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On natural Riemann extensions

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Abstract. A natural Riemann extension is a natural lift of a manifold with a symmetric affine connection to its cotangent bundle. The corresponding structure on the cotangent bundle is a pseudo-Riemannian metric. The classical Riemann extension has been studied by many authors. The broader (two-parameter) family of all natural Riemann extensions was found by the second author in 1987. We prove the equivariance property for the natural Riemann extensions. We also prove some theorems for Ricci curvature and scalar curvature.

Introduction

Riemann extensions of a manifold with a symmetric affine connection to its cotangent bundle in the form of a pseudo-Riemannian metric of type (n, n) have been studied by many authors. See *e.g.* (in the chronological order) [9], [15], [10], [17], [16], [12], [14]. From the more recent time, see also [1], [2], [3], [4], [5], [6], [7], and [13]. In [2], [7], the Riemann extension is related to the Osserman problem, in [9] to the so-called Walker manifolds, and in [4], [6] to the theory of differential equations. In [10], *general Riemann extensions* and in [2] *modified Riemann extensions* were defined. In this paper we study *natural Riemann extensions* which form a geometrically significant two-parameter subclass of both previous classes and still contain the"classical" Riemann extension ([9], [10]) as a special case (See [8] for the general concept of naturality). These Riemann extensions have been

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described by the second author in 1987 ([11]). For the natural Riemann extensions we prove an equivariance theorem as the main theorem. Further, we prove some results about Ricci tensor and scalar curvature. As a consequence we obtain a one-parameter family of Einstein spaces as natural Riemann extensions.

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1. Preliminaries

Let M be a smooth and connected manifold of dimension $n \geq 2$. Then the cotangent bundle T^*M over M consists of all pairs (x, w), where x is a point of M and w is a covector of M at x. We denote by p the natural projection of T^*M to M defined by $p(x, w) = x$. Let $(\mathcal{U}; x^1, x^2, \ldots, x^n)$ be a local coordinate system of M, and let w_i, $i = 1, 2, \ldots, n$, be real valued functions of $p^{-1}(\mathcal{U})$ which attain $w((\partial/\partial x^i)_x)$ to each point $(x, w) \in p^{-1}(\mathcal{U}) \subset T^*M$. Identifying the function $x^i \circ p$, on $p^{-1}(\mathcal{U})$ with x^i on $\mathcal{U}$, we define a local coordinate system $(p^{-1}(\mathcal{U}); x^1, x^2, \ldots, x^n, x^{1*}, x^{2*}, \ldots, x^{n*})$ of T^*M, where $x^{i*} = w_i$, $i = 1, 2, \ldots, n$. A set

$$\{(\partial_1)_{(x,w)}, (\partial_2)_{(x,w)}, \ldots, (\partial_n)_{(x,w)}, (\partial_{1*})_{(x,w)}, (\partial_{2*})_{(x,w)}, \ldots, (\partial_{n*})_{(x,w)}\}$$

at each point $(x, w) \in T^*M$ is a basis for the tangent space $(T^*M)_{(x,w)}$, where $\partial_i = \partial/\partial x^i$ and $\partial_{i*} = \partial/\partial w_i$, $i = 1, 2, \ldots, n$.

The *canonical vertical vector field* on T^*M is a vector field $\boldsymbol{W}$ defined, in terms of local coordinate systems, by

$$\boldsymbol{W} = \sum_{i=1}^{n} w_i \partial_{i*}.$$

This vector field does not depend on the choice of a local coordinate system and it is defined globally on T^*M. We denote by $\mathfrak{F}(M)$ and $\mathfrak{X}(M)$ the set of all smooth functions of M and the set of all smooth vector fields tangent to M, respectively. The *vertical lift* of $f \in \mathfrak{F}(M)$ to T^*M is a function f^v of T^*M defined by $f^v = f \circ p$. The *vertical lift* of $X \in \mathfrak{X}(M)$ to T^*M is a function X^v of T^*M whose value at each point $(x, w) \in T^*M$ is

$$X^v(x, w) = w(X_x). \tag{1.1}$$

In terms of local coordinate systems, we have $X^v(x, w) = \sum w_i \xi^i(x)$, where we put $X = \sum \xi^i \partial_i$. One can easily prove a very useful proposition saying that each vector field tangent to T^*M is determined by its effect on the functions of the form Z^v for all $Z \in \mathfrak{X}(M)$. More precisely, we have

Proposition 1.1 ([17])**.** *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be vector fields on T^*M. If $\boldsymbol{X}(Z^v) = \boldsymbol{Y}(Z^v)$ holds for all $Z \in \mathfrak{X}(M)$, then $\boldsymbol{X} = \boldsymbol{Y}$.*

The *vertical lift* of a differential one-form α of M is a vector field α^v tangent to T^*M defined by

$$\alpha^v(Z^v) = (\alpha(Z))^v \tag{1.2}$$

for all $Z \in \mathfrak{X}(Z)$. In terms of local coordinate systems, identifying $f^v = f \circ p \in \mathfrak{F}(T^*M)$ with $f \in \mathfrak{F}(M)$, we have

$$\alpha^v = \sum_{i=1}^n \alpha_i \partial_{i*}. \tag{1.3}$$

for all one-forms $\alpha = \sum \alpha_i \mathrm{d}x^i$. We have $\alpha^v(f^v) = 0$ for all $f \in \mathfrak{F}(M)$.

