Publ. Math. Debrecen
78/3-4 (2011), 557–568
DOI: 10.5486/PMD.2011.4784

Generalized Abel functional equations and numerical representability of semiorders

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Abstract. In the present paper we show an striking relationship between the existence of solutions of a generalization of the Abel functional equation, and the Scott–Suppes representability of semiordered structures. We analyze this relationship and, in addition, we study other functional equations related to the numerical representability of semiordered structures.

1. Introduction

Despite it had much earlier appeared in the works by Norbert Wiener under a different terminology (see e.g. [13]), and its intuition dates back to Poincaré as pointed out in [18], the concept of a semiorder is usually attributed to R. Duncan Luce who introduced that term in [16], in a way that is independent from Wiener's earlier work, in contexts related to Mathematical Economics and Psychology. Basically, the main reason to (re)-introduce this concept was due to the study of models where agents exhibit preferences with intransitive indifference.

As mentioned in [10], p. 62:

"A classical example, attributed to Armstrong [5] *considers a man that prefers a cup of coffee with a whole portion of sugar, to a cup of coffee with*

Mathematics Subject Classification: 39B22, 06A06, 39B52, 54F05, 91B16.
Key words and phrases: Abel equation, numerical representations of orderings, semiorders, Scott–Suppes representability, functional equations in several variables.
This paper is supported by the research project MTM2007-62499 (Spain). We want also to express our gratitude to the editor and three anonymous referees for their helpful suggestions.

> *no sugar at all. If such man is forced to declare his preference between a cup with no sugar at all and a cup with only one molecule of sugar, he will declare them indifferent. The same will occur if he compares a cup with n molecules and a cup with $n+1$ molecules of sugar. However, after a very large number of intermediate comparisons we would finally confront him with a cup that has a whole portion of sugar that he is able to discriminate from the cup with no sugar at all. Here, we observe a clear intransitivity of the indifference."*

Observe that in this example there is a "just noticeable difference" or "threshold of discrimination" (the cube of sugar) that allows us to detect a difference between the taste of two cups of coffee. It is a positive constant.

The importance of semiorders lies just in this fact: they constitute a tool to analyze situations in which, in order to distinguish an element from another, the quality being compared should trespass a *positive constant* threshold.

2. Definition of a semiorder

Definition 2.1. Let X be a nonempty set. Let $\prec$ be an asymmetric binary relation defined on X. Associated to $\prec$ we define the reflexive and total binary relation $\precsim$ given by $x \precsim y \iff \neg(y \prec x)$ $(x, y \in X)$, and the symmetric binary relation $\sim$, called *indifference* given by $x \sim y \iff [(\neg(x \prec y)) \wedge (\neg(y \prec x))]$ $(x, y \in X)$.

An *interval order* $\prec$ is an asymmetric binary relation such that $[(x \prec y) \wedge (z \prec t)] \implies [(x \prec t) \vee (z \prec y)]$ $(x, y, z, t \in X)$.

An interval order $\prec$ is said to be a *semiorder* if $[(x \prec y) \wedge (y \prec z)] \implies [(x \prec w) \vee (w \prec z)]$ for every $x, y, z, w \in X$.

To put an *example* of a semiorder, consider the relation $\prec$ defined on $\mathbb{R}$ by declaring that $x \prec y \iff x \leq y - 3$ $(x, y \in \mathbb{R})$.

3. Representability of semiorders

After introducing the definition in [16], the concepts of "*just noticeable difference*" and "*utility discrimination*" are discussed. Those seminal ideas were also developed in [20] trying to identify a semiorder $\prec$ defined on a (nonempty) set S to a subset of the real line $\mathbb{R}$ endowed with the binary relation $P_{\mathbb{R}}$ given by $aP_{\mathbb{R}}b \iff a+1 < b$ $(a, b \in \mathbb{R})$. The identification (if any) is made through a real-valued *utility* function $f : S \to \mathbb{R}$ such that $a \prec b \iff f(a)P_{\mathbb{R}}f(b) \iff$

$f(a) + 1 < f(b)$ $(a, b \in S)$. Here the constant 1 acts as a "*threshold of utility discrimination*".

In some of the classical studies on semiorders (see e.g. [4] for a modern revision of the theory) either necessary or else sufficient conditions for the existence of a numerical representation of a semiorder $\prec$ defined on a set S by means of a real-valued funtion $f : S \to \mathbb{R}$ such that $a \prec b \iff f(a) + 1 < f(b)$ $(a, b \in S)$ were obtained.

