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Weakly-peripherally multiplicative conditions and isomorphisms between uniform algebras

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Abstract. Suppose that A and B are uniform algebras on compact Hausdorff spaces X and Y, respectively. Let $\rho, \tau : \Lambda \to A$ and $S, T : \Lambda \to B$ be mappings on a non-empty set Λ. Suppose that $\rho(\Lambda), \tau(\Lambda)$ and $S(\Lambda), T(\Lambda)$ are closed under multiplications and contain $\exp A$ and $\exp B$ respectively and that $S(e_1) \in S(\Lambda)^{-1}$, $T(e_2) \in T(\Lambda)^{-1}$ with $|S(e_1)T(e_2)| = 1$ on $\mathrm{Ch}(B)$ for some fixed $e_1, e_2 \in A_1$ with $\rho(e_1) = \tau(e_2) = 1$. If $\sigma_\pi(S(f)T(g)) \cap \sigma_\pi(\rho(f)\tau(g)) \neq \emptyset$ for all $f, g \in \Lambda$ and there exists a first-countable dense subset D_B in $\mathrm{Ch}(B)$, or a first-countable dense subset D_A in $\mathrm{Ch}(A)$, then there exists an algebra isomorphism $\widetilde{S} : A \to B$ such that $\widetilde{S}(\rho(f)) = S(e_1)^{-1}S(f)$ and $\widetilde{S}(\tau(f)) = T(e_2)^{-1}T(f)$ for every $f \in \Lambda$.

1. Introduction

The search for sufficient conditions for mappings between Banach algebras to be algebra isomorphisms has a long and interesting history. Such results demonstrate that linear maps between Banach algebras that preserve the norm, the spectrum, or a subset of the spectrum must be multiplicative. For example, one of the corollaries of the classical theorem of Gleason–Kahane–Żelazko [Ze] states that a surjection $T : A \to B$ between uniform algebras is an algebra isomorphism if it is linear and preserves the spectra, i.e. $\sigma(T(f)) = \sigma(f)$ for all $f, g \in A$. A theorem by Kowalski and Słodkowski [K-S] considers alternative spectral conditions for not necessarily linear surjections.

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Molnár [Mo] have introduced an interesting spectral multiplicativity condition that contributes to the matter. In particular, he proved that if T is a surjection from the Banach algebra $C(X)$ of all complex-valued continuous functions on a first countable compact Hausdorff space X onto itself such that $\sigma(T(f)T(g)) = \sigma(fg)$ for all $f, g \in C(X)$, then T is an algebra isomorphism. In the case where T is a surjection from a uniform algebra A onto itself, this result was proven by Rao and Roy [RR1]. Hatori, Miura and Takagi [HMT06] showed that if $T : A \to B$ is a surjection between uniform algebras such that the range of $T(f)T(g)$ equals to that of fg for all $f, g \in A$, then $T(1)^{-1}T$ is an algebra isomorphism. Maps between uniform algebras and more general semi-simple commutative Banach algebras that satisfy $\sigma(T(f)T(g)) = \sigma(fg)$ [HMT07], [HMT], [RR2] or $\sigma_\pi(T(f)T(g)) = \sigma_\pi(fg)$ [G-T], [LT] were analyzed further (see also [Hon]).