The *complete lift* of a vector field X tangent to M is a vector field X^c tangent to T^*M defined by

$$X^c(Z^v) = [X, Z]^v \tag{1.4}$$

for all $Z \in \mathfrak{X}(M)$, where $[X, Z]$ is the Lie bracket of X and Z. In terms of local coordinate systems, we have at each point $(x, w) \in T^*M$

$$X^c{}_{(x,w)} = \sum_{i=1}^n \xi^i(x)(\partial_i)_{(x,w)} - \sum_{h,i=1}^n w_h(\partial_i \xi^h)(x)(\partial_{i*})_{(x,w)} \tag{1.5}$$

for all $X = \sum \xi^i \partial_i$. We also have $X^c(f^v) = (Xf)^v$ for all $f \in \mathfrak{F}(M)$.

The Lie bracket of T^*M is given by

$$\begin{aligned} &[X^c, Y^c] = [X, Y]^c, \qquad [X^c, \alpha^v] = (L_X \alpha)^v, \quad [\alpha^v, \beta^v] = 0, \\ &[X^c, \boldsymbol{W}] = 0, \qquad\qquad\; [\alpha^v, \boldsymbol{W}] = \alpha^v \end{aligned} \tag{1.6}$$

for all $X, Y \in \mathfrak{X}(M)$ and one-forms α, β of M, where L_X denotes the Lie derivative with respect to X.

2. Natural Riemann extensions and the invariance theorem

Let ∇ be a symmetric affine connection of M. Then the metrics $\bar{g}$ on T^*M *naturally lifted from* ∇ have been described by the second author in 1987. (See [11] for the classification theorem and [8] for the general theory of naturality.) Each such metric $\bar{g}$ is defined at $(x, w) \in T^*M$, in terms of classical lifts, by

$$\begin{aligned} \bar{g}_{(x,w)}(X^c, Y^c) &= -aw(\nabla_{X_x} Y + \nabla_{Y_x} X) + bw(X_x)w(Y_x), \\ \bar{g}_{(x,w)}(X^c, \beta^v) &= a\beta_x(X_x), \\ \bar{g}_{(x,w)}(\alpha^v, \beta^v) &= 0 \end{aligned} \tag{2.1}$$

for all vector fields X, Y and all one-forms α, β on M, where a and b are arbitrary constants. We can assume $a > 0$ without loss of generality. Equivalently, $\bar{g}$ is given, in terms of local coordinate systems, by

$$\begin{aligned} \bar{g}_{ij} &= -2a \sum_{h=1}^{n} w_h \Gamma_{ij}^h + bw_i w_j, \\ \bar{g}_{ij*} &= \bar{g}_{j*i} = a\delta_i^j, \\ \bar{g}_{i*j*} &= 0, \end{aligned} \tag{2.2}$$

where a and b are arbitrary constants, $\bar{g}_{IJ} = \bar{g}(\partial x^I, \partial x^J)$ for $I, J = 1, 2, \dots, n, 1*, 2*, \dots, n*$, and Γ_{ij}^h, $h, i, j = 1, 2 \dots, n$, are the local components of ∇ defined by $\nabla_{\partial_i}(\partial_j) = \sum_{h=1}^n \Gamma_{ij}^h \partial_h$. We call $(T^*M, \bar{g})$ the *natural Riemann extension* of (M, ∇). We shall show in Section 4 that the signature of $\bar{g}$ is (n, n). If $a = 1$ and $b = 0$, then $(T^*M, \bar{g})$ is the classical Riemann extension of (M, ∇) (see for example [9], [15]). On the other hand, the class of natural extensions is a special subclass of *general Riemann extensions* as defined in [10, p. 202].

Let ϕ be a (local) diffeomorphism of M. Then we define a lift Φ of ϕ to T^*M by

$$\Phi(x, w_x) = (\phi(x), (\phi^{-1})^*(w_x)) \tag{2.3}$$

for all $(x, w_x) \in T^*M$, where, for the sake of more explicit notation, we denote $(x, w) \in T^*M$ as $(x, w_x) \in T^*M$.

