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Volume form and its applications in Finsler geometry

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Abstract. We establish some volume comparison theorems for general volume forms, and they reduce to the same formulas as Riemannian case for extreme volume form (the maximal or minimal volume form) up to a cofactor. By using the extreme volume form, we are able to generalize Calabi–Yau's linear volume growth theorem, Milnor's results on curvature and fundamental group to Finsler manifolds. We also derive some McKean type estimations of the first eigenvalue for complete noncompact Finsler manifolds. Our results indicate that the extreme volume form is a good choice in comparison technique in Finsler geometry.

1. Introduction

Volume is an important geometric invariant in Riemannian geometry, and it is uniquely determined by the Riemannian metric. In Finsler geometry, however, there are different choices of volume forms. The frequently used volume forms are so-called Busemann–Hausdorff volume form and Holmes–Thompson volume form, and they are closely related to comparison theorems and the theory of Finsler submanifolds (see [9], [13], [14], [15], [16]). People select different volume form from different point of view. For example, for reversible Finsler manifolds, the Busemann–Hausdorff volume coincides the Hausdorff measure of the metric space

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induced by the Finsler metric (see [3], [4]). On the other hand, an example of [1] indicates that there are totally geodesic submanifolds which are not minimal for the Busemann–Hausdorff volume form, while all totally geodesic submanifolds must be minimal for the Holmes–Thompson volume form. So, the Holmes–Thompson volume form seems to be more advantage than Busemann–Hausdorff volume form when Finsler minimal submanifolds are discussed.

We considered the general volume form for Finsler manifold and then studied both the theory of submanifolds and comparison theorems for any given volume form in [17], [18], [19], [20]. In the present paper we would like to continue investigations in this direction. We establish some volume comparison theorems for general volume forms, and they reduce to the same formulas as Riemannian case for extreme volume form (the maximal or minimal volume form) up to a cofactor. We generalize Calabi–Yau's linear volume growth result [5], [23] to Finsler manifold and prove that with respect to the maximal volume form, any complete noncompact Finsler manifolds with non-negative Ricci curvature and finite reversibility must have infinite volume. Based on the volume comparison theorems for the maximal and minimal volume forms, we are able to obtain the Finsler version of Milnor's results on curvature and fundamental group. Our results remove the additional assumption on S-curvature which is needed in recent works (see e.g., [15], [20]). We also derive some McKean type estimations of the first eigenvalue for complete noncompact Finsler manifolds. We prove that with respect to extreme volume form, any complete noncompact and simply connected Finsler manifold with finite uniformity constant and flag curvature $\mathbf{K}(V;W) \leq c < 0$ has positive first eigenvalue. In summary, the extreme volume form is a good choice in comparison technique in Finsler geometry.

2. Finsler geometry

In this section, we give a brief description of basic quantities and fundamental formulas in Finsler geometry, for more details one is referred to see [7]. Throughout this paper, we shall use the Einstein convention, that is, repeated indices with one upper index and one lower index denotes summation over their range. Let (M,F) be a Finsler n-manifold with Finsler metric $F: TM \to [0,\infty)$. Let $(x,y) = (x^i, y^i)$ be the local coordinates on TM. Unlike in the Riemannian case, most Finsler quantities are functions of TM rather than M. For instance, the *fundamental tensor* g_{ij} and the *Cartan tensor* C_{ijk} are defined by

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}, \quad C_{ijk}(x,y) := \frac{1}{4}\frac{\partial^3 F^2(x,y)}{\partial y^i \partial y^j \partial y^k}.$$

Let $\Gamma^i_{jk}(x,y)$ be the Chern connection coefficients. Then the *first Chern curvature tensor* $R_j{}^i{}_{kl}$ can be expressed by

$$R_j{}^i{}_{kl} = \frac{\delta \Gamma^i_{jl}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^l} + \Gamma^i_{ks}\Gamma^s_{jl} - \Gamma^s_{jk}\Gamma^i_{ls},$$

where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - y^k \Gamma^j_{ik}\frac{\partial}{\partial y^j}$. Let $R_{ijkl} := g_{js}R_i{}^s{}_{kl}$, and write $\mathbf{g}_y = g_{ij}(x,y)dx^i \otimes dx^j, \mathbf{R}_y = R_{ijkl}(x,y)dx^i \otimes dx^j \otimes dx^k \otimes dx^l$. For a tangent plane $P \subset T_xM$, let

$$\mathbf{K}(P,y) = \mathbf{K}(y;u) := \frac{\mathbf{R}_y(y,u,u,y)}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,u) - [\mathbf{g}_y(y,u)]^2},$$

where $y, u \in P$ are tangent vectors such that $P = \mathrm{span}\{y,u\}$. We call $\mathbf{K}(P,y)$ the *flag curvature of* P *with flag pole* y. Let

$$\mathbf{Ric}(y) = \sum_i \mathbf{K}(y;e_i),$$

here $e_1, \ldots, e_n$ is a $\mathbf{g}_y$-orthogonal basis for the corresponding tangent space. We call $\mathbf{Ric}(y)$ the *Ricci curvature of* y.

Let $V = v^i\partial/\partial x^i$ be a non-vanishing vector field on an open subset $\mathcal{U} \subset M$. One can introduce a Riemannian metric $\widetilde{g} = \mathbf{g}_V$ and a linear connection ∇^V (called *Chern connection*) on the tangent bundle over $\mathcal{U}$ as follows:

$$\nabla^V_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} := \Gamma^k_{ij}(x,v)\frac{\partial}{\partial x^k}.$$

From the torsion freeness and $\mathbf{g}$-compatibility of Chern connection we have

$$\nabla^V_X Y - \nabla^V_Y X = [X,Y], \tag{2.1}$$

$$X \cdot \mathbf{g}_V(Y,Z) = \mathbf{g}_V(\nabla^V_X Y, Z) + \mathbf{g}_V(Y, \nabla^V_X Z) + 2\mathbf{C}_V(\nabla^V_X V, Y, Z), \tag{2.2}$$

here $\mathbf{C}_V = C_{ijk}(x,v)dx^i \otimes dx^j \otimes dx^k$, and it satisfies

$$\mathbf{C}_V(V,X,Y) = 0. \tag{2.3}$$

By (2.1)–(2.3) we see that the Chern connection ∇^V and the Levi–Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ are related by

$$\begin{aligned} \mathbf{g}_V(\nabla^V_X Y, Z) = \mathbf{g}_V(\widetilde{\nabla}_X Y, Z) &- \mathbf{C}_V(\nabla^V_X V, Y, Z) \\ &- \mathbf{C}_V(\nabla^V_Y V, X, Z) + \mathbf{C}_V(\nabla^V_Z V, X, Y). \end{aligned} \tag{2.4}$$

By (2.4) it is easy to see that $\nabla^V_V V = \widetilde{\nabla}_V V$, and consequently, V is a geodesic field of F if and only if it is a geodesic field of $\widetilde{g}$, and when V is a geodesic field, then $\nabla^V_V = \widetilde{\nabla}_V$, and for any plane P contain V, the flag curvature $\mathbf{K}(P,V)$ is just the sectional curvature $\widetilde{\mathbf{K}}(P)$ of $\widetilde{g}$ (see [12], [14]).