Maps T such that for some positive integers m and n, $\sigma_\pi(T(f)^m T(g)^n) \subset \sigma_\pi(f^m g^n)$, or, such that $\sigma_\pi(T(f)T(g))$ and $\sigma_\pi(fg)$ meet only, without being necessarily equal, were analyzed recently (see [HHMO], [LLT], [T.talk], [JLV]). Most recently, Tonev [T.talk], [T10] characterized a surjection $T : A \to B$ between function algebras, without assuming the existences of the units, such that $\sigma_\pi(T(f)T(g)) \cap \sigma_\pi(fg) \neq \emptyset$ and $\sigma_\pi(T(f)) = \sigma_\pi(f)$ for all $f, g \in A$. Hatori, Miura, Shindo and Takagi [HMST] have characterized maps $\rho, \tau : I \to A$ and $S, T : I \to B$ from a non-empty set into uniform algebras that satisfy $\sigma_\pi(S(f)T(g)) \subset \sigma_\pi(\rho(f)\tau(g))$ for all $f, g \in I$. In this paper, we analyze maps $\rho, \tau : \Lambda \to A$ and $S, T : \Lambda \to B$ from a non-empty set into uniform algebras such that $\sigma_\pi(S(f)T(g)) \cap \sigma_\pi(\rho(f)\tau(g)) \neq \emptyset$ for all $f, g \in \Lambda$ and give conditions for isomorphisms between uniform algebras.

2. Main result

We begin by providing definitions and notations. Let $C(X)$ be the space of all complex-valued continuous functions on a compact Hausdorff space X. $C(X)$ is Banach algebra with pointwise multiplication and the supremum norm $\|\cdot\|_\infty$. Let A be a uniform algebra on a compact Hausdorff space X. Denote by M_A the maximal ideal space of A, by $\sigma(f)$ the spectrum of $f \in A$, and by $\hat{f}$ the Gelfand transform of $f \in A$. Note that $\sigma(f) = \hat{f}(M_A)$ and $\sup\{|\lambda| : \lambda \in \sigma(f)\} = \|f\|_\infty$. The peripheral spectrum of an element $f \in A$ is the maximum modulus set of the spectrum of f, that is $\sigma_\pi(f) = \{\lambda \in \sigma(f) : |\lambda| = \|f\|_\infty\}$. If $\sigma_\pi(u) = \{1\}$ for $u \in A$, then u is called a peak function of A. In this case $u^{-1}(\{1\})$ is a peak set of A. For a fixed $x \in X$ denote by $P_A(x)$ the set of all peak functions u of A

with $u(x) = 1$. A point $x \in X$ that equals the intersection of peak sets is called a weak peak point of A. The set of all weak peak points of A is the Choquet boundary of A, denoted by $\mathrm{Ch}(A)$. It is known that $\mathrm{Ch}(A)$ is a boundary for A, that is $\|f\|_\infty = \max\{|f(x)| : x \in \mathrm{Ch}(A)\}$ for every $f \in A$. An $x \in X$ is said to be a peak point of A if $\{x\}$ is a peak set of A. Note that a weak peak point x which has a countable neighborhood basis is a peak point of A [Br, Lemma 2.3.1 and Theorem 2.3.4]. Denote by $\exp A$ the range of the exponential map on A. In the sequel we will need the following corollary of [HHMO, Proposition 2.2] (see also [HMST], [LL], [MHS], [S.Ce], [S.Me]):

Lemma 2.1. *If $x \in X$ is a peak point and $f(x) \neq 0$ for some $f \in A$, then there exists a $u \in P_A(x) \cap \exp A$ such that $\sigma_\pi(fu) = \{f(x)\}$ and $(fu)^{-1}(\{f(x)\}) = u^{-1}(\{1\}) = \{x\}$.*

Proof. By Proposition 2.2 in [HHMO], there exists a $u_1 \in P_A(x) \cap \exp A$ such that $\sigma_\pi(fu_1) = \{f(x)\}$. Since x is a peak set, there exists a $u' \in P_A(x)$ such that $u'^{-1}(\{1\}) = \{x\}$. Let $u_2 = (u' + 1)/2$. Then $u = u_1 u_2 \in P_A(x) \cap \exp A$ satisfies $\sigma_\pi(fu) = \{f(x)\}$ and $(fu)^{-1}(\{f(x)\}) = u^{-1}(\{1\}) = \{x\}$. □

Throughout this paper we assume that A and B are uniform algebras on compact Hausdorff spaces X and Y respectively and that Λ is a non-empty set. Denote by f^{-1} an inverse element of $f \in A$ and by E^{-1} the set of invertible elements of E. We will also use the following proposition, which is a corollary of [HMST, Proposition 2.3] (see also [S.Ce, Proposition 2.1]).

