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Algebras with orthogonal multiplication

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Introduction

In this paper we study real algebras, equipped also with an orthogonality relation, with the property that xy is both left and right orthogonal to x and y. The most known example of such an algebra is the 3-dimensional vector space with the vector product as multiplication. There are also other interesting examples which can be constructed from square matrices, Hilbert–Schmidt operators, flexible quadratic algebras and real associative algebras.

Algebras with orthogonal multiplication are always anticommutative and they are associative only in the trivial case. In this paper we focus our attention on algebras in which orthogonality arises from the inner product and most of the results are about such algebras. It is interesting that the multiplication in such a Hilbert algebra is automatically continuous and that such an algebra can be embedded into a quadratic Banach algebra. We also give a structure theorem for a certain subclass of Hilbert algebras with orthogonal multiplication.

1. Definitions

Let $\mathcal{A}$ be a real algebra and suppose that $\perp$ is an orthogonality relation on $\mathcal{A}$. The most well-known orthogonality is the one induced by the Hilbert space structure. However there are many papers devoted to the study of orthogonality relations in general Banach spaces and abstract orthogonalities. Among others we mention Birkhoff–James, Singer, Pythagorean and isosceles orthogonality. Some historically important papers on this topic are [1], [3], [4] and [5] while some recent papers are [6]–[8] and [10]–[12] where further references are available for the interested reader.

Different authors use slightly different definitions on what is an abstract orthogonality. In this paper we use the following concept:

Definition 1.1. Let $\perp$ be a binary relation on $\mathcal{A}$. Then $\mathcal{A}$ is a called an *algebra with orthogonal multiplication* if

(i) ${}^{\perp}\{x\} = \{y \in \mathcal{A}; y\perp x\}$ and $\{x\}^{\perp} = \{y \in \mathcal{A}; x\perp y\}$ are linear subspaces of $\mathcal{A}$. The first is called the *left orthogonal complement* and the second the *right orthogonal complement* of x.

(ii) ${}^{\perp}\{x\} = \mathcal{A}$ if and only if $x = 0$.

(iii) $\{x\}^{\perp} = \mathcal{A}$ if and only if $x = 0$.

(iv) For all $x, y \in \mathcal{A}$, we have $xy\perp x$, $xy\perp y$, $x\perp xy$ and $y\perp xy$.

Remark 1.2. We do not assume that the orthogonality relation is symmetric. Also we allow $x\perp x$ for nonzero $x \in \mathcal{A}$. However $\perp$ is both left and right nondegenerate by (ii) and (iii). We shall use the abbreviation OM-algebra instead of algebra with orthogonal multiplication.

2. Examples

Example 2.1. Let $\mathcal{A}$ be a real vector space. Define $xy = 0$ for all $x, y \in \mathcal{A}$ and $x\perp y$ if and only if $x = 0$ or $y = 0$. Then $\mathcal{A}$ is an OM-algebra which will be called a *trivial algebra*.

Example 2.2. Let $\mathcal{A} = \mathbb{R}^3$ be equipped with the usual inner product and vector product as multiplication. This Hilbert OM-algebra will be called a *classical algebra*. If we denote $i = (1, 0, 0)$, $j = (0, 1, 0)$ and $k = (0, 0, 1)$, then the multiplication table of $\mathcal{A}$ is the following:

$\cdot$	i	j	k
i	0	k	$-j$
j	$-k$	0	i
k	j	$-i$	0

Example 2.3. Let $n \geq 3$ and let $\mathcal{A}_n$ be the vector space of those real $n \times n$ matrices which are antisymmetric i.e. satisfy the identity $A^T = -A$ where A^T denotes the transpose matrix. If $A \circ B$ is the usual product of matrices A and B, the Lie product is defined by $AB = A \circ B - B \circ A$. In fact

$$(AB)^T = (A \circ B - B \circ A)^T = B^T \circ A^T - A^T \circ B^T =$$
$$= (-B) \circ (-A) - (-A) \circ (-B) = B \circ A - A \circ B = -AB.$$

We define the inner product in $\mathcal{A}_n$ by $\langle A, B\rangle = -\operatorname{trace}(A \circ B)$. Then we have

$$\langle AB, A\rangle = -\operatorname{trace}((A \circ B) \circ A) + \operatorname{trace}((B \circ A) \circ A) =$$
$$= -\operatorname{trace}((A \circ B) \circ A) + \operatorname{trace}(A \circ (B \circ A)) = 0.$$

Example 2.4. The above example can be generalized to the infinite dimensional case. Let Λ be some index set. Consider $A : \Lambda \times \Lambda \to \mathbb{R}$ such that

$$\sum_{i,j} |A(i,j)|^2 < \infty$$

with componentwise summation and multiplication by scalars. Define

$$\begin{aligned} A^T(i,j) &= A(j,i), \\ (A \circ B)(i,j) &= \sum_k A(i,k)B(k,j), \\ \langle A, B\rangle &= \sum_{i,j} A(i,j)B(i,j). \end{aligned}$$

Then $\mathcal{A}_\Lambda = \{A; A^T = -A\}$ together with the product $AB = A \circ B - B \circ A$ is a Hilbert OM-algebra.