In the rest of this section, after proving some preliminary facts, we shall prove that, if ϕ is an affine diffeomorphism (or, a local affine diffeomorphism, respectively) of (M, ∇), then Φ is an isometry (or, a local isometry, respectively) of $(T^*M, \bar{g})$.

Proposition 2.1. *We have $Z^v \circ \Phi = ((\phi^{-1})_* Z)^v$ for all $Z \in \mathfrak{X}(M)$.*

Proof. Evaluating $Z^v \circ \Phi$ at $(x, w_x) \in T^*M$, we have

$$\begin{aligned} (Z^v \circ \Phi)(x, w_x) &= Z^v(\Phi(x, w_x)) = Z^v((\phi(x), (\phi^{-1})^*(w_x)) = ((\phi^{-1})^*(w_x))(Z^v{}_{\phi(x)}) \\ &= w_x(((\phi^{-1})_*(Z^v)_{\phi(x)})) = w_x((\phi^{-1})_* Z)_x) = ((\phi^{-1})^v)(x, w_x). \end{aligned}$$

This implies the assertion. $\square$

Proposition 2.2. *We have $\Phi_*(X^c) = (\phi_* X)^c$ for all $X \in \mathfrak{X}(M)$.*

Proof. Let (x, w_x) be an arbitrary point of T^*M. Then, using Proposition 2.1, (1.4) and (1.1) we have for all $Z \in \mathfrak{X}(M)$

$$\begin{aligned}
&(\Phi_*(X^c))_{\Phi(x,w_x)}(Z^v) \\
&\qquad = X^c{}_{(x,w_x)}(Z^v \circ \Phi) = X^c{}_{(x,w_x)}(\left(\left(\phi^{-1}\right)_* Z\right)^v) \\
&\qquad = [X, \left(\phi^{-1}\right)_* Z]^v(x, w_x) = w_x([X, \left(\phi^{-1}\right)_* Z]_x).
\end{aligned} \tag{2.4}$$

On the other hand, using (1.4) and (1.1), we have

$$\begin{aligned}
&(\phi_* X)^c{}_{\Phi(x,w_x)}(Z^v) \\
&\quad = (\phi_* X)^c{}_{(\phi(x), \left(\phi^{-1}\right)^*(w_x))}(Z^v) = [\phi_* X, Z]^v(\phi(x), \left(\phi^{-1}\right)^*(w_x)) \\
&\quad = (\left(\phi^{-1}\right)^*(w_x))([\phi_* X, Z]_{\phi(x)}) = w_x(\left(\phi^{-1}\right)_{*\phi(x)}([\phi_* X, Z]_{\phi(x)})).
\end{aligned} \tag{2.5}$$

Since, in general, $\left(\phi^{-1}\right)_*([\phi_* X, Z]) = [X, \left(\phi^{-1}\right)_* Z]$ holds on M, using Proposition 1.1, (1.1) and (1.4), we have the assertion. □

Proposition 2.3. *We have $\Phi_*(\alpha^v) = (\left(\phi^{-1}\right)^*\alpha)^v$ for all one-forms α on M.*

Proof. Let (x, w_x) be an arbitrary point of T^*M. Then, using Proposition 2.1, (1.2) and (1.1), we have for all $Z \in \mathfrak{X}(M)$

$$\begin{aligned}
&\Phi_*(\alpha^v)(Z^v)(\Phi(x, w_x)) \\
&\quad = \alpha^v{}_{(x,w_x)}(Z^v \circ \Phi) = \alpha^v{}_{(x,w_x)}(\left(\left(\phi^{-1}\right)_* Z\right)^v) = (\alpha(\left(\phi^{-1}\right)_* Z))^v{}_{(x,w_x)} \\
&\quad = \alpha_x(\left(\left(\phi^{-1}\right)_* Z\right)_x) = (\left(\phi^{-1}\right)^*\alpha)_{\phi(x)}(Z_{\phi(x)}) = (\left(\phi^{-1}\right)^*\alpha)(Z)(\phi(x)) \\
&\quad = ((\left(\phi^{-1}\right)^*\alpha)^v(Z^v))(\Phi(x, w_x)).
\end{aligned}$$

This implies the assertion. □

Now we shall prove the main theorem of this Section.