Proposition 2.2. *Let $h_1, h_2 : \Lambda \to A$ and $H_1, H_2 : \Lambda \to B$ be mappings on Λ. Suppose that $h_1(\Lambda), h_2(\Lambda)$ and $H_1(\Lambda), H_2(\Lambda)$ are closed under multiplications and contain $\exp A$ and $\exp B$, respectively. If*

$$\|H_1(f)H_2(g)\|_\infty = \|h_1(f)h_2(g)\|_\infty,$$
$$\|H_1(f)\|_\infty = \|h_1(f)\|_\infty \text{ and } \|H_2(f)\|_\infty = \|h_2(f)\|_\infty$$

for all $f, g \in \Lambda$, then there exists a homeomorphism $\psi : \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ such that

$$|H_1(f)(y)| = |h_1(f)(\psi(y))| \quad \text{and} \quad |H_2(f)(y)| = |h_2(f)(\psi(y))|$$

for every $f \in \Lambda$ and $y \in \mathrm{Ch}(B)$.

Proposition 2.3. *Let $\rho, \tau : \Lambda \to A$ and $S, T : \Lambda \to B$ be mappings on Λ. Suppose that $\rho(\Lambda), \tau(\Lambda)$ and $S(\Lambda), T(\Lambda)$ are closed under multiplications and contain $\exp A$ and $\exp B$, respectively. Suppose that $S(e_1) \in S(\Lambda)^{-1}$, $T(e_2) \in$*

$T(\Lambda)^{-1}$ with $|S(e_1)T(e_2)| = 1$ on $\mathrm{Ch}(B)$ for some $e_1, e_2 \in \Lambda$ with $\rho(e_1) = \tau(e_2) = 1$. If

$$\sigma_\pi\left(S(f)T(g)\right) \cap \sigma_\pi\left(\rho(f)\tau(g)\right) \neq \emptyset$$

for all $f, g \in \Lambda$, then there exists a homeomorphism $\psi : \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ such that

$$|(S(e_1)^{-1}S(f))(y)| = |\rho(f)(\phi(y))|,$$
$$|(T(e_2)^{-1}T(f))(y)| = |\tau(f)(\phi(y))| \tag{2.1}$$

for every $f \in \Lambda$ and $y \in \mathrm{Ch}(B)$. If, in addition, $S(e_1)T(e_2) = 1$ and $y_0 \in \mathrm{Ch}(B)$ is a peak point of B, or $\phi(y_0)$ is a peak point of A, then

$$(S(e_1)^{-1}S(f))(y_0) = \rho(f)(\phi(y_0)),$$
$$(T(e_2)^{-1}T(f))(y_0) = \tau(f)(\phi(y_0))$$

for every $f \in \Lambda$.

Proof. Since $|S(e_1)T(e_2)| = 1$ on $\mathrm{Ch}(B)$, we obtain

$$\left\|S(e_1)^{-1}S(f)T(e_2)^{-1}T(g)\right\|_\infty = \|S(f)T(g)\|_\infty,$$

which implies that

$$\left\|S(e_1)^{-1}S(f)T(e_2)^{-1}T(g)\right\|_\infty = \|\rho(f)\tau(g)\|_\infty,$$
$$\left\|S(e_1)^{-1}S(f)\right\|_\infty = \|\rho(f)\|_\infty \text{ and } \left\|T(e_2)^{-1}T(g)\right\|_\infty = \|\tau(g)\|_\infty$$

for all $f, g \in \Lambda$. Then the mappings $\rho, \tau, S(e_1)^{-1}S$ and $T(e_2)^{-1}T$ satisfy the hypotheses of Proposition 2.2. Hence there exists a homeomorphism $\phi : \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ satisfying (2.1).