3. Volume form

A *volume form* $d\mu$ on Finsler manifold (M,F) is nothing but a global non-degenerate n-form on M. In local coordinates we can express $d\mu$ as $d\mu = \sigma(x)dx^1\wedge\cdots\wedge dx^n$. For $y\in T_xM\backslash 0$, define

$$\tau(y) := \log\frac{\sqrt{\det\left(g_{ij}(x,y)\right)}}{\sigma(x)}.$$

τ is called the *distortion* of $(M,F,d\mu)$. To measure the rate of distortion along geodesics, we define

$$\mathbf{S}(y) := \frac{d}{dt}\left[\tau(\dot\gamma(t))\right]_{t=0},$$

where $\gamma(t)$ is the geodesic with $\dot\gamma(0)=y$. $\mathbf{S}$ is called the *S-curvature* [14], [15].

The frequently used volume forms in Finsler geometry are so-called Busemann –Hausdorff volume form dV_{BH} and Holmes–Thompson volume form dV_{HT}. In local coordinates, dV_{BH} is expressed by

$$dV_{BH} = \sigma_{BH}(x)dx^1\wedge\cdots\wedge dx^n$$

with

$$\sigma_{BH}(x) := \frac{\mathrm{vol}(\mathbb{B}^n(1))}{\mathrm{vol}\left((y^i)\in\mathbb{R}^n : F(x,y^i\frac{\partial}{\partial x^i})<1\right)},$$

here $\mathbb{B}^n(1)$ denotes the Euclidean unit n-ball, and vol the standard Euclidean volume. On the other hand, the Holmes–Thompson volume form dV_{HT} is defined by

$$dV_{HT} = \sigma_{HT}(x)dx^1\wedge\cdots\wedge dx^n$$

with

$$\sigma_{HT}(x) = \frac{1}{V(S_xM)}\int_{S_xM}\sqrt{\det(g_{ij}(x,y))}dV_{S_xM},$$

here

$$dV_{S_xM} = \sqrt{\det(g_{ij}(x,y))}\sum_i(-1)^{i+1}\frac{y^i}{F}\frac{dy^1}{F}\wedge\cdots\wedge\widehat{\frac{dy^i}{F}}\wedge\cdots\wedge\frac{dy^n}{F}$$

is the induced volume form of $S_xM := \{y\in T_xM : F(x,y)=1\}$ from the Riemannian metric $\widehat{g} = g_{ij}(x,y)dy^i\otimes dy^j$ on the punctured tangent space $T_xM\backslash 0$, and

$$V(S_xM) = \int_{S_xM} dV_{S_xM}$$

is the corresponding volume of S_xM.

In the following we introduce the extreme volume form for Finsler manifold which plays an important role in the present paper. Let

$$dV_{\max} = \sigma_{\max}(x)dx^1 \wedge \cdots \wedge dx^n$$

and

$$dV_{\min} = \sigma_{\min}(x)dx^1 \wedge \cdots \wedge dx^n$$

with

$$\sigma_{\max}(x) := \max_{y\in T_xM\backslash 0} \sqrt{\det(g_{ij}(x,y))}, \quad \sigma_{\min}(x) := \min_{y\in T_xM\backslash 0} \sqrt{\det(g_{ij}(x,y))}.$$

Then it is easy to check that the n-forms $dV_{\max}$ and $dV_{\min}$ as well as the function $\nu := \frac{\sigma_{\max}}{\sigma_{\min}}$ are well-defined on M. We call $dV_{\max}$ and $dV_{\min}$ the *maximal volume form* and the *minimal volume form* of (M,F), respectively. Both maximal volume form and minimal volume form are called *extreme volume form*, and we shall denote by dV_{ext} the maximal or minimal volume form. Let $\mu : M \to \mathbb{R}$ be a function defined by

$$\mu(x) = \max_{y,z,u\in T_xM\backslash 0} \frac{\mathbf{g}_y(u,u)}{\mathbf{g}_z(u,u)}.$$

μ is called the *uniformity constant* [8]. It is clear that

$$\mu^{-1}F^2(u) \le \mathbf{g}_y(u,u) \le \mu F^2(u,u).$$

Proposition 3.1. *Let (M,F) be an n-dimensional Finsler manifold. Then*

(1) *F is Riemannian $\Leftrightarrow \nu = 1 \Leftrightarrow \mu = 1$;*

(2) *$\nu \le \mu^n$;*

(3) *Let $\tau_{\max}$ and $\tau_{\min}$ be the distortion of $dV_{\max}$ and $dV_{\min}$, respectively. Then*

$$-\log \nu \le \tau_{\max} \le 0 \le \tau_{\min} \le \log \nu.$$

Proof. (1) and (3) are obvious, here we only prove (2). For fixed $x \in M$, let $y, z \in T_xM\backslash 0$ be two vectors so that $\sigma_{\max}(x) = \sqrt{\det(g_{ij}(x,y))}$ and $\sigma_{\min}(x) = \sqrt{\det(g_{ij}(x,z))}$. Let $e_1, \ldots, e_n$ be an $\mathbf{g}_z$-orthogonal basis for T_xM such that they are eigenvectors of $(g_{ij}(x,y))$ with eigenvalues $\rho_1, \ldots, \rho_n$. Then

$$\rho_i = \mathbf{g}_y(e_i,e_i) \le \mu(x)\mathbf{g}_z(e_i,e_i) = \mu(x),$$

and consequently,

$$\nu(x) = \rho_1\rho_2\ldots\rho_n \le \mu(x)^n. \qquad \square$$

Example 3.2 (The Randers manifold). Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric on M, and $\beta = b_i(x)y^i$ the 1-from on M. It is well-know that $F = \alpha + \beta$ is a Finsler metric if and only if

$$\|\beta\|_\alpha(x) := \sup_{y \in T_xM\backslash 0} \frac{\beta(y)}{\alpha(y)} = \sqrt{a^{ij}b_ib_j} < 1, \quad (a^{ij}) = (a_{ij})^{-1}, \qquad \forall x \in M.$$

We call F a *Randers metric* on M, and call (M, F) a *Randers manifold.* Let dV_α be the Riemannian volume form of α, then $dV_{BH} = (1 - \|\beta\|_\alpha^2)^{\frac{n+1}{2}} dV_\alpha$ and $dV_{HT} = dV_\alpha$. Notice that

$$\det(g_{ij}) = \left(\frac{\alpha + \beta}{\alpha}\right)^{n+1} \det(a_{ij}),$$

the maximal volume form and the minimal volume form of Randers manifold are given by $dV_{\max} = (1 + \|\beta\|_\alpha)^{n+1} dV_\alpha$ and $dV_{\min} = (1 - \|\beta\|_\alpha)^{n+1} dV_\alpha$, respectively. Hence, $dV_{\min} \le dV_{BH} \le dV_{HT} \le dV_{\max}$.

