 2\right], \quad k, n \in \mathbb{N},$$

and $\tau_{f_0}^{n+1}(x)$ is increasing on $\left[\frac{3}{2}, 2\right]$. So, the intervals J_n, $n \geq 0$, are non-overlapping and consequently, $\bigcup_{n\geq 0} f_n$ is a solution of (1) on the interval $(0,2]$.

The following result is similar to Proposition 3.2, but with φ possessing only a single fixed point of φ in its domain.

Proposition 3.5. *Let $\psi : [p,q] \to [p,q]$, $0 \leq p < q < 1$, be continuous, g_0 on $J_0 = [a_1, a_0]$ forward ψ-extendable, with $g_0(a_0) = q$. Let $g = \bigcup_{n\geq 0} g_n$, and $r \in (q,1)$. Then there are a continuous $\varphi : [p,r] \to [p,r]$, and an $f \in \mathcal{S}(\varphi)$ such that*

$$\varphi|_{[p,q]} = \psi, f|_{(0,a_0]} = g,\ \varphi(y) < y \ \text{ for } y > q,\ f \text{ strictly increases on } [a_0, \infty),$$
$$\text{and} \lim_{x\to\infty} f(x) = r. \tag{15}$$

Proof. By [6], there is an increasing homeomorphism ϑ of $[q,r]$ such that, there is a strictly increasing function h with $R_h = (p,q)$, satisfying $h(xh(x)) = \vartheta(h(x))$ for $x > 0$. Let $b_0 > a_0$ be such that $\tau_h(b_0) = a_0$; such a b_0 exists by the continuity since $h < 1$. The remainder of the proof is similar as for Proposition 3.2. We take f such that $f|_{(0,a_0]} = g$, $f|_{[b_0,\infty)} = h$, and f is linear on $[a_0, b_0]$. Then $\varphi_{[p,q]} = \psi$, $\varphi|_{[h(b_0),r]} = \vartheta$, and $\varphi(y) = g(yf^{-1}(y))$ for $g(a_0) = q \leq y \leq h(b_0)$. □

Example 3.6. Applying Proposition 3.5 with $p = \frac{1}{\sqrt{6}}$, $q = \frac{1}{2}$, and $r = \frac{3}{4}$, to f_0 from Example 3.4, we obtain a solution f of (1) with $R_f = \left[\frac{1}{\sqrt{6}}, \frac{3}{4}\right)$ such that φ has on R_f a single periodic point – the fixed point $\frac{1}{2}$.

The following is the "dual"version of Proposition 3.5; its proof is omitted. Note that there is no "dual"version of Proposition 3.2.

Proposition 3.7. *Let $\psi : [p, q] \to [p, q]$, $0 \le p < q < 1$, be continuous, and g_0 on $J_0 = [a_1, a_0]$ forward ψ-extendable, with $g_0(a_0) = p$. Let $g = \bigcup_{n \ge 0} g_n$, and $s \in (0, p)$. Then there are a continuous $\varphi : [s, q] \to [s, q]$, and an $f \in \mathcal{S}(\varphi)$ such that*

$$\varphi|_{[p,q]} = \psi, f|_{(0,a_0]} = g,\ \varphi(y) > y \text{ for } y < p,\ f \text{ strictly decreases on } [a_0, \infty),$$
$$\text{and } \lim_{x \to \infty} f(x) = s.$$

The following theorem summarizes previous results.

Theorem 3.8. *Let $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi)$, and $f \in \mathcal{S}(\varphi)$ be a non-constant solution with $R_f \subseteq (0, 1]$. Then R_f contains at most two fixed points of φ, and there are four types of R_f.*

($T_\emptyset$) $\mathrm{Fix}(\varphi) \cap R_f = \emptyset$. *Then R_f is an open interval, with endpoints in* $\mathrm{Fix}(\varphi)$.

(T_1) $\mathrm{Fix}(\varphi) \cap R_f = \{1\}$. *Then R_f is an interval of the form $(u, 1]$.*

($T_{p,1}$) $\mathrm{Fix}(\varphi) \cap R_f = \{p, 1\}$, $p < 1$. *Then $R_f = (u, 1]$ or $R_f = [u, 1]$, and p is a boundary (i.e., $p = u$) or interior point of R_f.*

(T_p) $\mathrm{Fix}(\varphi) \cap R_f = \{p\}$, $p < 1$. *Then R_f is a non-closed interval, p can be interior or boundary point of R_f, and $\overline{R_f} \setminus R_f \subseteq \mathrm{Fix}(\varphi)$.*

Every type of behavior described above can be realized by suitable φ and $f \in \mathcal{S}(\varphi)$.

Proof. By Theorem 2.2, there are no more than four possible types of R_f, $T_\emptyset - T_p$, follow by Theorem 2.2. The case $T_\emptyset$ is described in [5] or [6]. The cases T_1 and $T_{p,1}$ follow by Theorem 2.8, Proposition 3.2, Example 3.4 and Remark 3.3. Finally, the case T_p follows by Theorems 2.5, 2.7, Propositions 3.5, 3.7, and Example 3.6. □

4. Concluding remarks

Remark 4.1. When constructing complicated solutions of (1) it suffices to look for a function $\psi : [p, q] \to [p, q]$ possessing a single periodic orbit of period 2, which allows an interval J, and a forward ψ-extendable function $f_0 : J \to [p, q]$. Propositions 3.2, 3.5 and 3.7 give an advantage that we need not look for maps f_0 which are also "backward"ψ-extendable; such functions, very likely, do not exist even when ψ has a single periodic orbit of period 2.

Remark 4.2. In [10] there is an example of a solution with $R_f = J \subset (0, 1)$ such that $\mathrm{Per}(\varphi) = J$, and all periodic points, except for one (fixed point), have period 2. This is the only known example of a solution possessing in its range a periodic point of period > 1. In [8] we posed the problem whether there exists a solution with a single φ-periodic orbit of period 2 in its range. Using results from Section 3, the authors spent considerable time to find such solution, but did not succeed. This and other arguments give credibility to the conjecture that no such solution is possible.

Remark 4.3. Based on the preceding remarks we conjecture that there are no solutions of (1) possessing in R_f φ-periodic points of other periods than 1 and 2, so that the answer to the Problem 3 from [8] is negative.

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