Suppose that $S(e_1)T(e_2) = 1$, that is $S(e_1)^{-1} = T(e_2)$. Let $f \in \Lambda$ and $y_0 \in \mathrm{Ch}(B)$. Note that $(S(e_1)^{-1}S(f))(y_0) = 0$ if and only if $\rho(f)(\phi(y_0)) = 0$ and that $(T(e_2)^{-1}T(f))(y_0) = 0$ if and only if $\tau(f)(\phi(y_0)) = 0$. If y_0 is a peak point of B and $(S(e_1)^{-1}S(f))(y_0) \neq 0$, then, by Lemma 2.1, there exists a $\mathfrak{p} \in P_B(y_0) \cap \exp B$ such that

$$\sigma_\pi\left(S(e_1)^{-1}S(f)\mathfrak{p}\right) = \{(S(e_1)^{-1}S(f))(y_0)\},$$
$$(S(e_1)^{-1}S(f)\mathfrak{p})^{-1}(\{(S(e_1)^{-1}S(f))(y_0)\}) = \mathfrak{p}^{-1}(\{1\}) = \{y_0\}. \tag{2.2}$$

Note that, by the hypotheses,

$$\sigma_\pi\left(S(e_1)^{-1}S(f)T(e_2)^{-1}T(g)\right) \cap \sigma_\pi\left(\rho(f)\tau(g)\right) \neq \emptyset$$

for every $g \in \Lambda$. Let $g_0 \in \Lambda$ with $T(e_2)^{-1}T(g_0) = \mathfrak{p}$. Since

$$\sigma_\pi \left(S(e_1)^{-1}S(f)\mathfrak{p}\right) \cap \sigma_\pi \left(\rho(f)\tau(g_0)\right) \neq \emptyset,$$

there exists a $y' \in \mathrm{Ch}(B)$ such that

$$(\rho(f)\tau(g_0))(\phi(y')) = (S(e_1)^{-1}S(f))(y_0).$$

We also have $\sigma_\pi(\mathfrak{p}) \cap \sigma_\pi(\tau(g_0)) \neq \emptyset$. Thus there exists a $y'' \in \mathrm{Ch}(B)$ such that $\tau(g_0)(\phi(y'')) = 1$. Equation (2.1) shows that

$$|(S(e_1)^{-1}S(f)\mathfrak{p})(y')| = |(\rho(f)\tau(g_0))(\phi(y'))| = |(S(e_1)^{-1}S(f))(y_0)|, \quad \text{and}$$
$$|\mathfrak{p}(y'')| = |\tau(g_0)(\phi(y''))| = 1.$$

Together with (2.2), we obtain $y' = y'' = y_0$. We conclude that

$$(S(e_1)^{-1}S(f))(y_0) = (\rho(f)\tau(g_0))(\phi(y_0)) = \rho(f)(\phi(y_0)).$$

If we consider the maps $T(e_2)^{-1}T$ and τ, the same arguments imply that

$$(T(e_2)^{-1}T(f))(y_0) = \tau(f)(\phi(y_0)).$$

Similar arguments complete the proof for the case where $\phi(y_0)$ is a peak point of A. □

We will now prove the main theorem by using Proposition 2.3.

Theorem 2.4. *Let $\rho, \tau : \Lambda \to A$ and $S, T : \Lambda \to B$ be mappings on Λ. Suppose that $\rho(\Lambda), \tau(\Lambda)$ and $S(\Lambda), T(\Lambda)$ are closed under multiplications and contain* $\exp A$ *and* $\exp B$ *respectively. Suppose that $S(e_1) \in S(\Lambda)^{-1}, T(e_2) \in T(\Lambda)^{-1}$ with $|S(e_1)T(e_2)| = 1$ on $\mathrm{Ch}(B)$ for some $e_1, e_2 \in \Lambda$ with $\rho(e_1) = \tau(e_2) = 1$. If there exists a first-countable dense subset D_B in $\mathrm{Ch}(B)$, or a first-countable dense subset D_A in $\mathrm{Ch}(A)$, and*