Theorem 2.4. *Let ϕ be an affine diffeomorphism (or, a local affine diffeomorphism, respectively) of a manifold M with a symmetric affine connection ∇. Then the metric $\bar{g}$ of the natural Riemann extension $(T^*M, \bar{g})$ of (M, ∇) is invariant by the lift Φ of ϕ defined by (2.3). In other words, Φ is an isometry (or, a local isometry, respectively) of $(T^*M, \bar{g})$.*

Proof. Firstly, because ϕ is an affine diffeomorphism (or, a local one) of (M,∇), we have $\phi_*(\nabla_X Y)=\nabla_{\phi_* X}(\phi_* Y)$ for all $X,Y\in\mathfrak{X}(M)$. We have, at each point $(x,w_x)\in T^*M$,

$$\bar g_{\Phi(x,w_x)}(\Phi_*(X^c),\Phi_*(Y^c))=\bar g_{(x,w_x)}(X^c,Y^c) \tag{2.6}$$

for all $X,Y\in\mathfrak{X}(M)$. In fact, using Proposition 2.2 and the first formula of (2.1), we have

$$\begin{aligned}
\bar g_{\Phi(x,w_x)}(\Phi_*(X^c),\Phi_*(Y^c)) &= \bar g_{(\phi(x),(\phi^{-1})^*(w_x))}((\phi_*X)^c,(\phi_*Y)^c)\\
&= -a(\big(\phi^{-1}\big)^*(w_x))((\nabla_{\phi_*X}(\phi_*Y)+\nabla_{\phi_*Y}(\phi_*X))_{\phi(x)})\\
&\quad + b(\big(\phi^{-1}\big)^*(w_x))((\phi_*X)_{\phi(x)})(\big(\phi^{-1}\big)^*(w_x))((\phi_*Y)_{\phi(x)})\\
&= -a(\big(\phi^{-1}\big)^*(w_x))(\phi_{*x}((\nabla_X Y+\nabla_Y X)_x))\\
&\quad + b(\big(\phi^{-1}\big)*(w_x))(\phi_{*x}(X_x))(\big(\phi^{-1}\big)^*(w_x))(\phi_{*x}(Y_x))\\
&= -aw_x((\nabla_X Y+\nabla_Y X)_x)+bw_x(X_x)w_x(Y_x)\\
&= \bar g_{(x,w_x)}(X^c,Y^c).
\end{aligned}$$

Secondly we have, at each point $(x,w_x)\in T^*M$,

$$\bar g_{\Phi(x,w_x)}(\Phi_*(X^c),\Phi_*(\beta^v))=\bar g_{(x,w_x)}(X^c,\beta^v) \tag{2.7}$$

for all $X\in\mathfrak{X}(M)$ and all one-forms α of M. In fact, by Propositions 2.2 and 2.3 and the second formula of (2.1), we have

$$\begin{aligned}
&\bar g_{\Phi(x,w_x)}(\Phi_*(X^c),\Phi_*(\beta^v))\\
&\quad= \bar g_{(\phi(x),\left(\phi^{-1}\right)^*(w_x))}((\phi_*X)^c,((\phi^{-1})^*\alpha)^v)=a(\big(\phi^{-1}\big)^*\beta)_{\phi(x)}((\phi_*X)_{\phi(x)})\\
&\quad= a(\big(\phi^{-1}\big)^*(\beta_x))(\phi_{*x}(X_x))=a\beta_x(X_x)=\bar g_{(x,w_x)}(X^c,\beta^v).
\end{aligned}$$

Finally we notice that, by definition (2.1), $\bar g_{(x,w)}(\alpha^v,\beta^v)=0$ for all one-forms α and β of M.

Thus we proved the assertion. □

Remark. If we start with a Riemannian metric g on M (and its Levi-Civita connection ∇) then there is the canonical isomorphism of T^*M onto TM induced by g. In this situation, the classical Riemann extension, say $(T^*M,\tilde g)$, determines a unique pseudo-Riemannian metric of signature (n,n) on TM. According to [12] this new metric is the complete lift g^c of g to the tangent bundle (see [16] for the definition of g^c). If (M,g) is homogeneous with the isometry group G then (TM,g^c) is homogeneous with respect to the tangent group TG (see [16]). Hence, $(T^*M,\tilde g)$ is also homogeneous with respect to TG. But it is never homogeneous with respect to the isometry group $\mathrm{I}(T^*M,\tilde g)$.

3. The Riemannian curvature

In the subsequent sections we shall need additional conventions and formulas. Let T be a $(1,1)$-tensor field on M and (x,w) a point of T^*M. Then the contracted vector field $C_w(T)$ is a vector field tangent to T^*M and given, at each point $(x,w)\in T^*M$, by its value on any vertical lift (*cf.* Proposition 1.1) as follows:

$$(C_w(T))(X^v)(x,w)=(TX)^v(x,w)=w((TX)_x) \tag{3.1}$$

for all $X\in\mathfrak{X}(M)$. In terms of local coordinate systems, we have

$$(C_w(T))_{(x,w)}=\sum_{h,i=1}^{n} w_h T_i^h(x)(\partial_{i*})_{(x,w)} \tag{3.2}$$

at each point $(x,w)\in T^*M$, where we put $T=\sum T_i^h\partial_h\otimes \mathrm{d}x^i$. For the Lie bracket, we have

$$[X^c,C_w(T)]=C_w(L_XT),\quad [C_w(S),C_w(T)]=C_w([S,T]) \tag{3.3}$$

for all $X\in\mathfrak{X}(M)$ and $(1,1)$-tensor fields S, T of M.