4. The singular Riemannian metrics and polar coordinates

Let (M, F) be a Finsler manifold. Fix $p \in M$, let $I_p = \{v \in T_pM : F(v) = 1\}$ be the indicatrix at p. For $v \in I_p$, the *cut-value* $c(v)$ is defined by

$$c(v) := \sup\{t > 0 : d_F(p, \exp_p(tv)) = t\}.$$

Then, we can define the *tangential cut locus* $\mathbf{C}(p)$ of p by $\mathbf{C}(p) := \{c(v)v : c(v) < \infty,\ v \in I_p\}$, the *cut locus* $C(p)$ of p by $C(p) = \exp_p \mathbf{C}(p)$, and the *injectivity radius* i_p at p by $i_p = \inf\{c(v) : v \in I_p\}$, respectively. It is known that $C(p)$ has zero Hausdorff measure in M. Also, we set $\mathbf{D}_p = \{tv : 0 \le t < c(v),\ v \in I_p\}$ and $D_p = \exp_p \mathbf{D}_p$. It is known that $\mathbf{D}_p$ is the largest domain, which is starlike with respect to the origin of T_pM for which $\exp_p$ restricted to that domain is a diffeomorphism, and $D_p = M\backslash C(p)$.

Let $\hat{V}$ be the unit radial vector field on $T_pM\backslash\{0\}$ which is defined by $\hat{V}|_y = y/F(y), \forall y \in T_pM\backslash\{0\}$, here we have identified $T_y(T_pM)$ with T_pM in the natural way. The Finsler metric F induces a singular Riemannian metric $\hat{g} = \mathbf{g}_{\hat{V}}$ on $T_pM\backslash\{0\}$. Let $\theta^\alpha, \alpha = 1, \dots, n-1$ be the local coordinates that are intrinsic to I_p. The *polar coordinates* of $y \in T_pM\backslash\{0\}$ is $(r, \theta^1(u), \dots, \theta^{n-1}(u)) := (r, \theta)$, here $r = F(y), u = y/F(y)$. Consider the diffeomorphism $\Phi : (0, \infty) \times I_p \to T_pM\backslash\{0\}$ which is defined by $\Phi(r, u) = ru$. Then the polar coordinate vector fields are

$$d\Phi\left(\frac{\partial}{\partial r}\right) = \hat{V}, \quad d\Phi\left(\frac{\partial}{\partial \theta^\alpha}\right) = r\frac{\partial}{\partial \theta^\alpha}.$$

It is well-known that $\hat{V}$ is orthogonal to $\frac{\partial}{\partial\theta^\alpha}$ with respect to $\hat{g}$, and we can express $\hat{g}$ in terms of polar coordinates as

$$\hat{g} = dr^2 + r^2\dot{g}_{\alpha\beta}d\theta^\alpha d\theta^\beta, \quad \dot{g}_{\alpha\beta} = \mathbf{g}_{\hat{V}}\left(\frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\theta^\beta}\right),$$

and the induced Rirmannian metric on I_p is

$$\dot{g} = \dot{g}_{\alpha\beta}d\theta^\alpha d\theta^\beta.$$

The Riemannian volume forms of $\hat{g}$ and $\dot{g}$ are given by

$$dV_{\hat{g}} = r^{n-1}\sqrt{\det(\dot{g}_{\alpha\beta})}dr \wedge d\theta, \quad dV_{\dot{g}} = \sqrt{\det(\dot{g}_{\alpha\beta})}d\theta,$$

here $d\theta = d\theta^1 \wedge \cdots \wedge \theta^{n-1}$. Define the *density* Θ_p at $p \in M$ by [15]

$$\Theta_p = \frac{\mathrm{vol}_{\dot{g}}(I_p)}{\mathrm{vol}(\mathbb{S}^{n-1}(1))}.$$

Θ_p can be controlled by the uniformity constant as following.

Proposition 4.1. *The density Θ_p satisfies*

$$\frac{1}{\mu(p)^{\frac{n}{2}}} \le \Theta_p \le \mu(p)^{\frac{n}{2}},$$

Proof. Let $B_F = \{y \in T_pM : F(y) < 1\}$, then

$$\mathrm{vol}_{\hat{g}}(B_F) = \int_{B_F} dV_{\hat{g}} = \frac{1}{n}\mathrm{vol}_{\dot{g}}(I_p).$$

Recall that $\mathrm{vol}(\mathbb{B}^n(1)) = \frac{1}{n}\mathrm{vol}(\mathbb{S}^{n-1}(1))$, we get

$$\Theta_p = \frac{\mathrm{vol}_{\hat{g}}(B_F)}{\mathrm{vol}(\mathbb{B}^n(1))}.$$

Let $u \in I_p$ be a unit vector in T_pM such that

$$\sqrt{\det(g_{ij}(p,u))} = \max_{y \in I_p}\sqrt{\det(g_{ij}(p,y))},$$

namely, $dV_{\max} = dV_{\mathbf{g}_u}$. By the definition of uniformity constant, one can easily check that $B_F \subset B^n(\sqrt{\mu(p)})$, here $B^n(\sqrt{\mu(p)}) = \{y \in T_pM : \mathbf{g}_u(y,y) < \mu(p)\}$ denotes the ball of radius $\sqrt{\mu(p)}$ in T_pM with respect to $\mathbf{g}_u$. Hence,

$$\mathrm{vol}_{\hat{g}}(B_F) \le \mathrm{vol}_{\mathbf{g}_u}(B_F) \le \mathrm{vol}_{\mathbf{g}_u}(B^n(\sqrt{\mu(p)})) = \mu(p)^{\frac{n}{2}}\mathrm{vol}(\mathbb{B}^n(1)).$$

This proves that $\Theta_p \le \mu(p)^{\frac{n}{2}}$. Similarly we can verify that $\Theta_p \ge \frac{1}{\mu(p)^{\frac{n}{2}}}$. □