$$\sigma_\pi \left(S(f)T(g)\right) \cap \sigma_\pi \left(\rho(f)\tau(g)\right) \neq \emptyset$$

for all $f, g \in \Lambda$, then $S(e_1)T(e_2) = 1$ and there exists a homeomorphism $\phi : \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ such that

$$(S(e_1)^{-1}S(f))(y) = \rho(f)(\phi(y)), \qquad (T(e_2)^{-1}T(f))(y) = \tau(f)(\phi(y)) \qquad (2.3)$$

for every $f \in \Lambda$ and $y \in \mathrm{Ch}(B)$. Moreover, there exist an algebra isomorphism $\widetilde{S} : A \to B$ and a homeomorphism $\Phi : M_B \to M_A$ satisfying

$$\widehat{\widetilde{S}(f)} = \hat{f} \circ \Phi$$

for every $f \in A$,

$$\widetilde{S}(\rho(f)) = S(e_1)^{-1}S(f) \quad \text{and} \quad \widetilde{S}(\tau(f)) = T(e_2)^{-1}T(f)$$

for every $f \in \Lambda$.

Proof. Applying Proposition 2.3 to $\rho, \tau, S(e_1)^{-1}S$ and $T(e_2)^{-1}T$, there exists a homeomorphism $\phi : \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ such that

$$|(S(e_1)^{-1}S(f))(y)| = |\rho(f)(\phi(y))|,$$
$$|(T(e_2)^{-1}T(f))(y)| = |\tau(f)(\phi(y))|$$

for every $f \in \Lambda$ and $y \in \mathrm{Ch}(B)$.

We will prove that $S(e_1)T(e_2) = 1$ if there exists a first-countable dense subset D_B in $\mathrm{Ch}(B)$. Let $y \in \mathrm{Ch}(B)$. If $y \in D_B$, then y is a peak point of B. By Lemma 2.1, there exist $T(u_1) \in P_B(y) \cap \exp B$ such that

$$\sigma_\pi\left(S(e_1)T(u_1)\right) = \{S(e_1)(y)\},$$
$$(S(e_1)T(u_1))^{-1}(\{S(e_1)(y)\}) = T(u_1)^{-1}(\{1\}) = \{y\} \tag{2.4}$$

and $S(u_2) \in P_B(y) \cap \exp B$ such that

$$\sigma_\pi\left(S(u_2)T(e_2)\right) = \{T(e_2)(y)\},$$
$$(S(u_2)T(e_2))^{-1}(\{T(e_2)(y)\}) = S(u_2)^{-1}(\{1\}) = \{y\}. \tag{2.5}$$

Since

$$\sigma_\pi\left(S(e_1)T(u_1)\right) \cap \sigma_\pi\left(\rho(e_1)\tau(u_1)\right) \neq \emptyset$$

and

$$\sigma_\pi\left(S(u_2)T(e_2)\right) \cap \sigma_\pi\left(\rho(u_2)\tau(e_2)\right) \neq \emptyset,$$

there exist $y_1, y_2 \in \mathrm{Ch}(B)$ such that

$$\tau(u_1)(\phi(y_1)) = S(e_1)(y) \quad \text{and} \quad \rho(u_2)(\phi(y_2)) = T(e_2)(y).$$

Note that

$$|(S(e_1)T(u_1))(y_1)| = |\tau(u_1)(\phi(y_1))| = |S(e_1)(y)|$$

and

$$|(S(u_2)T(e_2))(y_2)| = |\rho(u_2)(\phi(y_2))| = |T(e_2)(y)|.$$

By (2.4) and (2.5), we obtain $y_1 = y_2 = y$. Since

$$\sigma_\pi\left(S(u_2)T(u_1)\right) \cap \sigma_\pi\left(\rho(u_2)\tau(u_1)\right) \neq \emptyset,$$

there exists a $y_3 \in \mathrm{Ch}(B)$ such that $(\rho(u_2)\tau(u_1))(\phi(y_3)) = 1$. We also have