We assume that $a>0$ in (2.1) and the $n\times n$-matrix $[\bar g_{ij}]$ is nonsingular. Then the $(2n)\times(2n)$-matrix $[\bar g_{IJ}]$ has the inverse matrix $[\bar g^{IJ}]$, which is given by

$$\begin{aligned}
\bar g^{ij}&=0,\\
\bar g^{ij*}&=\bar g^{j*i}=\frac{1}{a}\delta^i_j,\\
\bar g^{i*j*}\Big(=-\frac{1}{a^2}\bar g_{ij}\Big)&=\frac{2}{a}\sum_{h=1}^{n}w_h\Gamma^h_{ij}-\frac{b}{a^2}w_iw_j,
\end{aligned} \tag{3.4}$$

where δ^i_j is the Kronecker's delta. Using (2.2) and (3.4), we obtain easily the Christoffel symbols $\bar\Gamma^H_{IJ}$, $H,I,J=1,2,\dots,n,1*,2*,\dots,n*$, of the Levi-Civita connection $\bar\nabla$ of $\bar g$ as follows:

$$\begin{aligned}
\bar\Gamma^h_{ij}&=\Gamma^h_{ij}-\frac{c}{2}(\delta^h_iw_j+w_i\delta^h_j),\\
\bar\Gamma^{h*}_{ij}&=\sum_{l=1}^{n}w_l\Big(-\frac{\partial\Gamma^l_{jh}}{\partial x^i}-\frac{\partial\Gamma^l_{ih}}{\partial x^j}+\frac{\partial\Gamma^l_{ij}}{\partial x^h}+2\sum_{k=1}^{n}\Gamma^l_{hk}\Gamma^k_{ij}\Big)\\
&\quad-c\sum_{l=1}^{n}w_l\big(\Gamma^l_{hi}w_j+\Gamma^l_{hj}w_i+\Gamma^l_{ij}w_h\big)+c^2w_hw_iw_j,
\end{aligned}$$

$$\begin{aligned} \bar\Gamma^{h}_{i*j} &= 0, \qquad \bar\Gamma^{h*}_{i*j} = -\Gamma^{i}_{jh} + \frac{c}{2}(\delta^{i}_{j} w_h + \delta^{i}_{h} w_j), \\ \bar\Gamma^{h}_{i*j*} &= 0, \qquad \bar\Gamma^{h*}_{i*j*} = 0 \end{aligned} \tag{3.5}$$

for $h, i, j = 1, 2, \ldots, n$, where $c = b/a$. Using (1.3), (1.5) and (3.5), we have the formulas for the Levi-Civita connection $\bar\nabla$ of $\bar g$ given in terms of lifts:

$$\begin{aligned} (\bar\nabla_{X^c} Y^c)_{(x,w)} &= (\nabla_X Y)^c_{(x,w)} + C_w\big((\nabla X)(\nabla Y) + (\nabla Y)(\nabla X)\big)_{(x,w)} \\ &\quad + C_w\big(R_x(\cdot, X)Y + R_x(\cdot, Y)X\big)_{(x,w)}, \\ &\quad - \frac{c}{2}\big\{w(Y)X^c + w(X)Y^c + 2w(Y)C_w(\nabla X) + 2w(X)C_w(\nabla Y) \\ &\quad + w(\nabla_X Y + \nabla_Y X)\boldsymbol{W}\big\}_{(x,w)} + c^2 w(X)w(Y)\boldsymbol{W}_{(x,w)}, \\ (\bar\nabla_{X^c}\beta^v)_{(x,w)} &= (\nabla_X\beta)^v_{(x,w)} + \frac{c}{2}\big\{w(X)\beta^v + \beta(X)\boldsymbol{W}\big\}_{(x,w)}, \\ (\bar\nabla_{\alpha^v} Y^c)_{(x,w)} &= -(\iota_\alpha(\nabla Y))^v_{(x,w)} + \frac{c}{2}\big\{w(Y)\alpha^v + \alpha(Y)\boldsymbol{W}\big\}_{(x,w)}, \\ (\bar\nabla_{\alpha^v}\beta^v)_{(x,w)} &= 0 \end{aligned} \tag{3.6}$$

for all $X, Y \in \mathfrak{X}(M)$ and all one-forms α, β of M. Here, for a $(1,1)$-tensor field T and a one-form α on M, $\iota_\alpha(T)$ is a one-form of M defined by $(\iota_\alpha(T))(X) = \alpha(TX)$ for all $X \in \mathfrak{X}(M)$. Moreover, for $X \in \mathfrak{X}(M)$, ∇X is a $(1,1)$-tensor field of M defined by $(\nabla X)Y = \nabla_Y X$ for all $Y \in \mathfrak{X}((M)$. In particular, $((\nabla X)(\nabla Y))Z = \nabla_{\nabla_Z Y} X$ for all $X, Y, Z \in \mathfrak{X}(M)$.