In the following we consider the polar coordinates on $D(p)$. For any $q \in D(p)$, the polar coordinates of q are defined by $(r, \theta) = (r(q), \theta^1(q), \ldots, \theta^{n-1}(q))$, where $r(q) = F(v), \theta^\alpha(q) = \theta^\alpha(u)$, here $v = \exp_p^{-1}(q)$ and $u = v/F(v)$. Then by the Gauss lemma (see [2], page 140), the unit radial coordinate vector $\partial r = d(\exp_p)\left(\frac{\partial}{\partial r}\right)$ is $\mathbf{g}_{\partial r}$-orthogonal to coordinate vectors ∂_α which is defined by

$$\partial_\alpha|_{\exp_p(ru)} = d(\exp_p)\left(\frac{\partial}{\partial\theta^\alpha}\right)\Bigg|_{\exp_p(ru)} = d(\exp_p)_{ru}\left(r\frac{\partial}{\partial\theta^\alpha}\right) = rd(\exp_p)_{ru}\left(\frac{\partial}{\partial\theta^\alpha}\right)$$

for $\alpha = 1, \ldots, n-1$, and consequently, $\nabla r = \partial r$. Consider the singular Riemannian metric $\widetilde{g} = \mathbf{g}_{\partial r}$ on $D(p)$, then it is clear that

$$\widetilde{g} = dr^2 + \widetilde{g}_{\alpha\beta}d\theta^\alpha d\theta^\beta, \quad \widetilde{g}_{\alpha\beta} = \mathbf{g}_{\partial r}(\partial_\alpha, \partial_\beta) = r^2\dot{\widetilde{g}}_{\alpha\beta},$$

$$\dot{\widetilde{g}}_{\alpha\beta} = \mathbf{g}_{\partial r}\left(d(\exp_p)_{ru}\left(\frac{\partial}{\partial\theta^\alpha}\right), d(\exp_p)_{ru}\left(\frac{\partial}{\partial\theta^\alpha}\right)\right).$$

For fixed $0 < r < i_p, \dot{\widetilde{g}} = \dot{\widetilde{g}}_{\alpha\beta}d\theta^\alpha d\theta^\beta$ can be viewed as a Riemannian metric on I_p. Recall that $d(\exp_p)_0 = id_{T_pM}$, we have $\dot{\widetilde{g}} \to \dot{g}(r \to 0)$ (see Lemma 3.1 in [15]). The volume form of $\widetilde{g}$ is given by

$$dV_{\widetilde{g}} = \widetilde{\sigma}(r, \theta)dr \wedge d\theta, \quad \widetilde{\sigma}(r, \theta) = r^{n-1}\sqrt{\det(\dot{\widetilde{g}}_{\alpha\beta})}. \tag{4.1}$$

5. Volume comparison theorems

In this section we shall obtain some volume comparison theorems for Finsler manifold which are different from some recent works (compare to [15], [20]). For this purpose, let us first recall some notations.

Given a Finsler manifold (M, F), the *dual Finsler metric* F^* on M is defined by

$$F^*(\xi_x) := \sup_{Y \in T_xM \setminus 0} \frac{\xi(Y)}{F(Y)}, \quad \forall \xi \in T^*M,$$

and the corresponding fundamental tensor is defined by

$$g^{*kl}(\xi) = \frac{1}{2}\frac{\partial^2 F^{*2}(\xi)}{\partial\xi_k\partial\xi_l}.$$

The *Legendre transformation* $l : TM \to T^*M$ is defined by

$$l(Y) = \begin{cases} \mathbf{g}_Y(Y, \cdot), & Y \neq 0 \\ 0, & Y = 0. \end{cases}$$

It is well-known that for any $x \in M$, the Legendre transformation is a smooth diffeomorphism from $T_xM\backslash 0$ onto $T_x^*M\backslash 0$, and it is norm-preserving, namely, $F(Y) = F^*(l(Y)), \forall Y \in TM$. Consequently, $g^{ij}(Y) = g^{*ij}(l(Y))$.

Now let $f : M \to \mathbb{R}$ be a smooth function on M. The *gradient* of f is defined by $\nabla f = l^{-1}(df)$. Thus we have

$$df(X) = \mathbf{g}_{\nabla f}(\nabla f, X), \quad X \in TM.$$

Let $\mathcal{U} = \{x \in M : \nabla f\mid_x \neq 0\}$. We define the *Hessian* $H(f)$ of f on $\mathcal{U}$ as follows:

$$H(f)(X, Y) := XY(f) - \nabla_X^{\nabla f} Y(f), \quad \forall X, Y \in TM\mid_{\mathcal{U}} .$$

It is known that $H(f)$ is symmetric, and it can be rewritten as (see [20])

$$H(f)(X, Y) = g_{\nabla f}(\nabla_X^{\nabla f} \nabla f, Y).$$

It should be noted that the notion of Hessian defined here is different from that in [Sh1-2]. In that case $H(f)$ is in fact defined by

$$H(f)(X, X) = X \cdot X \cdot (f) - \nabla_X^X X(f),$$

and there is no definition for $H(f)(X, Y)$ if $X \neq Y$.

In order to study the volume we need the following result which can be verified directly.

Lemma 5.1. *Let f, g are two positive integrable functions of r. Suppose that*

(1) *f/g is monotone increasing (resp. decreasing). Then the function*

$$\frac{\displaystyle\int_0^r f(t)dt}{\displaystyle\int_0^r g(t)dt}$$

is also monotone increasing (resp. decreasing).

(2) *f/g is monotone decreasing, then for any $0 < r < R$ the following holds:*

$$\frac{\displaystyle\int_0^r f(t)dt}{\displaystyle\int_0^r g(t)dt} \geq \frac{\displaystyle\int_r^R f(t)dt}{\displaystyle\int_r^R g(t)dt}.$$

Let $B_p(R)$ be the forward geodesic ball of M with radius R centered at p, and $d\mu$ a volume form of (M,F). By definition, $B_p(R) = r^{-1}([0,R))$, here $r = d_F(p,\cdot) : M \to \mathbb{R}$ is the distance function from p. The volume of $B_p(R)$ with respect to $d\mu$ is defined by

$$\mathrm{vol}(B_p(R)) = \int_{B_p(R)} d\mu.$$

For $r > 0$, let $\mathbf{D}_p(r) \subset I_p$ be defined by

$$\mathbf{D}_p(r) = \{v \in I_p : rv \in \mathbf{D}_p\}.$$

It is easy to see that $\mathbf{D}_p(r_1) \subset \mathbf{D}_p(r_2)$ for $r_1 > r_2$ and $\mathbf{D}_p(r) = I_p$ for $r < i_p$. Write $d\mu = \sigma(r,\theta)dr \wedge d\theta^1 \wedge \cdots \wedge \theta^{n-1} := \sigma(r,\theta)dr \wedge d\theta$. Since $C(p)$ has zero Hausdorff measure in M, we have

$$\begin{aligned}\mathrm{vol}(B_p(R)) &= \int_{B_p(R)} d\mu = \int_{B_p(R)\cap D_p} d\mu \\ &= \int_{\exp_p^{-1}(B_p(R))\cap \mathbf{D}_p} \exp_p^*(d\mu) = \int_0^R dr \int_{\mathbf{D}_p(r)} \sigma(r,\theta)d\theta. \qquad (5.1)\end{aligned}$$