$$|(S(u_2)T(u_1))(y_3)| = |(\rho(u_2)\tau(u_1))(\phi(y_3))| = 1.$$

Equations (2.4) and (2.5) imply that $y_3 = y$. Hence we obtain

$$S(e_1)(y)T(e_2)(y) = \tau(u_1)(\phi(y))\rho(u_2)(\phi(y)) = 1.$$

Consequently, $S(e_1)T(e_2) = 1$ on D_B. Since D_B is dense in $\mathrm{Ch}(B)$ and $\mathrm{Ch}(B)$ is a boundary for B, we obtain $S(e_1)T(e_2) = 1$. Similar arguments show that the equation $S(e_1)T(e_2) = 1$ holds for the case where there exists a first-countable dense subset D_A in $\mathrm{Ch}(A)$.

If there exists a first-countable dense subset D_B in $\mathrm{Ch}(B)$, then by Proposition 2.3 we obtain (2.3) for every $f \in \Lambda$ and $y \in D_B$, that is for every $f \in \Lambda$ and $y \in \mathrm{Ch}(B)$. Similar arguments imply (2.3) for the case where there exists a first-countable dense subset D_A in $\mathrm{Ch}(A)$. The existence of $\widetilde{S} : A \to B$ and $\Phi : M_B \to M_A$ is follows by the arguments from the proof of Theorem 3.6 in [HMT06] (see also [MHS]). □

Below we give examples of topological spaces that have first-countable dense subsets.

Example 1 (cf. [S-S]).

(1) Let $[0,1]$ be the unit interval and p a fixed point of $[0,1]$. Consider the topology on $[0,1]$ consisting of open set G such that:
   (a) G excludes p; or
   (b) G contains all but a finite number of the points of $[0,1]$.

   We call this space Fort space [S-S, Part II, 24]. The subset $[0,1] \setminus \{p\}$ is first-countable and dense in the Fort space $[0,1]$.

(2) For the square $[0,1] \times [0,1]$, we define the topology by taking as a neighborhood basis of all points (a,b) off the diagonal $\Delta = \{(x,x) : x \in [0,1]\}$ the intersection of $X - \Delta$ with an open vertical line segment centered at (a,b) : $N_\varepsilon(a,b) = \{(a,y) \in X - \Delta : |b-y| < \varepsilon\}$. Neighborhoods of points $(a,a) \in \Delta$ are defined by the intersection with X of open horizontal strips less a finite number of vertical lines: $N_\varepsilon(a,a) = \{(x,y) \in X : |y-a| < \varepsilon,\ x \neq x_0, x_1, \ldots, x_n\}$. We call this space the Alexandroff square [S-S, Part II, 101]. The subset $[0,1] \times [0,1] \setminus \Delta$ is first-countable and dense in $[0,1] \times [0,1]$.

(3) Let $\beta\mathbb{N}$ be the Stone–Čech compactification of the set $\mathbb{N}$ of natural numbers. Then $\mathbb{N}$ is first-countable and dense in $\beta\mathbb{N}$.

3. Applications

In this section we give some corollaries of Theorem 2.4.

Corollary 3.1. *Suppose that the sets $A_0 \subset A$ and $B_0 \subset B$ are closed under multiplications and contain* $\exp A$ *and* $\exp B$*, respectively. Let $\mathcal{T}$ be a surjection from A_0 onto B_0 and m, n positive integers such that*

$$\sigma_\pi\left(\mathcal{T}(f)^m \mathcal{T}(g)^n\right) \cap \sigma_\pi\left(f^m g^n\right) \neq \emptyset \tag{3.1}$$

for all $f, g \in A_0$.