In addition we have

$$\begin{aligned} (\bar\nabla_{X^c}\boldsymbol{W})_{(x,w)} &= -C_w(\nabla X)_{(x,w)} + cw(X)\boldsymbol{W}_{(x,w)}, \qquad (\bar\nabla_{\alpha^v}\boldsymbol{W})_{(x,w)} = \alpha^v_{(x,w)}, \\ (\bar\nabla_{X^c}(C_w(T)))_{(x,w)} &= C_w\big(L_X T - T(\nabla X)\big)_{(x,w)} \\ &\quad + \frac{c}{2}\big\{w(TX)\boldsymbol{W} + w(X)C_w(T)\big\}_{(x,w)}, \\ (\bar\nabla_{\alpha^v}(C_w(T)))_{(x,w)} &= \big(\iota_\alpha(T)\big)^v_{(x,w)}, \\ (\bar\nabla_{\boldsymbol{W}}\boldsymbol{W})_{(x,w)} &= \boldsymbol{W}_{(x,w)}, \qquad (\bar\nabla_{C_w(T)}\boldsymbol{W})_{(x,w)} = C_w(T)_{(x,w)}, \\ (\bar\nabla_{\boldsymbol{W}}(C_w(T)))_{(x,w)} &= C_w(T)_{(x,w)}, \\ (\bar\nabla_{C_w(S)}(C_w(T)))_{(x,w)} &= C_w(ST)_{(x,w)}. \end{aligned} \tag{3.7}$$

After long calculations using (3.6) and (3.7), we obtain the Riemannian curvature tensor $\bar{R}$ of $(T^*M, \bar{g})$. It is given in terms of lifts as follows:

$$\begin{aligned}
\bar{R}_{(x,w)}(X^c, Y^c)Z^c &= (R(X,Y)Z)^c{}_{(x,w)} + \frac{c}{2}\Big\{\big(w(\nabla_Z Y) + \frac{c}{2}w(Y)w(Z)\big)X^c \\
&- \big(w(\nabla_Z X) + \frac{c}{2}w(X)w(Z)\big)Y^c - w([X,Y])Z^c\Big\}_{(x,w)} \\
&+ C_w\big((\nabla_X R)(\cdot, Y)Z + (\nabla_X R)(\cdot, Z)Y - (\nabla_Y R)(\cdot, X)Z - (\nabla_Y R)(\cdot, Z)X \\
&- [\nabla X, R(\cdot, Z)Y] + [\nabla Y, R(\cdot, Z)X] - (\nabla Z)R(\cdot, X)Y + (\nabla Z)R(\cdot, Y)X \\
&- (R(\cdot, X)Y)(\nabla Z) + (R(\cdot, Y)X)(\nabla Z)\big)_{(x,w)} \\
&+ cw(X_x)C_w\big(R(\cdot, Z)Y - (\nabla Y)(\nabla Z) - (\nabla Z)(\nabla Y)\big)_{(x,w)} \\
&- cw(Y_x)C_w\big(R(\cdot, Z)X - (\nabla X)(\nabla Z) - (\nabla Z)(\nabla X)\big)_{(x,w)} \\
&- cw(Z_x)C_w\big(R(X, \cdot)Y - R(Y, \cdot)X - 2[\nabla X, \nabla Y]\big)_{(x,w)} \\
&- \frac{c}{2}\big\{w(X)w([Y,Z]) - w(Y)w([X,Z]) - 2w([X,Y])w(Z)\big\}\boldsymbol{W}_{(x,w)},
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\bar{R}_{(x,w)}(X^c, Y^c)\gamma^v &= -(\iota_\gamma(R(X,Y)))^v{}_{(x,w)} + \frac{c}{2}w([X,Y])\gamma^v{}_{(x,w)} \\
&- \frac{c}{2}\big\{\gamma(Y)C_w(\nabla X) - \gamma(X)C_w(\nabla Y)\big\}_{(x,w)}, \\
&+ \frac{c^2}{4}\big\{\gamma(Y)w(X) - \gamma(X)w(Y)\big\}\boldsymbol{W}_{(x,w)},
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\bar{R}_{(x,w)}(X^c, \beta^v)Z^c &= -(\iota_\beta(R(\cdot, Z)X))^v{}_{(x,w)} + \frac{c}{2}\big\{\beta(Z)X^c + \beta(X)Z^c\big\}_{(x,w)} \\
&+ \frac{c}{2}\big\{w(\nabla_X Z) - \frac{c}{2}w(X)w(Z)\big\}\beta^v{}_{(x,w)} \\
&+ \frac{c}{2}\big\{\beta(Z)C_w(\nabla X) + 2\beta(X)C_w(\nabla Z)\big\}_{(x,w)} \\
&- \frac{c^2}{2}\big\{\beta(X)w(Z) + \frac{1}{2}\beta(Z)w(X)\big\}\boldsymbol{W}_{(x,w)},
\end{aligned} \tag{3.10}$$