Consider the Riemannian metric $\widetilde{g} = \mathbf{g}_{\partial r}$ on $\dot{B}_p(R) = B_p(R) \cap D_p \backslash \{p\}$ as defined in §4. The corresponding volume from of $\widetilde{g}$ is given by (4.1). Notice that the distortion of $d\mu$ along ∂r is

$$\tau(\partial r) = \log \frac{\widetilde{\sigma}}{\sigma},$$

which together with (5.1) yields

$$\begin{aligned}\mathrm{vol}(B_p(R)) &= \int_0^R dr \int_{\mathbf{D}_p(r)} \sigma(r,\theta)d\theta = \int_0^R dr \int_{\mathbf{D}_p(r)} e^{-\tau(\partial r)}\widetilde{\sigma}(r,\theta)d\theta \\ &\geq e^{-\Lambda} \int_0^R dr \int_{\mathbf{D}_p(r)} \widetilde{\sigma}(r,\theta)d\theta = e^{-\Lambda} \int_{B_p(R)} dV_{\widetilde{g}} = e^{-\Lambda}\mathrm{vol}_{\widetilde{g}}(B_p(R)), \qquad (5.2)\end{aligned}$$

here

$$\Lambda = \sup_{x\in B_p(R)} \tau(\partial r(x)).$$

Let

$$\mathrm{s}_c(t) = \begin{cases} \dfrac{\sin(\sqrt{c}t)}{\sqrt{c}}, & c > 0 \\ t, & c = 0 \\ \dfrac{\sinh(\sqrt{-c}t)}{\sqrt{-c}}, & c < 0, \end{cases} \qquad (5.3)$$

$$V_{c,n}(R) = \mathrm{vol}(\mathbb{S}^{n-1}(1)) \int_0^R \mathrm{s}_c(t)^{n-1} dt. \tag{5.4}$$

The geometric meaning of $V_{c,n}(R)$ is that it equals to $\mathrm{vol}(\mathbb{B}_c^n(R))$ when $R \leq i_c$, here $\mathbb{B}_c^n(R)$ denotes the geodesic ball of radius R in space form of constant c, and i_c the corresponding injectivity radius. Now we are ready to prove the following

Theorem 5.2. *Let $(M, F, d\mu)$ be a complete Finsler n-manifold which satisfies $\mathbf{K}(V;W) \leq c$ and $\tau \leq \Lambda$. Then*

$$\mathrm{vol}(B_p(R)) \geq e^{-\Lambda}\Theta_p \mathrm{vol}(\mathbb{B}_c^n(R))$$

for any $R \leq i_p$, here i_p is the injectivity radius of p.

Proof. Recall that $\partial r = \nabla r$ is a geodesic field, and

$$[\partial r, \partial_\alpha] = \left[d(\exp_p)\left(\frac{\partial}{\partial r}\right), d(\exp_p)\left(\frac{\partial}{\partial \theta^\alpha}\right)\right] = 0,$$

by (2.1) and (2.2) we have

$$\begin{aligned}\frac{\partial \widetilde{g}_{\alpha\beta}}{\partial r} &= \partial r \cdot \mathbf{g}_{\partial r}(\partial_\alpha, \partial_\beta) = \mathbf{g}_{\partial r}(\nabla^{\partial r}_{\partial r}\partial_\alpha, \partial_\beta) + \mathbf{g}_{\partial r}(\partial_\alpha, \nabla^{\partial r}_{\partial r}\partial_\beta) \\ &= \mathbf{g}_{\partial r}(\nabla^{\partial r}_{\partial_\alpha}\partial r, \partial_\beta) + \mathbf{g}_{\partial r}(\partial_\alpha, \nabla^{\partial r}_{\partial_\beta}\partial r) = 2H(r)(\partial_\alpha, \partial_\beta).\end{aligned}$$

Consequently,

$$\frac{\partial}{\partial r}\log\widetilde{\sigma} = \frac{1}{2}\widetilde{g}^{\alpha\beta}\frac{\partial \widetilde{g}_{\alpha\beta}}{\partial r} = \mathrm{tr}_{\mathbf{g}_{\partial r}} H(r).$$

Since $\mathbf{K}(V;W) \leq c$, by Hessian comparison theorem [20] it follows that

$$\frac{\partial}{\partial r}\log\widetilde{\sigma} \geq (n-1)\mathrm{ct}_c(r) = \frac{d}{dr}\log\left(\mathrm{s}_c(r)^{n-1}\right), \tag{5.5}$$

here

$$\mathrm{ct}_c(r) = \begin{cases} \sqrt{c}\cdot\mathrm{cotan}(\sqrt{c}r), & c > 0 \\ \dfrac{1}{r}, & c = 0 \\ \sqrt{-c}\cdot\mathrm{cotanh}(\sqrt{-c}r), & c < 0. \end{cases}$$

From (5.5) we see that the function

$$\frac{\displaystyle\int_{I_p}\widetilde{\sigma}(r,\theta)d\theta}{\mathrm{vol}(\mathbb{S}^{n-1})\mathrm{s}_c(r)^{n-1}}$$

is monotone increasing about $r(\leq i_p)$, and thus by Lemma 5.1 (1) the function

$$\frac{\displaystyle\int_0^R \int_{I_p} \widetilde{\sigma}(r,\theta)drd\theta}{\displaystyle\mathrm{vol}(\mathbb{S}^{n-1})\int_0^R \mathrm{s}_c(r)^{n-1}dr} = \frac{\mathrm{vol}_{\widetilde{g}}(B_p(R))}{\mathrm{vol}(\mathbb{B}_c^n(R))}$$

is also monotone increasing for $R \leq i_p$. Using (4.1), and noticing that $\dot{\widetilde{g}} \to \dot{g}(R \to 0)$, we have