This kind of numerical representations of semiordered structures is also encountered in a wide range of applications, as choice theory under risk (see [12]) or decision-making under risk (see [19]), or modellization of choice with errors (see [3]) among many others.

Definition 3.1. If $(X, \precsim)$ is a preordered set then a real-valued function $f : X \to \mathbb{R}$ is said to be an *order-isomorphism* for $\precsim$ if, for every $x, y \in X$, it holds that $[x \precsim y \iff f(x) \leq f(y)]$. A total preorder $\precsim$ on X is said to be *representable* if there exists an order-isomorphism for $\precsim$.

Definition 3.2. An interval order $\prec$ defined on X is said to be *representable* (as an interval order, see e.g. [11]) if there exist two real valued maps $f, g : X \longrightarrow \mathbb{R}$ such that $x \prec y \iff g(x) < f(y)$ $(x, y \in X)$.

Also, a semiorder $\prec$ defined on X is said to be *representable* (now, as a semiorder!) *in the sense of Scott and Suppes* [20] if there exist a real-valued map $f : X \to \mathbb{R}$ such that $x \prec y \iff f(x) + 1 < f(y)$ $(x, y \in X)$. (Here the choice of 1 is immaterial. Any positive value could be used).

There exist interval orders that fail to be representable (as interval orders). However interval orders on countably infinite sets are always representable, as well as interval orders on finite sets (see [6]). Also, semiorders on finite sets are always representable in the sense of Scott and Suppes. In the infinite case, there exist semiorders that are not representable, even if the set is countably infinite. (See Proposition 8 on pp. 237–239 in [17]).

Another *example* of a non-representable semiorder is the binary relation $\prec$ on $\mathbb{R}$ given by $x \prec y \iff 3|x| < 2|y|$ $(x, y \in \mathbb{R})$. See also [10] for further details.

The representability of a semiorder is characterized by means of the existence of solutions of a particular kind of functional equation in two variables.

Lemma 3.1 (See [10]). *Let X be a nonempty set endowed with a semiorder $\prec$. The following conditions are equivalent:*

a) *The semiordered structure $(X, \prec)$ is representable in the sense of Scott and Suppes.*

b) *There exists a bivariate map* $G : X \times X \to \mathbb{R}$ *such that* $x \prec y \iff G(x,y) > 0$ $(x, y \in X)$ *and, in addition,* $G(x,y) + G(y,z) = G(x,z) + G(t,t)$ *for every* $x, y, z, t \in X$.

The following lemma is an old result, obtained by the Norwegian mathematician Niels Heinrik Abel early in 1824. It gave raise to the name *"Abel equation"* to call the functional equation in one real variable that is involved in its statement. Suitable variations of this equation and its subsequent lemma will furnish important information about how to cope with semiordered structures.

Lemma 3.2. *Let* I *be an open real interval. Let* $h : I \to I$ *a continuous and strictly increasing map such that*[1] $x < h(x)$ *for every* $x \in I$. *Then there exists a continuous and strictly increasing map* $f : I \to \mathbb{R}$ *that satisfies the equation* $f(h(x))) = f(x) + 1$ $(x \in I)$.

Proof. See Theorem 2.1 in [15], or alternatively pp. 133 and ff. in [2]. □

4. Generalized Abel equations

In order to look for characterizations of the representability of semiorders, a technique used by several authors (see e.g. [14]) consists in first considering the semiorder as an interval order, then representing it as an interval order (when possible), and finally trying to modify that representation to get a new representation (now, as a semiorder) in the sense of Scott and Suppes.

Related to this fact, the *Abel equation* has been used to characterize (whenever possible) the existence of Scott–Suppes representations of semiorders (see [8]). In the present paper we analyze the relationship between different kinds of functional equations involved in the representability of interval orders and semiorders, paying a special attention to *generalized Abel equations*.

We prove that the existence of solutions of a natural generalization of Abel equation is closely related to the existence of representations (in the sense of Scott and Suppes) of semiordered structures.

Let I be an interval of the real line $\mathbb{R}$. Let $h : I \to I$ be a function. The original *Abel equation* is defined as $f(h(x)) = f(x)+1$ $(x \in I)$, where the function $f : I \to \mathbb{R}$ is (a priori) unknown. A suitable function f of this kind (if any) is said to be a *solution* of the Abel equation.

[1]This last condition, namely $x < h(x)$ for every $x \in I$, was missed in the statement of Lemma 3.4 in [8]. It should be added there in order for the statement to be correct.

As mentioned before in Lemma 3.2, when h is continuous, increasing, and such that $x < h(x)$ for every $x \in I$, then there is a solution f that is also continuous and strictly increasing. However, in what follows we shall be interested in solutions that, unless otherwise stated, may or may not be continuous or strictly increasing. To start with, we are now ready to introduce a generalization of Abel functional equation.