$$\bar{R}_{(x,w)}(X^c, \beta^v)\gamma^v = -\frac{c}{2}\big\{\gamma(X)\beta^v + \beta(X)\gamma^v\big\}_{(x,w)}, \tag{3.11}$$

$$\bar{R}_{(x,w)}(\alpha^v, \beta^v)Z^c = 0, \tag{3.12}$$

$$\bar{R}_{(x,w)}(\alpha^v, \beta^v)\gamma^v = 0 \tag{3.13}$$

for all $X, Y, Z \in \mathfrak{X}(M)$ and one-forms α, β, γ of M.

Remark. For classical Riemann extension, the formula for covariant derivatives and the curvature tensor coincides with those of [16, pp. 269–270]. Yet, there is a sign misprint in the fourth formula of Proposition 10.2 of [16].

4. The Ricci curvature

We choose a point (x,w) of T^*M and fix it. Let $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ be a basis for the space M_x^* of covectors at x such that $\alpha_1=w$, and let $\{E_1,E_2,\dots,E_n\}$ be the basis for M_x dual to $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$. We denote the parallel extension of each E_i (along geodesics starting at x) to a normal neighborhood of x in M by the same symbol E_i, $i=1,2,\dots,n$. In this case, the first formula of (2.1) is reduced to the form $\bar g_{(x,w)}({E_i}^c,{E_j}^c)=bw(E_i(x))w(E_j(x))$, $i,j=1,2,\dots,n$.

Now we put at $(x,w)\in T^*M$ that

$$\begin{aligned}
\boldsymbol{E}_1&=\frac{1}{\sqrt{s(s+b)}}\big((s+b){E_1}^c+2a{\alpha_1}^v\big)\\
\boldsymbol{E}_k&=\frac{1}{\sqrt{2a}}\big({E_k}^c+{\alpha_k}^v\big),\quad k=2,3,\dots,n,\\
\boldsymbol{E}_{1*}&=\frac{1}{\sqrt{s(s-b)}}\big((s-b){E_1}^c-2a{\alpha_1}^v\big)\\
\boldsymbol{E}_{k*}&=\frac{1}{\sqrt{2a}}\big({E_k}^c-{\alpha_k}^v\big),\quad k=2,3,\dots,n,
\end{aligned}\tag{4.1}$$

where $s=\sqrt{4a^2+b^2}$. Then, $\{\boldsymbol{E}_1,\boldsymbol{E}_2,\dots,\boldsymbol{E}_n,\boldsymbol{E}_{1*},\boldsymbol{E}_{2*},\dots,\boldsymbol{E}_{n*}\}$ is an orthonormal basis for the tangent space $(T^*M)_{(x,w)}$ such that $\bar g(\boldsymbol{E}_i,\boldsymbol{E}_i)=1$, $\bar g(\boldsymbol{E}_{i*},\boldsymbol{E}_{i*})=-1$, $i=1,2,\dots,n$. Thus, the signature of $\bar g$ is (n,n).

Let $\overline{\mathrm{Ric}}$ be the Ricci tensor of $\bar g$. Then we have

$$\begin{aligned}
\overline{\mathrm{Ric}}(\boldsymbol{X},\boldsymbol{Y})&=\sum_I\epsilon_I\bar g(\bar R(\boldsymbol{E}_I,\boldsymbol{X})\boldsymbol{Y},\boldsymbol{E}_I)\\
&=\frac{1}{a}\sum_{i=1}^n\big\{\bar g(\bar R({E_i}^c,\boldsymbol{X})\boldsymbol{Y},{\alpha_i}^v)+\bar g(\bar R({\alpha_i}^v,\boldsymbol{X})\boldsymbol{Y},{E_i}^c)\big\}\\
&\quad-\frac{c}{a}\bar g(\bar R({\alpha_1}^v,\boldsymbol{X})\boldsymbol{Y},{\alpha_1}^v).
\end{aligned}$$

for all $\boldsymbol{X},\boldsymbol{Y}\in\mathfrak{X}(T^*M)$, where $\epsilon_I=\bar g(\boldsymbol{E}_I,\boldsymbol{E}_I)$, $I=1,2,\dots,n,1*,2*,\dots,n*$. After long calculations using (4.1) and (3.8)–(3.13), we obtain