Example 2.5. Let $\mathcal{B}$ be a (nonassociative) algebra with the identity e such that for each $x \in \mathcal{B}$ there exist $\alpha, \beta \in \mathbb{R}$ with $x^2 + \alpha x + \beta = 0$. Such algebras are called *quadratic* (see [8] page 49). Examples of such algebras are the algebras obtained via the Cayley–Dickson process (see [8] page 45). Suppose that $\mathcal{B}$ is also flexible i.e. $(xy)x = x(yx)$ holds for all $x, y \in \mathcal{B}$.

Define $\mathcal{A} = \{x \in \mathcal{B};\ x = 0 \text{ or } (x \notin \mathbb{R}e \text{ and } x^2 \in \mathbb{R}e)\}$. Then $\mathcal{A}$ is a linear vector space with $\mathcal{B} = \mathbb{R}e \oplus \mathcal{A}$. If $x, y \in \mathcal{A}$, then $xy = B(x,y)e + x \circ y$ defines a bilinear form $B : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ and a product $\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. If we assume in addition that $B(x,y)$ is nondegenerate in both factors, then we can define an OM-algebra $(\mathcal{A}, \circ, \perp)$ with $x \perp y$ if and only if $B(x,y) = 0$. Orthogonality in this case is automatically symmetric.

The proof of this statement can be done by some simple calculations. Take some $x, y \in \mathcal{A}$. Since $\mathcal{B}$ is quadratic $(x+y)^2 + \alpha(x+y) \in \mathbb{R}e$ for some real α. Thus

$$xy + yx + \alpha x + \alpha y \in \mathbb{R}e. \tag{1}$$

Also $(x-y)^2 + \beta(x-y) \in \mathbb{R}e$ for some real β and thus

$$-xy - yx + \beta x - \beta y \in \mathbb{R}e. \tag{2}$$

If we add (1) and (2) we obtain $(\alpha+\beta)x + (\alpha-\beta)y \in \mathbb{R}e$. Suppose first that $\{x, y, e\}$ are linearly independent. Then $\alpha + \beta = \alpha - \beta = 0 = \alpha = \beta$ and so $x + y \in \mathcal{A}$. This further implies

$$xy + yx \in \mathbb{R}e. \tag{3}$$

If $y = \gamma x + \delta e$, the fact that $y^2 \in \mathbb{R}e$ implies $\gamma^2 x^2 + 2\delta\gamma x + \delta^2 e \in \mathbb{R}e$ and since $x^2 \in \mathbb{R}e$ we have $2\delta\gamma x = 0$. If $x = 0$, then $y = 0$ since $y \notin \mathbb{R}e$ and there is nothing to prove. If $x \neq 0$, then $\delta = 0$ or $\gamma = 0$ and since the second equality is not possible, we have $y = \gamma x$ and thus $x + y \in \mathcal{A}$ and $xy + yx \in \mathbb{R}e$. Note that (3) also implies $x \circ y + y \circ x = 0$.

Let x be nonzero. Then we have, using the flexibility of $\mathcal{B}$,

$$(xy)x = x(yx),$$
$$(B(x, y)e + x \circ y)x = x(B(y, x)e + y \circ x),$$
$$B(x, y)x+B(x \circ y, x)e + (x \circ y) \circ x = B(y, x)x + B(x, y \circ x)e+x \circ (y \circ x).$$

Since $x \circ (y \circ x) = -(y \circ x) \circ x = (x \circ y) \circ x$, we obtain

$$(B(x, y) - B(y, x))x = (B(x, y \circ x) - B(x \circ y, x))e.$$

Since $x \notin \mathbb{R}e$, we finally get $B(x, y) = B(y, x)$ and

$$B(x \circ y, x) = B(x, y \circ x) = B(y \circ x, x) = -B(x \circ y, x) = 0.$$

If $x = 0$, the above equality obviously holds.

Example 2.6. Let $\mathcal{A}$ be an associative real algebra. Define $\mathrm{Comm}(\mathcal{A})$ to be a linear subspace spanned by all commutators and $\mathrm{Der}(\mathcal{A})$ a linear subspace spanned by the images of all linear derivations acting on $\mathcal{A}$. If $\mathrm{Comm}(A) \subset \mathcal{J} \subset \mathrm{Der}(\mathcal{A})$, then we can define an orthogonality relation $a \perp b$ if and only if $ab \in \mathcal{J}$. Then $a \perp b$ implies $b \perp a$ since $ba - ab \in \mathcal{J}$ and so $ba = (ba - ab) + ab \in \mathcal{J}$. Define a new product $a \circ b = ab - ba$ on $\mathcal{A}$. Then $(\mathcal{A}, \circ, \perp)$ is an OM-algebra since

$$(a \circ b)a = aba - ba^2 = a(ba) - (ba)a \in \mathcal{J},$$
$$(a \circ b)b = ab^2 - bab = (ab)b - b(ab) \in \mathcal{J}.$$

Of course $\perp$ must also be nondegenerate to fulfil the requirements of Definition 1.1. If $\mathcal{A}$ is the algebra of $n \times n$ real matrices, then it is well-known that $\mathrm{Comm}(\mathcal{A}) = \mathrm{Der}(\mathcal{A}) = \mathcal{J}$ and that $\mathcal{J} = \{a; \mathrm{trace}(a) = 0\}$. This means that $a \perp b$ and only if $\mathrm{trace}(ab) = 0$. Since $a \neq 0$ implies $\mathrm{trace}(aa^T) \neq 0$, in this case $\perp$ is automatically nondegenerate.