If there exists a first-countable dense subset D_B in $\mathrm{Ch}(B)$*, or a first-countable dense subset D_A in* $\mathrm{Ch}(A)$*, then $\mathcal{T}(1)^{m+n} = 1$ and there exist an algebra isomorphism $\widetilde{S} : A \to B$ and a homeomorphism $\Phi : M_B \to M_A$ satisfying $\widehat{\widetilde{S}(f)} = \hat{f} \circ \Phi$ for every $f \in A$ and $\widetilde{S}(f)^d = \mathcal{T}(1)^{-d}\mathcal{T}(f)^d$ for every $f \in A_0$, where d is the greatest common divisor of m and n. If, in addition, $d = 1$ and $\mathcal{T}(1) = 1$, then $\mathcal{T}$ can be extended to an algebra isomorphism.*

For the case where $m = n = 1$, $A_0 = A$ and $B_0 = B$, this result is proven in [S.Ni, Theorem 4.6]. If, moreover, $A = B = C(X)$, then it generalizes the result proven by Molnár [Mo].

Proof. By (3.1), we have

$$\|\mathcal{T}(f)^m \mathcal{T}(g)^n\|_\infty = \|f^m g^n\|_\infty$$

for all $f, g \in A_0$. In particular, $\|\mathcal{T}(f)^{m+n}\|_\infty = \|f^{m+n}\|_\infty$, that is

$$\|\mathcal{T}(f)\|_\infty = \|f\|_\infty, \|\mathcal{T}(f)^m\|_\infty = \|f^m\|_\infty \quad \text{and} \quad \|\mathcal{T}(f)^n\|_\infty = \|f^n\|_\infty$$

for every $f, g \in A_0$. Proposition 2.2 shows that there exists a homeomorphism $\phi : \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ such that $|\mathcal{T}(f)(y)| = |f(\phi(y))|$ for every $f \in A_0$ and $y \in \mathrm{Ch}(B)$. In particular, $|\mathcal{T}(1)| = 1$ on $\mathrm{Ch}(B)$. By similar arguments to the second paragraph of the proof of Theorem 2.4, we obtain $\mathcal{T}(1)^{m+n} = 1$, that is $\mathcal{T}(1) \in B_0^{-1}$. According to Theorem 2.4 for $S(f) = \mathcal{T}(f)^m$, $T(f) = \mathcal{T}(f)^n$, $\rho(f) = f^m$, and $\tau(f) = f^n$, we obtain the conclusion. □

Corollary 3.2. *Suppose that the sets $A_1 \subset A^{-1}$ and $B_1 \subset B^{-1}$ are closed under multiplications and contain* $\exp A$ *and* $\exp B$*, respectively. Let $\mathcal{T}$ be a surjection from A_1 onto B_1 and k, l are non-zero integers such that*

$$\sigma_\pi\left(\mathcal{T}(f)^k \mathcal{T}(g)^l\right) \cap \sigma_\pi\left(f^k g^l\right) \neq \emptyset \tag{3.2}$$

for all $f, g \in A_1$. *If* $\mathcal{T}(1) \in B_0^{-1}$, $|\mathcal{T}(1)| = 1$ *on* $\mathrm{Ch}(B)$ *and there exists a first-countable dense subset* D_B *in* $\mathrm{Ch}(B)$, *or a first-countable dense subset* D_A *in* $\mathrm{Ch}(A)$, *then* $\mathcal{T}(1)^{k+l} = 1$ *and there exist an algebra isomorphism* $\widetilde{S} : A \to B$ *and a homeomorphism* $\Phi : M_B \to M_A$ *satisfying* $\widehat{\widetilde{S}(f)} = \hat{f} \circ \Phi$ *for every* $f \in A$ *and* $\widetilde{S}(f)^d = \mathcal{T}(1)^{-d}\mathcal{T}(f)^d$ *for every* $f \in A_1$, *where* d *is the greatest common divisor of* k *and* l. *If, in addition,* $d = 1$ *and* $\mathcal{T}(1) = 1$, *then* $\mathcal{T}$ *can be extended to an algebra isomorphism.*

This follows from Theorem 2.4 with $S(f) = \mathcal{T}(f)^k$, $T(f) = \mathcal{T}(f)^l$, $\rho(f) = f^k$, and $\tau(f) = f^l$.