Definition 4.1. Let X be a nonempty set, and $h : X \to X$ a (fixed) map. We say that a real-valued function $f : X \to \mathbb{R}$ satisfies the *generalized Abel equation* if it holds that $f(h(x)) = f(x) + 1$, for every $x \in X$.

Next Proposition 4.1 is an easy result showing that solutions of generalized Abel equations induce representable semiorders.

Proposition 4.1. *Let X be a nonempty set. Let $h : X \to X$ be a map, and $f : X \to \mathbb{R}$ a solution of the generalized Abel equation $f(h(x)) = f(x) + 1$. Then the binary relation $\prec$ defined on X by $x \prec y \iff f(x) + 1 < f(y)$ is a semiorder. Moreover, this semiorder is representable in the sense of Scott and Suppes, through the real-valued function f and the positive threshold* 1.

Now we point out a relationship between the generalized Abel equation and a functional relation in two variables, known as the *separability equation.*

Proposition 4.2. *Let X be a nonempty set. Let $h : X \to X$ be a map. A solution of a generalized Abel equation $f(h(x)) = f(x) + 1$ $(x \in X)$ has associated a solution of the separability equation $F(x, y) + F(y, z) = F(x, z) + F(y, y)$ $(x, y, z \in X)$ such that $F(t, t) = -1$ $(t \in X)$.*

Proof. Consider the map $F : X \times X \to \mathbb{R}$ given by $F(x, y) = f(x) - f(h(y))$ $(x, y \in X)$. It follows that: $F(x, y) + F(y, z) = f(x) - f(h(y)) + f(y) - f(h(z)) = f(x) - f(h(z)) + f(y) - f(h(y)) = F(x, z) + F(y, y)$ $(x, y, z \in X)$. Thus F satisfies the separability equation. Moreover, $F(t, t) = f(t) - f(h(t)) = -1$ $(t \in X)$, by hypothesis. □

The next result tells us what we can expect about the existence of solutions of the generalized Abel equation.

Theorem 4.1. *Let X be a (nonempty) set. Let $h : X \to X$ be a map. Then, the generalized Abel equation $f(h(x)) = f(x) + 1$ $(x \in X;\ f : X \to \mathbb{R})$ has a solution if and only if the following conditions hold:*

a) *the set X is infinite (no matter if countably or uncountably infinite),*
b) *for any $m \neq n \in \mathbb{N}$ and $x \in X$ it holds that $h^m(x) \neq h^n(x)$. (Consequently, the map h has no cycles. In particular, h has no fixed point).*

Proof. Assume first that f is a solution of the generalized Abel equation. Then, if X is finite, it is plain that h must have cycles, so that there exist an element $x \in X$ and a natural number $k > 0$ such that $h^k(x) = x$. But, recurrently, we should have $f(h^k(x)) = f(x)+k$, so that $f(x) = f(x)+k$, which is a contradiction. Therefore, X is infinite. In the same way, if there exist $m \neq n \in \mathbb{N}$ and $x \in X$ such that $h^m(x) = h^n(x)$, it follows that $f(h^m(x)) = f(x) + m$; $f(h^n(x)) = f(x) + n$, so that $f(x) + m = f(x) + n$ and $m = n$, which contradicts the hypothesis.

Conversely, let us assume that both conditions in the statement of Theorem 4.1 hold on the set X. Define on X the equivalence relation $\mathcal{R}$ given by $x\mathcal{R}y$ if there exist $m, n \in \mathbb{N}$ such that $h^m(x) = h^n(y)$ $(x, y \in X)$. Let $X/\mathcal{R}$ stand for the corresponding quotient space. For each $t \in X$, consider the equivalence class $\{x \in X : x\mathcal{R}t\}$. Denote by t^*, a representative element of this class and define by $X^* \subseteq X$ the subset of X which consists of these selected elements. To get a solution f of the generalized Abel equation we could define, for instance, $f(t^*) = 0$ for every $t^* \in X^*$, and then extend f to the whole X in the natural way, namely, given $x \in X$ we consider the element $x^* \in X^*$ such that $x\mathcal{R}x^*$ and $h^m(x) = h^n(x^*)$ for some $m, n \in \mathbb{N}$, and we define $f(x) = min\{n - m; h^m(x) = h^n(x^*)\}$. It is straightforward to see that f is well-defined and satisfies the generalized Abel equation $f(h(x)) = f(x) + 1$ $(x \in X)$. □