3. Two general results

There is not much that can be said about general OM-algebras. However we present two results which are perhaps of some interest.

Proposition 3.1. *Let $\mathcal{A}$ be an* OM-*algebra. Then $y^2 = 0$ holds for all $y \in \mathcal{A}$ and so $\mathcal{A}$ is anticommutative.*

Proof. Take some $x, y \in \mathcal{A}$. From $(x + y)^2 \perp (x + y)$ and $(x - y)^2 \perp (x - y)$ we obtain

$$x^2 + xy + yx + y^2 \perp x + y, \tag{4}$$
$$x^2 - xy - yx + y^2 \perp x - y, \tag{5}$$

We also know that xy, $yx\perp x, y$ and so

$$xy + yx\perp\ x + y, \tag{6}$$

$$-xy - yx\perp\ x - y. \tag{7}$$

From (4) and (6) we get

$$x^2 + y^2\perp\ x + y. \tag{8}$$

From (5) and (7) we get

$$x^2 + y^2\perp\ x - y. \tag{9}$$

Then (8) and (9) together imply $x^2 + y^2 \perp 2x$ and also $x^2 + y^2 \perp x$. Finally $x^2 \perp x$ implies $x^2 \perp y$. If we fix y and allow x to pass through all of $\mathcal{A}$, we obtain $\{y^2\}^{\perp} = \mathcal{A}$. The nondegeneracy of $\perp$ tells us that $y^2 = 0$ for all $y \in \mathcal{A}$. Thus $xy + yx = (x + y)^2 - x^2 - y^2 = 0$. □

Remark 3.2. Since $\mathcal{A}$ is anticommutative, all left or right ideals of $\mathcal{A}$ are automatically two-sided ideals. Thus we shall simply use the term *ideal of* $\mathcal{A}$.

Lemma 3.3. *Let $\mathcal{A}$ be an* OM-*algebra and $xy\perp z$ for some $x, y, z \in \mathcal{A}$. Then $xz\perp y$ holds. Also $z\perp xy$ implies $y\perp xz$.*

PROOF. Suppose that $xy\perp z$. From this and $xz\perp z$ we obtain

$$xy + xz\ \perp\ z. \tag{10}$$

From $x(y + z) = xy + xz\ \perp\ y + z$ and (10) it follows

$$xy + xz\ \perp\ y. \tag{11}$$

Since $xy\perp y$, we finally obtain $xz\perp y$. The rest can be proved in a similar way. □

Proposition 3.4. *Let $\mathcal{A}$ be an* OM-*algebra and $\mathcal{J}$ an ideal of $\mathcal{A}$. Then $\mathcal{A}(\mathcal{J}^{\perp}) \subset {}^{\perp}\mathcal{J}$ and $\mathcal{A}({}^{\perp}\mathcal{J}) \subset \mathcal{J}^{\perp}$ holds. In particular ${}^{\perp}\mathcal{J} \cap \mathcal{J}^{\perp}$ is also an ideal of $\mathcal{A}$.*

PROOF. Take some $a \in \mathcal{A}$, $b \in \mathcal{J}$ and $x \in \mathcal{J}^{\perp}$. Since $ab \in \mathcal{J}$, we have $ab\perp x$. According to Lemma 3.3 $ax\perp b$ holds and thus $ax \in {}^{\perp}\mathcal{J}$ (we fix a and x and allow b to pass through all of $\mathcal{J}$). Therefore $\mathcal{A}(\mathcal{J}^{\perp}) \subset {}^{\perp}\mathcal{J}$. The rest can be proved in a similar way. □

4. Hilbert algebras

Much more than in the general case can be said about a Hilbert OM-algebra i.e. an algebra $\mathcal{A}$ which is a Hilbert space with respect to the inner product $\langle\ ,\ \rangle$ such that $\langle xy, x\rangle = \langle xy, y\rangle = 0$ for all $x, y \in \mathcal{A}$. The results of the third section can be reformulated into

Lemma 4.1. *Let $\mathcal{A}$ be a Hilbert* OM*-algebra. Then $\langle xy, z\rangle = -\langle xz, y\rangle$ for all $x, y \in \mathcal{A}$.*

Proof. This follows from

$$0 = \langle x(y+z), y+z\rangle = \langle xy, y\rangle + \langle xy, z\rangle + \\ + \langle xz, y\rangle + \langle xz, z\rangle = \langle xy, z\rangle + \langle xz, y\rangle. \qquad \square$$

Proposition 4.2. *Let $\mathcal{A}$ be a Hilbert* OM*-algebra. If $\mathcal{J}$ is an ideal of $\mathcal{A}$, then $\mathcal{J}^\perp$ is also an ideal of $\mathcal{A}$ and $\mathcal{A} = \mathcal{J} \oplus \mathcal{J}^\perp$.*

Our next result concerns the question whether a Hilbert OM-algebra is a Banach algebra. A nonassociative algebra $\mathcal{A}$ with a Banach space structure is called a *Banach algebra* if $\|xy\| \le \|x\|\,\|y\|$ holds for all $x, y \in \mathcal{A}$.