$$\begin{aligned}
\lim_{R\to 0} \frac{\mathrm{vol}_{\widetilde{g}}(B_p(R))}{\mathrm{vol}(\mathbb{B}_c^n(R))} &= \lim_{R\to 0} \frac{\displaystyle\int_0^R r^{n-1}dr \int_{I_p} \sqrt{\det(\dot{\widetilde{g}}_{\alpha\beta})}d\theta}{\displaystyle\mathrm{vol}(\mathbb{S}^{n-1})\int_0^R \mathrm{s}_c(r)^{n-1}dr} \\
&= \lim_{R\to 0} \frac{\displaystyle R^{n-1}\int_{I_p} \sqrt{\det(\dot{\widetilde{g}}_{\alpha\beta})}d\theta}{\mathrm{vol}(\mathbb{S}^{n-1})\mathrm{s}_c(R)^{n-1}} = \lim_{R\to 0} \frac{\displaystyle R^{n-1}\int_{I_p} \sqrt{\det(\dot{g}_{\alpha\beta})}d\theta}{\mathrm{vol}(\mathbb{S}^{n-1})\mathrm{s}_c(R)^{n-1}} \\
&= \frac{\mathrm{vol}_{\dot{g}}(I_p)}{\mathrm{vol}(\mathbb{S}^{n-1})} \lim_{R\to 0} \frac{R^{n-1}}{\mathrm{s}_c(R)^{n-1}} = \Theta_p,
\end{aligned}$$

thus it follows from (5.2) that

$$\mathrm{vol}(B_p(R)) \geq e^{-\Lambda}\mathrm{vol}_{\widetilde{g}}(B_p(R)) \geq e^{-\Lambda}\Theta_p \mathrm{vol}(\mathbb{B}_c^n(R)),$$

and so we are done. □

The following two theorems can be deduced similarly by using Lemma 5.1 (1) and comparison results for $\mathrm{tr}_{\mathbf{g}_{\partial r}} H(r)$ (see the proofs of Theorems 5.2 and 5.3 in [20]).

Theorem 5.3. *Let $(M, F, d\mu)$ be a complete and simply connected Finsler n-manifold with nonpositive flag curvature. If the Ricci curvature of M satisfies $\mathbf{Ric}_M \leq c < 0$ and $\tau \leq \Lambda$, then*

$$\mathrm{vol}(B_p(R))) \geq e^{-\Lambda}\frac{\mathrm{vol}_{\widetilde{g}}(B_p(1))}{\mathrm{vol}(\mathbb{B}_c^2(1))}\mathrm{vol}(\mathbb{B}_c^2(R)), \quad \forall R \geq 1.$$

Theorem 5.4. *Let $(M, F, d\mu)$ be a complete Finsler n-manifold. Suppose that*

$$\mathbf{Ric}_M \geq (n-1)c, \quad \tau \geq \Lambda.$$

Then

$$\mathrm{vol}(B_p(R)) \leq e^{-\Lambda}\Theta_p \mathrm{vol}(\mathbb{B}_c^n(R)).$$

Compare Theorems 5.2–5.4 to corresponding results in [15], [20], we replace the boundedness of the S-curvature by the boundedness of the distortion. Furthermore, since by Proposition 3.1, the distortion $\tau_{\max}$ of the maximal volume form is non-positive, while the distortion $\tau_{\min}$ of the minimal volume form is non-negative, and Θ_p is controlled by $\mu(p)$, we have the following volume comparison theorem which remove the assumption on S-curvature (compare to the recent works of [15], [20]).

Theorem 5.5. *Let (M,F) be a complete Finsler n-manifold. We have*

(1) *If the flag curvature of M satisfies $\mathbf{K}(V;W) \leq c$, then*

$$\mathrm{vol}_{\max}(B_p(R)) \geq \frac{1}{\mu(p)^{\frac{n}{2}}} \cdot \mathrm{vol}(\mathbb{B}^n_c(R))$$

for any $R \leq i_p$;

(2) *If the flag curvature of M is non-positive, the Ricci curvature of M satisfies $\mathbf{Ric}_M \leq c < 0$, and M is simply connected, then*

$$\mathrm{vol}_{\max}(B_p(R)) \geq \frac{\mathrm{vol}_{\widetilde{g}}(B_p(1))}{\mathrm{vol}(\mathbb{B}^2_c(1))} \mathrm{vol}(\mathbb{B}^2_c(R)), \quad \forall R \geq 1;$$

(3) *If the Ricci curvature of M satisfies $\mathbf{Ric}_M \geq (n-1)c$, then*

$$\mathrm{vol}_{\min}(B_p(R)) \leq \mu(p)^{\frac{n}{2}} \cdot \mathrm{vol}(\mathbb{B}^n_c(R)).$$

Here $\mathrm{vol}_{\max}$ and $\mathrm{vol}_{\min}$ are the volume with respect to $dV_{\max}$ and $dV_{\min}$, respectively.

A theorem due to Calabi and Yau states that the volume of any complete noncompact Riemannian manifold with nonnegative Ricci curvature has at least linear growth [5], [23]. Using the extreme volume form, we can generalize this result to Finsler manifolds. For this purpose we need the notion of reversibility for Finsler manifolds. For a given Finsler manifold (M,F), the *reversibility* λ_F of (M,F) is defined by (see [12])

$$\lambda_F = \max_{X \in TM \setminus 0} \frac{F(X)}{F(-X)}.$$

(M,F) is called *reversible* if $\lambda_F = 1$. It is clear that the induced distance function d_F of F satisfies

$$d_F(p,q) \leq \lambda_F d_F(q,p), \quad \forall p,q \in M. \tag{5.6}$$

Now we can prove

Theorem 5.6. *Let (M, F) be a complete noncompact Finsler manifold with nonnegative Ricci curvature and finite reversibility, then the volume* $\mathrm{vol}_{\max}(B_p(R))$ *of the forward geodesic ball has at least linear growth:*

$$\mathrm{vol}_{\max}(B_p(R)) \geq c(p)R. \tag{5.7}$$

Consequently, with respect to the maximal volume form, any complete noncompact Finsler manifolds with non-negative Ricci curvature and finite reversibility constant must have infinite volume.

PROOF. Since M is complete and noncompact, there is a geodesic $\gamma : (-\infty, 0] \to M$ such that $\gamma(0) = p, d_F(\gamma(-t_2), \gamma(-t_1)) = t_2 - t_1, \forall t_2 > t_1 > 0$. By (5.6) and the triangle inequality it is easy to see that

$$B_p(1) \subset B_{\gamma(-t)}(t+1) \setminus B_{\gamma(-t)}(t - \lambda_F), \quad \forall t > \lambda_F. \tag{5.8}$$