$$\begin{aligned}
\overline{\mathrm{Ric}}_{(x,w)}(X^c,Y^c)&=\mathrm{Ric}_x(X,Y)+\mathrm{Ric}_x(Y,X)\\
&\qquad+\frac{1}{2}\cdot\frac{\big(4a+(n-1)b\big)b}{a^2}w(X_x)w(Y_x),
\end{aligned}\tag{4.2}$$

$$\overline{\mathrm{Ric}}_{(x,w)}(X^c,\gamma^v)=\frac{1}{2}\cdot\frac{(n+1)b}{a}\gamma_x(X_x),\tag{4.3}$$

$$\overline{\mathrm{Ric}}_{(x,w)}(\beta^v, \gamma^v) = 0. \tag{4.4}$$

First we have

Theorem 4.1. *Let M be a manifold of dimension n with a symmetric affine connection ∇ whose Ricci tensor is skew-symmetric. Then, there is a one-parameter family of Einstein metrics*

$$\bar{g} = a\tilde{g} + 2a\theta^2, \quad a > 0,$$

as natural Riemann extensions $(T^*M, \bar{g})$. The Ricci tensor of each $\bar{g}$ is

$$\overline{\mathrm{Ric}} = \frac{n+1}{a}\bar{g}$$

and the scalar curvature is $2n(n+1)/a$.

Proof. If the Ricci tensor Ric of ∇ is skew-symmetric, then (4.2)–(4.4) can be written as

$$\overline{\mathrm{Ric}}(X^c, Y^c) = \frac{1}{2} \cdot \frac{4a + (n-1)b}{a^2}\bar{g}(X^c, Y^c),$$

$$\overline{\mathrm{Ric}}(X^c, \gamma^v) = \frac{1}{2} \cdot \frac{(n+1)b}{a^2}\bar{g}(X^c, \gamma^v),$$

$$\overline{\mathrm{Ric}}(\beta^v, \gamma^v) = 0 = \bar{g}(\beta^v, \gamma^v)$$

for all vector fields X, Y and one-forms β, γ of M. The metric $\bar{g}$ is Einstein if and only if $b = 2a$. As concerns the scalar curvature, see the formula (5.1) in the next Section. $\square$

Next we have

Theorem 4.2. *Let M be a manifold with a symmetric affine connection ∇, and let $(T^*M, \bar{g})$ be its cotangent bundle with a natural Riemann extension $\bar{g} = a\tilde{g} + b\theta^2$ where $a > 0$ and b are constants. Then, $(T^*M, \bar{g})$ is never Ricci flat if $b \neq 0$.*

For the Riemann extension, that is, a natural Riemann extension with $a = 1$ and $b = 0$, we have a well-known result (see for example Paterson–Walker [9]):

Theorem 4.3. *The Riemann extension of a manifold with symmetric affine connection ∇ is Ricci flat if and only if the Ricci tensor of ∇ is skew symmetric.*

5. The scalar curvature

Let $\overline{\mathrm{Sc}}(\bar g)$ be the scalar curvature of $\bar g$. Then we have

$$\overline{\mathrm{Sc}}(\bar g) = \sum_I \epsilon_I \overline{\mathrm{Ric}}(\boldsymbol{E}_I, \boldsymbol{E}_I) = \frac{2}{a}\sum_{i=1}^{n} \overline{\mathrm{Ric}}({E_i}^c, {\alpha_i}^v),$$

where $\epsilon_I = \bar g(\boldsymbol{E}_I, \boldsymbol{E}_I)$, $I = 1, 2, \dots, n, 1*, 2*, \dots, n*$. Now, using (4.3), we obtain

$$\overline{\mathrm{Sc}}(\bar g) = \frac{n(n+1)b}{a^2}. \tag{5.1}$$

Hence we obtain

Theorem 5.1. *Let (M, ∇) be a manifold with a symmetric affine connection ∇, and let $(T^*M, \bar g)$ be its cotangent bundle with a natural Riemann extension $\bar g = a\tilde g + b\theta^2$ where $a > 0$ and b are constants. Then, the scalar curvature $\overline{\mathrm{Sc}}(\bar g)$ of $\bar g$ is constant.*

For the Riemann extension, that is, a natural Riemann extension with $a = 1$ and $b = 0$, we have a well-known result (see for example Paterson-Walker [9]):

Theorem 5.2. *The Riemann extension of a manifold with a symmetric affine connection ∇ is a space of constant scalar curvature* 0.

Since b/a can take arbitrary constant, we have

Theorem 5.3. *Let M be a manifold with a symmetric affine connection ∇. Then, there exists a natural Riemann extension $(T^*M, \bar g)$ of (M, ∇) whose scalar curvature is a preassigned constant.*

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