Proposition 4.3. *Let $\mathcal{A}$ be a Hilbert* OM*-algebra. Then the multiplication of $\mathcal{A}$ is automatically continuous and there exists a new inner product, equivalent to the original one, such that $\mathcal{A}$ with this new inner product is both a Hilbert* OM*-algebra and a Banach algebra.*

Proof. Define $L_a : \mathcal{A} \to \mathcal{A}$ by $L_a(x) = ax$. We shall prove first that L_a is continuous. Suppose that $x_n \to 0$ and $L_a(x_n) \to y$. Then we have

$$\begin{aligned} \|y\|^2 = \langle y, y\rangle &= \langle \lim_{n\to\infty}(ax_n), y\rangle = \\ &= \lim_{n\to\infty}\langle ax_n, y\rangle = -\lim_{n\to\infty}\langle ay, x_n\rangle = \\ &= -\langle ay, \lim_{n\to\infty} x_n\rangle = 0. \end{aligned}$$

We used Lemma 4.1 in the above computation. By the closed graph theorem, L_a is continuous. Now we can apply more or less standard arguments from the theory of Banach algebras. Let $\mathcal{S} = \{a; \|a\| \le 1\}$. Take some $a \in \mathcal{S}$. For all $x \in \mathcal{A}$ we have

$$\|L_a(x)\| = \|ax\| = \|-ax\| = \|xa\| = \|L_x(a)\| \le \|L_x\| = M(x) < \infty$$

and so, by the uniform boundedness principle, it follows $\|L_a\| < M < \infty$ for all $a \in \mathcal{S}$. Thus $\|ab\| = \|L_a(b)\| \le M\|a\|\,\|b\|$ for all $a, b \in \mathcal{A}$. If we define a new inner product $\langle a, b\rangle_1 = M^2\langle a, b\rangle$, then $\mathcal{A}$ with this product clearly satisfies the requirements. $\square$

For every nonassociative algebra $\mathcal{A}$ its *annihilator*

$$\mathrm{Ann}(\mathcal{A}) = \{a \in \mathcal{A}; a\mathcal{A} = \mathcal{A}a = \{0\}\}$$

is an ideal of $\mathcal{A}$. If $\mathcal{A}$ is a Hilbert OM-algebra, then we shall say that $\mathcal{A}$ is proper if $\mathrm{Ann}(\mathcal{A}) = \{0\}$. Using Proposition 4.2, we get

Proposition 4.4. *Let $\mathcal{A}$ be a Hilbert* OM*-algebra. Then $\mathcal{A}$ is uniquely expressible as an orthogonal sum $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{B}$ where $\mathcal{A}_0$ is an algebra with zero multiplication and $\mathcal{B}$ a proper Hilbert* OM*-algebra.*

Remark. $\mathcal{A}_0$ is not necessarily a trivial algebra in the sense of Example 2.1. While $xy = 0$ for all $x, y \in \mathcal{A}_0$ it is not true that $x \perp y$ implies $x = 0$ or $y = 0$. Thus the orthogonality relation on $\mathcal{A}_0$ may not be trivial.

Proof. Let $\mathcal{A}_0 = \mathrm{Ann}(\mathcal{A})$ and $\mathcal{B} = \mathcal{A}_0^{\perp}$. If we take into account Proposition 4.2, it remains to prove that $\mathcal{B}$ is proper. Let $x\mathcal{B} = \{0\}$ for some $x \in \mathcal{B}$. Then $x\mathcal{A} = x\mathcal{B} + x\mathcal{A}_0 = \{0\}$ implies $x \in \mathrm{Ann}(\mathcal{A})$ (recall that $\mathcal{A}$ is anticommutative and that $x\mathcal{A} = -\mathcal{A}x$). Therefore $x \in \mathcal{A}_0 \cap \mathcal{A}_0^{\perp} = \{0\}$. $\square$

If $\mathcal{J}$ is some ideal of $\mathcal{A}$, we can define its annihilator with respect to $\mathcal{A}$ as

$$\mathrm{Ann}(\mathcal{J}) = \{x \in \mathcal{A}; x\mathcal{J} = \mathcal{J}x = \{0\}\}.$$

In the sequel we shall need the following

Proposition 4.5. *Let $\mathcal{A}$ be a proper Hilbert* OM-*algebra and $\mathcal{J}$ its closed ideal. Then* $\mathrm{Ann}(\mathcal{J}) = \mathcal{J}^{\perp}$ *holds.*