For fixed $t > \lambda_F$, consider the Riemannian metric $\widetilde{g} = \mathbf{g}_V$ on $M \backslash (\{p\} \cup C(p))$, here $V = \nabla r$ with $r = d_F(\gamma(-t), \cdot)$. Let (r, θ) be the polar coordinates centered at $\gamma(-t)$, and write $dV_{\widetilde{g}} = \widetilde{\sigma}(r, \theta) dr \wedge d\theta$ as before. Since $\mathbf{Ric}_M \geq 0$, the function

$$\frac{\displaystyle\int_{I_{\gamma(-t)}} \widetilde{\sigma}(r, \theta) d\theta}{r^{n-1}}$$

is monotone decreasing about r, thus by Lemma 5.1 (2) we see that

$$\mathrm{vol}_{\widetilde{g}}(B_{\gamma(-t)}(r)) \geq \frac{r^n}{R^n - r^n} (\mathrm{vol}_{\widetilde{g}}(B_{\gamma(-t)}(R)) - \mathrm{vol}_{\widetilde{g}}(B_{\gamma(-t)}(r))) \tag{5.9}$$

holds for all $R > r > 0$. (5.8) and (5.9) yields

$$\begin{aligned}
\mathrm{vol}_{\max}&(B_{\gamma(-t)}(t-1)) \geq \mathrm{vol}_{\max}(B_{\gamma(-t)}(t - \lambda_F)) \geq \mathrm{vol}_{\widetilde{g}}(B_{\gamma(-t)}(t - \lambda_F)) \\
&\geq \frac{(t-\lambda_F)^n}{(t+1)^n - (t-\lambda_F)^n} (\mathrm{vol}_{\widetilde{g}}(B_{\gamma(-t)}(t+1)) - \mathrm{vol}_{\widetilde{g}}(B_{\gamma(-t)}(t - \lambda_F))) \\
&\geq \frac{(t-\lambda_F)^n}{(t+1)^n - (t-\lambda_F)^n} \mathrm{vol}_{\widetilde{g}}(B_p(1)) \geq \frac{(t-\lambda_F)^n}{(t+1)^n - (t-\lambda_F)^n} \mathrm{vol}_{\min}(B_p(1)).
\end{aligned}$$

Since

$$\lim_{t \to +\infty} \frac{(t-\lambda_F)^n}{t((t+1)^n - (t-\lambda_F)^n)} = \frac{1}{n(1+\lambda_F)},$$

there is a constant $\delta > 0$ such that

$$\frac{(t-\lambda_F)^n}{(t+1)^n - (t-\lambda_F)^n} \geq \delta t, \quad \forall t > \lambda_F,$$

and consequently,

$$\mathrm{vol}_{\max}(B_{\gamma(-t)}(t-1)) \geq \delta t \cdot \mathrm{vol}_{\min}(B_p(1)).$$

On the other hand, let $B_p^-(r) = \{x \in M : d_F(x,p) < r\}$ be the backward geodesic ball of radius r centered at p, by (5.6) and the triangle inequality we easily see that

$$B_{\gamma(-t)}(t-1) \subset B^-_{\gamma(-t)}(\lambda_F(t-1)) \subset B_p^-(2\lambda_F t) \subset B_p(2\lambda_F^2 t),$$

and thus

$$\mathrm{vol}_{\max}(B_p(2\lambda_F^2 t)) \geq \mathrm{vol}_{\max}(B_{\gamma(-t)}(t-1)) \geq \delta t \cdot \mathrm{vol}_{\min}(B_p(1)) := c(p) \cdot 2\lambda_F^2 t,$$

here $c(p)$ is a constant depending on p. Letting $R = 2\lambda_F^2 t$ we obtain (5.7). □

6. Curvature and fundamental group

In this section we shall use the volume comparison theorems to derive the Finsler version of Milnor's results on curvature and fundamental group. In 1968 Milnor [11] studied the curvature and fundamental group of Riemannian manifold and obtained two estimations for the growth order of fundamental group. The key in the proof is that the fundamental group can be identified with the deck transformation group of the universal covering space, and any geodesic ball in the universal covering space can be covered by the union of a number of translate of the fundamental domain. Combining with the estimate of the volume growth Milnor was able to obtain his results. His results were generalized in [21], [22]. The Finsler version of Milnor's results were obtained by [15] and recently by [20], but an additional assumption on S-curvature was required there. By Theorem 5.5 we can remove this additional assumption, namely, we have the following Finsler version of Milnor's results:

Theorem 6.1. *Let (M,F) be a complete Finsler n-manifold with nonnegative Ricci curvature and bounded uniformity constant. Then the fundamental group of M has polynomial growth of order $\leq n$.*

Theorem 6.2. *Let (M,F) be a compact Finsler n-manifold. Suppose that one of the following two conditions holds:*

(i) *the flag curvature of M satisfies $\mathbf{K}(V;W) \leq c < 0$;*

(ii) *M has nonpositive flag curvature and $\mathbf{Ric}_M \leq c < 0$.*

Then the fundamental group of M grows at least exponentially.

7. McKean type theorems for the first eigenvalue

In this section we shall study the first eigenvalue on Finsler manifolds and prove some McKean type theorems. Let us first recall the definition of the first eigenvalue for non-compact Finsler manifolds. Let $(M, F, d\mu)$ be a Finsler n-manifold, $\Omega \subset M$ a domain with compact closure and nonempty boundary $\partial\Omega$. The first eigenvalue $\lambda_1(\Omega)$ of Ω is defined by (see [14], p. 176)

$$\lambda_1(\Omega) = \inf_{f \in L^2_{1,0}(\Omega)\backslash\{0\}} \left\{ \frac{\int_\Omega (F^*(df))^2 \, d\mu}{\int_\Omega f^2 d\mu} \right\},$$

where $L^2_{1,0}(\Omega)$ is the completion of C^∞_0 with respect to the norm

$$\|\varphi\|^2_\Omega = \int_\Omega \varphi^2 d\mu + \int_\Omega (F^*(d\varphi))^2 \, d\mu.$$

If $\Omega_1 \subset \Omega_2$ are bounded domains, then $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2) \geq 0$. Thus, if $\Omega_1 \subset \Omega_2 \subset \cdots \subset M$ be bounded domains so that $\bigcup \Omega_i = M$, then the following limit

$$\lambda_1(M) = \lim_{i\to\infty} \lambda_1(\Omega_i) \geq 0$$

exists, and it is independent of the choice of $\{\Omega_i\}$.

