Publ. Math. Debrecen
78/3-4 (2011), 687–689
DOI: 10.5486/PMD.2011.4978

The number of maximal subgroups of a solvable group

By PÁL HEGEDŰS (Budapest)

Abstract. A new proof is given for Wall's theorem, that the number of maximal subgroups of a finite solvable group G is at most $|G|-1$.

The following theorem was proved by G. E. Wall [W] with the help of the Eulerian function of a finite group.

Theorem. *If G is a solvable group then the number of maximal subgroups of G is at most $|G|-1$. Equality holds (only) for elementary Abelian 2-groups.*

It has been widely conjectured that the result is valid even without assuming solvability.

In this note I present a proof that is much shorter than the one in [W] and that uses a different tool: character theory. The argument depends on an observation (equation (1) below) that was probably first made by Aschbacher and Guralnick [AG]. They used it to prove that the number of conjugacy classes of maximal subgroups of a solvable group is smaller than the number of conjugacy classes of the group, see Corollary 2 to Theorem B in [AG].

It also extends easily to cover the case when the smallest prime divisor of the group order is bounded from below, a theorem of Cook, Wiegold and Williamson [CWW]:

Theorem. *Let G be a solvable group and p the smallest prime divisor of its order $|G|$. Then the number of maximal subgroups of G is at most $\frac{|G|-1}{p-1}$. Equality holds (only) for elementary Abelian p-groups.*

Mathematics Subject Classification: 20C15, 20D10.
Key words and phrases: maximal subgroups, solvable groups.
The research was partly supported by OTKA (NK 72523).

The proof of this theorem is presented below.

Let $\mathcal{N}$ denote the maximal subgroups of G that are normal and let $\mathcal{M}$ denote a full set of representatives from the conjugacy classes of non-normal maximal subgroups of G. The number of maximal subgroups in the class of $M \in \mathcal{M}$ is $|G : M|$. So the number of maximal subgroups of G is $|\mathcal{N}| + \sum_{M\in\mathcal{M}} |G : M|$. We prove that $|G| - 1 \geq (p-1)(|\mathcal{N}| + \sum_{M\in\mathcal{M}} |G : M|)$ to finish the proof.

The following key observation was made first by Ore [O, Theorem 14].

Theorem. *If G is solvable and M, $N < G$ are maximal subgroups then either $MN = G$ or M and N are conjugate.*

Let M, N be non-conjugate maximal subgroups of G. Then the following holds by Frobenius reciprocity, by Ore's Theorem and by Mackey's Theorem,

$$(1_M^G,\, 1_N^G) = (1_M,\, 1_N^G|_M) = (1_M,\, 1_{M\cap N}^M) = (1_{M\cap N},\, 1_{M\cap N}) = 1. \tag{1}$$

(Equation $(1_M^G, 1_N^G) = 1$ first appeared as Corollary 1 to Theorem B in [AG].)

Let $\tau(M) \subseteq \mathrm{Irr}(G)$ be the set of nonprincipal constituents of the permutation character 1_M^G. By equation (1), $\tau(M_1)$ and $\tau(M_2)$ are disjoint for non-conjugate maximal subgroups M_1 and M_2.

As is well known, all characters of primitive permutation representations are multiplicity-free; if a point stabilizer is normal, all irreducible constituents of the character have degree 1, while if a point stabilizer is not normal, then all non-trivial irreducible constituents have degree greater than 1 and therefore greater than or equal to p.

Based on this and the fact that $|G : M| \geq p$ for maximal subgroups $M < G$ we obtain the following two bounds. If $M \ntriangleleft G$ then

$$\sum_{\xi\in\tau(M)} \xi^2(1) \geq p \sum_{\xi\in\tau(M)} \xi(1) = p(|G : M| - 1) \geq (p-1)|G : M|.$$

However, if $M \triangleleft G$ then

$$\sum_{\xi\in\tau(M)} \xi^2(1) = |G : M| - 1 \geq p - 1.$$

By the above equation (1), every nonprincipal irreducible character occurs in at most one $\tau(M)$. So

$$|G| - 1 = \sum_{1_G\neq\xi\in\mathrm{Irr}(G)} \xi^2(1) \geq \sum_{M\in\mathcal{N}} \sum_{\xi\in\tau(M)} \xi^2(1) + \sum_{M\in\mathcal{M}} \sum_{\xi\in\tau(M)} \xi^2(1).$$

For these last sums we use the bounds established above to get

$$|G|-1 \geq \sum_{M\in\mathcal{N}}(p-1) + \sum_{M\in\mathcal{M}}(p-1)|G:M| = (p-1)\Big(|\mathcal{N}| + \sum_{M\in\mathcal{M}}|G:M|\Big),$$

as claimed.

Two concluding remarks are due. First, it seems difficult to provide a good description for those $\xi \in \mathrm{Irr}(G)$ such that $\xi \in \cup_{M\in\mathcal{M}}\ \tau(M)$. For example, in S_4 one of the 3-dimensional irreducibles does, the other does not correspond to a class of maximal subgroups. But these characters are indistinguishable from each other based only on the *matrix* of the character table.

Second, changing slightly the course of the above proof provides the strict bound $\sum_{\xi\in\mathrm{Irr}(G)}\xi(1)$ for the number of maximal subgroups of a solvable group G. Based on computational evidence this seems to be the proper formulation for a bound for nonsolvable groups, too. Of course, this would imply Wall's theorem for arbitrary finite groups.

References

[AG] Michael Aschbacher and Robert M. Guralnick, Solvable generation of groups and Sylow subgroups of the lower central series, *J. Algebra* **77**, no. 1 (1982), 189–201.

[CWW] Roger J. Cook, James Wiegold and Alan G. Williamson, Enumerating subgroups, *J. Austral. Math. Soc. Ser. A* **43**, no. 2 (1987), 220–223.

[O] Oystein Ore, Contributions to the theory of groups of finite order, *Duke Math. J.* (1939), 431–460.

[W] G. E. Wall, Some applications of the Eulerian functions of a finite group, *J. Austral Math. Soc.* **2** (1961/1962), 35–59.

PÁL HEGEDŰS
DEPARTMENT OF MATHEMATICS
CENTRAL EUROPEAN UNIVERSITY
NÁDOR UTCA 9.
H-1051, BUDAPEST
HUNGARY

E-mail: HegedusP@ceu.hu

(Received July 29, 2010; revised October 19, 2010)