Proof. Since $\mathcal{J}^{\perp}$ is also an ideal of $\mathcal{A}$ by Proposition 4.2, we have $\mathcal{J}\mathcal{J}^{\perp} \subset \mathcal{J} \cap \mathcal{J}^{\perp} = \{0\}$ and so $\mathcal{J}^{\perp} \subset \mathrm{Ann}(\mathcal{J})$. If $a \in \mathrm{Ann}(\mathcal{J})$, we may decompose it into a sum $a = x + y$ with $x \in \mathcal{J}$ and $y \in \mathcal{J}^{\perp}$. Then we have, using the facts that $x\mathcal{J}^{\perp} \in \mathcal{J} \cap \mathcal{J}^{\perp} = \{0\}$ and $a\mathcal{J} \in \mathcal{J}^{\perp} \cap \mathcal{J} = \{0\}$,

$$x\mathcal{A} = x\mathcal{J} + x\mathcal{J}^{\perp} = x\mathcal{J} = (a - y)\mathcal{J} = -y\mathcal{J} \in \mathcal{J}\mathcal{J}^{\perp} = \{0\}$$

and thus $x \in \mathrm{Ann}(\mathcal{A})$ which is zero by the assumption. Then $a = y \in \mathcal{J}^{\perp}$ follows. $\square$

Proposition 4.6. *Let $\mathcal{A}$ be a 3-dimensional Hilbert* OM-*algebra. Then $\mathcal{A}$ is either an algebra with zero multiplication or isomorphic to the classical algebra.*

Remark 4.7. The classical algebra is defined in Example 2.2.

Proof. Let $\{i, j, k\}$ be some orthonormal basis of $\mathcal{A}$. Since ij is orthogonal to i and j, there exists some real α such that $ij = \alpha k$. Then we have, using Lemma 4.1,

$$\langle ik, j\rangle = -\langle ij, k\rangle = -\alpha\langle k, k\rangle = -\alpha$$

and so $ik = -\alpha j$. Note that ik is orthogonal to i and k and is therefore a scalar multiple of j. A similar computation

$$\langle jk, i\rangle = -\langle ji, k\rangle = \langle ij, k\rangle = \alpha\langle k, k\rangle = \alpha$$

enables us to fill in the multiplication table

$\cdot$	i	j	k
i	0	αk	$-\alpha j$
j	$-\alpha k$	0	αi
k	αj	$-\alpha i$	0

If $\alpha = 0$, then the multiplication of $\mathcal{A}$ is zero. Otherwise the elements $i' = \frac{1}{\alpha}i$, $j' = \frac{1}{\alpha}j$ and $k' = \frac{1}{\alpha}k$ have the same multiplication table as the classical algebra. $\square$

We shall continue this section by establishing a simple criterion which tells us whether some multiplication table generates a Hilbert OM-algebra or not.

Proposition 4.8. *Let $\mathcal{A}$ a Hilbert* OM-*algebra and $\mathcal{B}$ a closed subspace of $\mathcal{A}$. Then $(\mathcal{B}, \circ)$ can be structured as a Hilbert* OM-*algebra if we set $x \circ y = P(xy)$ where $P : \mathcal{A} \to \mathcal{A}$ denotes orthogonal projection on $\mathcal{B}$.*

Proof. Take $x, y \in \mathcal{B}$. It is well-known that orthogonal projections are self-adjoint and so $\langle P(a), b\rangle = \langle a, P(b)\rangle$ for all $a, b \in \mathcal{A}$. Now we have, using the fact that P acts as identity on the subspace B,

$$\langle x \circ y, x\rangle = \langle P(xy), x\rangle = \langle xy, P(x)\rangle = \langle xy, x\rangle = 0.$$

In a similar way we prove that $x \circ y$ is orthogonal to y. $\square$

Criterion 4.9. *Let $\mathcal{A}$ be a Hilbert algebra with jointly continuous product. Let $\{e_\alpha\}$ be some orthonormal basis of $\mathcal{A}$. For $\mathcal{S} \subset \{e_\alpha\}$ we define $\mathcal{A}_\mathcal{S}$ to be the closed linear hull of $\mathcal{S}$ together with the product defined in Proposition 4.8. Then $\mathcal{A}$ is a Hilbert* OM-*algebra if and only if $\mathcal{A}_\mathcal{S}$ is a Hilbert* OM-*algebra for all $\mathcal{S}$ with* $\mathrm{card}(S) \leq 3$.

Proof. Take some $\mathcal{S} = \{e_\alpha, e_\beta, e_\gamma\}$ not necessarily distinct. Then we have, using the fact that $\mathcal{A}_\mathcal{S}$ is a Hilbert OM-algebra,

$$\langle e_\alpha e_\beta, e_\gamma\rangle + \langle e_\gamma e_\beta, e_\alpha\rangle = \langle e_\alpha e_\beta, P_\mathcal{S}(e_\gamma)\rangle + \langle e_\gamma e_\beta, P_\mathcal{S}(e_\alpha)\rangle =$$
$$= (P_\mathcal{S}(e_\alpha e_\beta), e_\gamma\rangle + \langle P_\mathcal{S}(e_\gamma e_\beta), e_\alpha\rangle = \langle e_\alpha \circ e_\beta, e_\gamma\rangle + \langle e_\gamma \circ e_\beta, e_\alpha\rangle = 0.$$