Now let $B_p(R)$ be the forward geodesic ball of M with radius R centered at p, and $R < i_p$, where i_p denotes the injectivity radius about p. For $R > \varepsilon > 0$, let $\Omega_\varepsilon = B_p(R)\backslash\overline{B_p(\varepsilon)}$. Then $r = d_F(p, \cdot)$ is smooth on Ω_ε, and thus $V = \nabla r$ is a unit geodesic vector field on Ω_ε, and we can consider the Riemannian metric $\widetilde{g} = \mathbf{g}_V$ on Ω_ε. Since the Legendre transformation $l : TM \to T^*M$ is norm-preserving, and thus it also preserves the uniformity constant. Hence, for any $f \in C^\infty_0(\Omega_\varepsilon)$,

$$\begin{aligned}(F^*(df))^2(x) &= g^{*ij}(x, df)\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} \geq \frac{1}{\mu^*(x)} g^{*ij}(x, l(V(x)))\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} \\ &= \frac{1}{\mu(x)} g^{ij}(x, V(x))\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} = \frac{1}{\mu(x)}\|df\|^2_{\widetilde{g}}(x). \end{aligned} \tag{7.1}$$

Using (7.1), we get

$$\frac{\int_{\Omega_\varepsilon} (F^*(df))^2 \, d\mu}{\int_{\Omega_\varepsilon} f^2 d\mu} = \frac{\int_{\Omega_\varepsilon} e^{-\tau(V)}(F^*(df))^2 dV_{\widetilde{g}}}{\int_{\Omega_\varepsilon} e^{-\tau(V)} f^2 dV_{\widetilde{g}}} \geq \frac{1}{\Lambda e^{\Xi}} \frac{\int_{\Omega_\varepsilon} \|df\|^2_{\widetilde{g}} dV_{\widetilde{g}}}{\int_{\Omega_\varepsilon} f^2 dV_{\widetilde{g}}},$$

here

$$\Lambda = \max_{x \in B_p(R)} \mu(x), \quad \Xi = \max_{y_1, y_2 \in TB_p(R) \setminus 0} (\tau(y_1) - \tau(y_2)). \tag{7.2}$$

As the result, we have

$$\lambda_1(\Omega_\varepsilon) \geq \frac{1}{\Lambda e^{\Xi}} \widetilde{\lambda}_1(\Omega_\varepsilon), \tag{7.3}$$

where $\widetilde{\lambda}_1(\Omega_\varepsilon)$ is the first eigenvalue of Ω_ε with respect to $\widetilde{g}$. Now we can prove

Theorem 7.1. *Let $(M, F, d\mu)$ be a Finsler n-manifold whose flag curvature satisfies $\mathbf{K}(V; W) \leq c$ for any $V, W \in TM$. Let $B_p(R)$ be the forward geodesic ball of M with radius R centered at p, and $R < i_p$, where i_p denotes the injectivity radius about p. Then*

$$\lambda_1(B_p(R)) \geq \frac{(n-1)^2 (\mathrm{ct}_c(R))^2}{\Lambda e^{\Xi}},$$

here

$$\mathrm{ct}_c(t) = \begin{cases} \sqrt{c} \cdot \mathrm{cotan}(\sqrt{c}t), & c > 0 \\ \dfrac{1}{t}, & c = 0 \\ \sqrt{-c} \cdot \mathrm{cotanh}(\sqrt{-c}t), & c < 0, \end{cases}$$

and Λ, Ξ are given by (7.2). In particular, when $d\mu = dV_{\max}$ or $d\mu = dV_{\min}$,

$$\lambda_1(B_p(R)) \geq \frac{(n-1)^2 (\mathrm{ct}_c(R))^2}{\Lambda^{n+1}}.$$

Proof. First we recall that $V = \nabla r$ is also a unit geodesic vector field on M with respect to $\widetilde{g}$, as we have pointed out in the end of §2. From the define of gradient,

$$dr(X) = \mathbf{g}_V(V, X) = \widetilde{g}(V, X) = \widetilde{g}(\widetilde{\nabla} r, X),$$

namely, $\nabla r = \widetilde{\nabla} r$, here $\widetilde{\nabla} r$ is the gradient of r with respect to $\widetilde{g}$. Furthermore, by (2.3) and (2.4) we see that $\nabla^V_X V = \widetilde{\nabla}_X V$ for any $X \in TM$, and thus

$$\widetilde{H}(r)(X, Y) = \mathbf{g}_V(\widetilde{\nabla}_X V, Y) = \mathbf{g}_V(\nabla^V_X V, Y) = H(r)(X, Y),$$

here $\widetilde{H}$ is the Hessian of $\widetilde{g}$. Let $\widetilde{\Delta}$ and $\widetilde{\mathrm{div}}$ be the Laplacian and divergence with respect to $\widetilde{g}$, respectively. Since $\mathbf{K}(V; W) \leq c$ for any $V, W \in TM$, the Hessian comparison theorem in [20] yields

$$\widetilde{\Delta} r = \widetilde{\mathrm{div}} \widetilde{\nabla} r = \mathrm{tr}_{\widetilde{g}} \widetilde{H}(r) = \mathrm{tr}_{\mathbf{g}_V} H(r) \geq (n-1)\mathrm{ct}_c(r),$$

by applying Lemma 7.2 of [20] to vector field V on Ω_ε with respect to $\widetilde{g}$ we get

$$\widetilde{\lambda}_1(\Omega_\varepsilon) \geq (n-1)^2 (\mathrm{ct}_c(R))^2.$$

Now letting $\varepsilon \to 0$, by (7.3) and Proposition 3.1 we easily get the result. $\square$

By Theorem 7.1 we can generalize MCKEAN's result [10] into Finsler manifolds as following.

Theorem 7.2. *Let $(M, F, d\mu)$ be a complete noncompact and simply connected Finsler n-manifold with finite unifirmity constant $\mu \leq \Lambda$ and flag curvature $\mathbf{K}(V; W) \leq -a^2$ $(a > 0)$. If $\sup_{y_1, y_2 \in TM\setminus 0}(\tau(y_1) - \tau(y_2)) \leq \Xi$, then*

$$\lambda_1(M) \geq \frac{(n-1)^2 a^2}{\Lambda e^{\Xi}}.$$

In particular, when $d\mu = dV_{\max}$ or $d\mu = dV_{\min}$,

$$\lambda_1(M) \geq \frac{(n-1)^2 a^2}{\Lambda^{n+1}}.$$

Corollary 7.3. *With respect to the extreme volume form, any complete noncompact and simply connected Finsler manifold with finite uniformity constant and flag curvatureq $\mathbf{K}(V; W) \leq c < 0$ has positive first eigenvalue.*

The following result can be viewed as another Finsler version of McKean's theorem in term of the Ricci curvature which can be verified similarly as Theorem 7.1.

Theorem 7.4. *Let $(M, F, d\mu)$ be a complete noncompact and simply connected Finsler n-manifold with finite uniformity constant $\mu \leq \Lambda$ and nonpositive flag curvature. If $\mathbf{Ric}_M \leq -a^2 (a > 0)$ and $\sup_{y_1, y_2 \in TM\setminus 0}(\tau(y_1) - \tau(y_2)) \leq \Xi$, then*

$$\lambda_1(M) \geq \frac{a^2}{\Lambda e^{\Xi}}.$$

In particular, when $d\mu = dV_{\max}$ or $d\mu = dV_{\min}$,

$$\lambda_1(M) \geq \frac{a^2}{\Lambda^{n+1}}.$$

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