Remark 3.1. Let A_1, B_1, k and l be as in Corollary 3.2. Suppose $\mathcal{T}$ is a surjection from A_1 onto B_1 satisfying (3.2) for all $f, g \in A_1$. By (3.2), we have

$$\left\|\mathcal{T}(f)^k\mathcal{T}(g)^l\right\|_\infty = \left\|f^k g^l\right\|_\infty$$

for all $f, g \in A_1$. If $kl > 0$, then $\left\|\mathcal{T}(f)^{k+l}\right\|_\infty = \left\|f^{k+l}\right\|_\infty$, that is

$$\left\|\mathcal{T}(f)^{(k+l)/|k+l|}\right\|_\infty = \left\|f^{(k+l)/|k+l|}\right\|_\infty,$$

$$\left\|\mathcal{T}(f)^k\right\|_\infty = \left\|f^k\right\|_\infty \quad \text{and} \quad \left\|\mathcal{T}(f)^l\right\|_\infty = \left\|f^l\right\|_\infty$$

for every $f \in A_1$. Proposition 2.2 implies that there exists a homeomorphism $\phi : \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ with $|\mathcal{T}(f)(y)| = |f(\phi(y))|$ for every $f \in A_1$ and $y \in \mathrm{Ch}(B)$. In particular, $|\mathcal{T}(1)| = 1$ on $\mathrm{Ch}(B)$. If, in addition, there exists a first-countable dense subset $D_B \subset \mathrm{Ch}(B)$ or $D_A \subset \mathrm{Ch}(A)$, then, by similar arguments to the second paragraph of the proof of Theorem 2.4, we obtain $\mathcal{T}(1)^{k+l} = 1$.

This shows that Corollary 3.2 holds in the case when $kl > 0$ or $k + l = 0$ without assuming that $|\mathcal{T}(1)| = 1$ on $\mathrm{Ch}(B)$.

Corollary 3.3. *Let* $\mathcal{T}$ *be a surjection from* A *onto* B *such that*

$$\sigma_\pi\left(\mathcal{T}(f)\exp\mathcal{T}(g)\right) \cap \sigma_\pi\left(f\exp g\right) \neq \emptyset \tag{3.3}$$

for all $f, g \in A$. *If* $\mathcal{T}(1) \in B^{-1}$, $|\mathcal{T}(1)\exp\mathcal{T}(0)| = 1$ *on* $\mathrm{Ch}(B)$ *and there exists a first-countable dense subset* D_B *in* $\mathrm{Ch}(B)$, *or a first-countable dense subset* D_A *in* $\mathrm{Ch}(A)$, *then* $\mathcal{T}(1)\exp\mathcal{T}(0) = 1$ *and there exists a homeomorphism* $\Phi : M_B \to M_A$ *such that*

$$\widehat{\mathcal{T}(1)}^{-1}\widehat{\mathcal{T}(f)} = \hat{f} \circ \Phi$$

$$\widehat{\exp\mathcal{T}(0)}^{-1}\widehat{\exp\mathcal{T}(f)} = \widehat{\exp f} \circ \Phi \tag{3.4}$$

for every $f \in A$. *Moreover,* $\mathcal{T}(0) = 0$ *and* $\mathcal{T}(1) = 1$, *hence* $\mathcal{T}$ *is an algebra isomorphism.*

Proof. According to Theorem 2.4 for $S(f) = \mathcal{T}(f)$, $T(f) = \exp \mathcal{T}(f)$, $\rho(f) = f$, and $\tau(f) = \exp f$, we obtain

$$\mathcal{T}(1) \exp \mathcal{T}(0) = 1$$

and equation (3.4). In particular, $\widehat{\mathcal{T}(1)}^{-1}\widehat{\mathcal{T}(0)} = 0$, that is $\mathcal{T}(0) = 0$. Consequently, $\mathcal{T}(1) = \mathcal{T}(1) \exp \mathcal{T}(0) = 1$. □

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