The last equality follows from Lemma 4.1. Take some $x, y \in \mathcal{A}$. Since $x = \sum \lambda_\alpha e_\alpha$ and $y = \sum \mu_\beta e_\beta$ we get (also by the joint continuity of the product in $\mathcal{A}$ and the fact that the below series are absolutely convergent)

$$\langle xy, x\rangle = \sum_{\alpha,\beta,\gamma} \lambda_\alpha \mu_\beta \lambda_\gamma \langle e_\alpha e_\beta, e_\gamma\rangle = \sum_\beta \mu\beta \left(\sum_{\alpha,\gamma} \lambda_\alpha\lambda_\gamma \langle e_\alpha e_\beta, e_\gamma\rangle\right) =$$
$$= \sum_\beta \mu\beta \left(\sum_{\alpha\neq\gamma} \lambda_\alpha\lambda_\gamma\langle e_\alpha e_\beta, e_\gamma\rangle + \sum_\alpha \lambda_\alpha^2 \langle e_\alpha e_\beta, e_\alpha\rangle\right) =$$
$$= \sum_\beta \mu_\beta \left(\sum_{\alpha\neq\gamma} \lambda_\alpha\lambda_\gamma\langle e_\alpha e_\beta, e_\gamma\rangle\right).$$

The last sum can be decomposed into pairs

$$\lambda_\alpha\lambda_\gamma(\langle e_\alpha e_\beta, e_\gamma\rangle + \langle e_\gamma e_\beta, e_\alpha\rangle) = 0$$

by the above paragraph and thus $\langle xy, x\rangle = 0$. In a similar way we can prove that $\langle xy, y\rangle = 0$. $\square$

Next we shall prove that every Hilbert OM-algebra can be embedded in some flexible quadratic Banach algebra in the sense of Example 2.5. According to the Proposition 4.3 we may assume that $\|ab\| \le \|a\|\,\|b\|$ holds for all $a, b \in \mathcal{A}$.

Lemma 4.10. *Let $\mathcal{A}$ be a Hilbert* OM-*algebra. Then $\|ab\|^2 + \langle a, b\rangle^2 \le \|a\|^2\|b\|^2$ holds for all $a, b \in \mathcal{A}$.*

Proof. If $a = 0$, the above inequality is obvious. If $a \neq 0$, we define $c = -\dfrac{\langle a, b\rangle}{\|a\|^2}a + b$ and then observe the inequality $\|ac\|^2 \le \|a\|^2\|c\|^2$. By Proposition 3.1, we have $a^2 = 0$ and so $ac = ab$. Thus we obtain

$$\|ab\|^2 = \|ac\|^2 \le \|a\|^2 \left\| -\frac{\langle a, b\rangle}{\|a\|^2}a + b \right\|^2 =$$
$$= \|a\|^2 \left(\frac{\langle a, b\rangle^2}{\|a\|^4}\|a\|^2 + \|b\|^2 - 2\frac{\langle a, b\rangle}{\|a\|^2}\langle a, b\rangle \right) =$$
$$= \langle a, b\rangle^2 + \|a\|^2\|b\|^2 - 2\langle a, b\rangle^2 = \|a\|^2|b\|^2 - \langle a, b\rangle^2$$

and from this the desired inequality easily follows. $\square$

Theorem 4.11. *Let $(\mathcal{C}, \circ)$ be a Hilbert* OM-*algebra. Then $\mathcal{B} = \mathbb{R}e \oplus \mathcal{C}$ with the inner product $\langle \alpha e + a, \beta e + b\rangle = \alpha\beta + \langle a, b\rangle$ and the algebra product $(\alpha e + a)(\beta e + b) = (\alpha\beta - \langle a, b\rangle)e + \alpha b + \beta a + a \circ b$ is a flexible quadratic Banach algebra.*

Proof. First we have

$$e(\alpha e + a) = (e + 0)(\alpha e + a) =$$
$$= (\alpha - \langle 0, a\rangle)e + a + \alpha \cdot 0 + 0 \circ a = \alpha e + a$$

and so e is a left identity of $\mathcal{B}$. In a similar way we see that e is also a right identity of $\mathcal{B}$. Take some $x = \alpha e + a$ and $y = \beta e + b$ from $\mathcal{B}$. From the definition of the multiplication of $\mathcal{B}$, using also Lemma 4.10, we see that

$$\|xy\|^2 = (\alpha\beta - \langle a, b\rangle)^2 + \|\alpha b + \beta a + a \circ b\|^2 = \alpha^2\beta^2 + \langle a, b\rangle^2 - 2\alpha\beta\langle a, b\rangle +$$
$$+ \alpha^2\|b\|^2 + \beta^2\|a\|^2 + 2\alpha\beta\langle a, b\rangle + \|a \circ b\|^2 \le$$
$$\le \alpha^2\beta^2 + \alpha^2\|b\|^2 + \beta^2\|a\|^2 + \|a\|^2\|b\|^2 = \|x\|^2\|y\|^2$$

holds and thus $\mathcal{B}$ is a Banach algebra. Take any $x = \alpha e + \alpha$ and compute

$$x^2 - 2\alpha x + \|x\|^2 =$$
$$= [(\alpha^2 - \|a\|^2)e + 2\alpha a + a \circ a] - [2\alpha^2 e + 2\alpha a] + [\alpha^2 + \|a\|^2]e = 0.$$

In this computation we used Proposition 3.1 which says that $a \circ a = 0$. Therefore every element of $\mathcal{B}$ is a zero of some polynomial of degree 2 and hence $\mathcal{B}$ is quadratic. The verification that $\mathcal{A}$ (in the notation of Example 2.5) is isomorphic to $\mathcal{C}$ is very easy and will be omitted. The flexibility of $\mathcal{B}$ follows, using the fact that e is the identity element of $\mathcal{B}$, from

$$\begin{gathered}(xy)x - x(yx) = \\ = (\alpha e + a)(\beta e + b) \cdot (\alpha e + a) - (\alpha e + a) \cdot (\beta e + b)(\alpha e + a) = \\ = (ab)a - a(ba) = (-\langle a, b\rangle e + a \circ b)a - a(-\langle a, b\rangle e + b \circ a) = \\ = (a \circ b)a - a(b \circ a) = -\langle a \circ b, a\rangle + (a \circ b) \circ a + \langle a, b \circ a\rangle - a \circ (b \circ a).\end{gathered}$$

Since $(\mathcal{C}, \circ)$ is an OM-algebra, the first and the third term of the last equality are zero. Using the anticommutativity of $\mathcal{C}$ (see Proposition 3.1) we finally obtain

$$\begin{gathered}(xy)x - x(yx) = (a \circ b) \circ a - a \circ (b \circ a) = \\ = -a \circ (a \circ b) - a \circ (b \circ a) = a \circ (b \circ a) - a \circ (b \circ a) = 0. \qquad \square\end{gathered}$$

5. Hilbert OM-algebras and comtrans identity

In the classical vector product algebra the following comtrans identity for the double vector product

$$(x \times y) \times z = \langle z, x\rangle y - \langle z, y\rangle x \tag{12}$$

holds for all vectors x, y, z. A similar equality $(xy)z = \langle z, x\rangle y - \langle z, y\rangle x$ makes sense in a general Hilbert OM-algebra since the element $\langle z, x\rangle y - \langle z, y\rangle x$ is always orthogonal to z and xy.

In this section we prove two results: first we show that the comtrans identity characterizes the classical algebra among Hilbert OM-algebras and next we give a structure theorem for Hilbert OM-algebras satisfying a weaker form of this identity.

Proposition 5.1. *Let $\mathcal{A}$ be a Hilbert* OM-*algebra with zero annihilator and suppose that $(xy)z = \langle z, x\rangle y - \langle z, y\rangle x$ holds for all $x, y, x \in \mathcal{A}$. Then $\mathcal{A}$ is isomorphic to the classical algebra.*

Proof. We can see that $\mathrm{Ann}(\mathcal{A}) = \mathcal{A}$ if the dimension of $\mathcal{A}$ is 1 or 2. This follows from the following observation: take any $i, j \in \mathcal{A}$. If they are linearly dependent, then $ij = 0$ because of Proposition 3.1. If they are linearly independent, then they span $\mathcal{A}$. Since $ij \in \{i, j\}^{\perp} = \{0\}$, the statement is proved.

Thus $\dim(\mathcal{A}) \geq 3$. Let $i, j \in \mathcal{A}$ be orthonormal elements and define $k = ij$. Then by the anticommutativity of $\mathcal{A}$

$$\begin{aligned} ik &= i(ij) = (ji)i = \langle i, j\rangle j - \langle i, i\rangle j = -j, \\ jk &= j(ij) = (ji)j = \langle j, j\rangle i - \langle i, j\rangle j = i \end{aligned}$$

imply that $\{i, j, k\}$ generate a classical subalgebra of $\mathcal{A}$. Take any $x \in \{i, j, k\}^{\perp}$. Then we have first

$$(ix)j = \langle j, i\rangle x - \langle j, x\rangle i = 0.$$

Using this and Lemma 4.1, we obtain

$$\begin{aligned} 0 = ((ix)j)j &= \langle j, ix\rangle j - \langle j, j\rangle ix = \\ &= -\langle ij, x\rangle j - ix = -\langle k, x\rangle j - ix = xi \end{aligned}$$

and finaly

$$0 = (ix)i = \langle i, i\rangle x - \langle i, x\rangle i = x.$$

Hence $\{i, j, k\}^{\perp} = \{0\}$ and so $\mathcal{A} = \mathrm{Lin}\{i, j, k\}$. $\square$

Our last result is about a Hilbert OM-algebra $\mathcal{A}$ in which there exists some fixed element a with $\|a\| = 1$ such that (13) holds for all $x, y \in \mathcal{A}$ and a in place of z.

Theorem 5.2. *Let $\mathcal{A}$ be a Hilbert* OM-*algebra and suppose that $a \in \mathcal{A}$ is some fixed norm one element such that $(xy)a = \langle x, a\rangle y - \langle y, a\rangle x$ holds for all $x, y \in \mathcal{A}$. Then $\mathcal{A}$ can be decomposed into an orthogonal sum $\mathcal{A} = \mathbb{R}a \oplus \mathcal{B} \oplus \mathcal{C}$ such that $a\mathcal{B} \subset \mathcal{C}$, $a\mathcal{C} \subset \mathcal{B}$ and $\mathcal{B}\mathcal{B} = \mathcal{C}\mathcal{C} = \mathcal{B}\mathcal{C} = \{0\}$. Moreover for each nonzero $b \in \mathcal{B}$ the elements $\{a, b, ab\}$ generate a classical subalgebra. Also* $\dim(\mathcal{B}) = \dim(\mathcal{C})$ *and so if $\mathcal{A}$ is finite dimensional then its dimension is odd.*

The proof will be divided into three steps.

Lemma 5.3. *Let $i \in \mathcal{A}$ be some norm one element orthogonal to a. Then* $\mathrm{Lin}\{a, i, ai\}$ *is a classical subalgebra of $\mathcal{A}$.*

Proof. Denote $j = ai$. First we have

$$ja = (ai)a = \langle a, a\rangle i - \langle a, i\rangle a = i.$$

Using the fact that $\mathcal{A}$ is an OM-algebra, we obtain

$$\langle ij, a\rangle = -\langle ia, j\rangle = \langle ai, j\rangle = \langle j, j\rangle = \|j\|^2$$

and so $ij = \|j\|^2 a + z$ where $z \in \{a, i, j\}^{\perp}$. From Proposition 3.1 it follows that $a^2 = 0$ and so $(ij)a = za$. Thus

$$za = (ij)a = \langle i, a\rangle j - \langle j, a\rangle i = -\langle ai, a\rangle i = 0$$

and finally

$$0 = (za)a = \langle a, z\rangle a - \langle a, a\rangle z = -z$$

implies $ij = \|j\|^2 a$. However by Lemma 4.1

$$\|j\|^2 = \langle ij, a\rangle = -\langle ji, a\rangle = \langle ja, i\rangle = \langle i, i\rangle = 1. \qquad \square$$

Lemma 5.4. *Let $\{a, i, j\}$ generate a classical subalgebra of $\mathcal{A}$ (compare the above lemma) and $k \in \{a, i, j\}^\perp$ with $\|k\| = 1$. Then $\{a, i, j, k, \ell = ak\}$ is an orthonormal subbase of $\mathcal{A}$. Moreover $ik = i\ell = jk = j\ell = 0$.*

Proof. First we have to prove that ℓ is orthogonal to $\{a, i, j, k\}$. Since $\mathcal{A}$ is a Hilbert OM-algebra, $\langle a, \ell\rangle = \langle k, \ell\rangle = 0$ is obvious. Next we have

$$\langle i, \ell\rangle = \langle i, ak\rangle = -\langle ai, k\rangle = -\langle j, k\rangle = 0,$$
$$\langle j, \ell\rangle = \langle j, ak\rangle = -\langle aj, k\rangle = -\langle i, k\rangle = 0$$

which completes the proof of the first assertion (we have also taken into account Lemma 5.3). From $(ik)a = \langle a, i\rangle k - \langle a, k\rangle i = 0$ it follows that

$$0 = ((ik)a)a = \langle a, ik\rangle a - \langle a, a\rangle ik = -ik - \langle ia, k\rangle a = -ik - \langle j, k\rangle a = -ik.$$

In a similar way we prove that $i\ell = jk = j\ell = 0$. $\square$

Proof of the Theorem 5.2. Define $\mathcal{A}_1 = \{a\}^\perp$ and $\phi : \mathcal{A}_1 \to \mathcal{A}_1$ with $\phi(x) = ax$. Then Lemma 5.3 tells us that $\phi^2(x) = -x$ for all $x \in \mathcal{A}_1$ and also, since $\mathcal{A}$ is an OM-algebra, $\langle \phi(x), x\rangle = 0$. Thus ϕ is some sort of real anticonjugation and by standard methods we can obtain that $\mathcal{A}_1 = \mathcal{B} \oplus \mathcal{C}$ where $\mathcal{B} \perp \mathcal{C}$ and $\phi(\mathcal{B}) = \mathcal{C}$. We can for instance apply the Zorn lemma to the collection $\{\mathcal{B} \le \mathcal{A}_1; \mathcal{B} \perp \phi(\mathcal{B})]$.

Now it remains to prove that $\mathcal{B}\mathcal{B} = \mathcal{C}\mathcal{C} = \mathcal{B}\mathcal{C} = \{0\}$. Take $x, y \in \mathcal{B}$. If they are linearly dependent, $xy = 0$ follows from Proposition 3.1. Otherwise $x = \alpha i + \beta k$, $y = \gamma i + \delta k$ for some $i, k \in \mathcal{B}$ with $\|i\| = \|k\| = 1$ and $\langle i, k\rangle = 0$. Since $j = ai = \phi(i) \in \mathcal{C} \perp \mathcal{B}$, we are in the situation of Lemma 5.4 and thus $ik = 0$. From the anticommutativity of $\mathcal{A}$, $xy = 0$ easily follows. In a similar way we prove that $\mathcal{C}\mathcal{C} = \mathcal{B}\mathcal{B} = \{0\}$. $\square$

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