Remark 1. Let X be a (nonempty) set. In the light of Theorem 4.1, and looking for some converse of Proposition 4.2, it is now evident that a solution $F : X \times X \to \mathbb{R}$ of the separability equation $F(x, y) + F(y, z) = F(x, z) + F(y, y)$ $(x, y \in X)$, even if $F(t, t) = -1$ for every $t \in X$, *may fail to have associated* (in a natural way) *a solution of a generalized Abel equation.* For instance, if X is a finite set and we define $F : X \times X \to \mathbb{R}$ as the constant map $F(x, y) = -1$ $(x, y \in X)$, it is obvious that F is a solution of the separability equation. However, no matter which map $h : X \to X$ we consider, the generalized Abel equation $f(h(x)) = f(x)+1$ $(x \in X)$ has no solutions by the first condition in Theorem 4.1, because X is finite.

But, in some special cases, it could still happen that a solution of a suitable separability equation generates a solution of a generalized Abel equation. To see this, suppose that $F : X \times X \to \mathbb{R}$ satisfies the separability equation and there exists $K < 0$ such that $F(t, t) = K$ $(t \in X)$. Fix an element $a \in X$. We observe that $F(x, a) + F(a, x) = F(x, x) + F(a, a) = 2K$ $(x \in X)$. Dividing by $2K$ we obtain $\frac{F(x.a)}{2K} = -\frac{F(a,x)}{2K} + 1$ $(x \in X)$.

Assume, in addition, that for every $t \in X$ there exists an element $h(t) \in X$, unique, such that $F(a, h(t)) = -F(t, a)$ $(t \in X)$.

Under this *extra* hypothesis, if we define $f : X \to \mathbb{R}$ by $f(t) = -\frac{F(a,t)}{2K}$

($t \in \mathbb{R}$) we immediately check that f satisfies the generalized Abel equation: $f(h(x)) = f(x) + 1$ ($x \in X$).

5. Scott–Suppes representable semiorders and generalized Abel equations

In the present section we analyze the relationship between semiorders that are representable in the sense of Scott and Suppes and the generalized Abel equations just introduced in Definition 4.1. To start with, we recall that a solution of a generalized Abel equation gives raise to a Scott–Suppes representable semiorder.

Due to the first condition in Theorem 4.1, if $\prec$ is a semiorder defined on a *finite* (nonempty) set X, no generalized Abel equation on X admits a solution, so that, even if $\prec$ is representable in the sense of Scott and Suppes, it fails to have associated (directly and in a natural way) a generalized Abel equation with solutions.

Nevertheless, *if we enlarge the set* X where a Scott–Suppes representable semiorder $\prec$ has been defined, *we can finally arrive at solutions of suitable generalized Abel equations, related to the given semiordered structure* $(X, \prec)$. Next Theorem 5.1 explains the process.

Theorem 5.1. *Let X be a nonempty set and $\prec$ a semiorder defined on X. Then $\prec$ is representable in the sense of Scott and Suppes if and only if there exists a superset $\bar{X}$ of X and a map $h : \bar{X} \to \bar{X}$ such that the generalized functional equation $\bar{f}(h(\bar{x})) = \bar{f}(\bar{x}) + 1$ ($\bar{x} \in \bar{X}; \bar{f} : \bar{X} \to \mathbb{R}$) has a solution and, in addition, $x \prec y \iff \bar{f}(x) + 1 < \bar{f}(y) \iff \bar{f}(h(x)) < \bar{f}(y)$ for every $x, y \in X \subseteq \bar{X}$.*

Proof. Assume that $\prec$ admits a representation by means of a real-valued map $f : X \to \mathbb{R}$ such that $x \prec y \iff f(x) + 1 < f(y)$ ($x, y \in X$). Four cases may appear:

Case 1. In the particular case in which f is a *bijection* between X and $\mathbb{R}$, we observe that given $x \in X$ the element $f^{-1}(f(x) + 1) \in X$ is well-defined. Let $h : X \to X$ be defined by $h(x) = f^{-1}(f(x) + 1)$ ($x \in X$). By definition of h, we have $f(h(x)) = f(x) + 1$ ($x \in X$) so that the function f satisfies a suitable generalized Abel equation. It is now plain that $x \prec y \iff f(x)+1 < f(y) \iff f(h(x)) < f(y)$ ($x, y \in X$).

Case 2. If f is surjective (i.e. $f(X) = \mathbb{R}$) but it is *not* injective, an equivalence relation $\mathcal{R}$ can be immediately defined on X by declaring $a\mathcal{R}b \iff f(a) = f(b)$ ($a, b \in X$). Let $X_{\mathcal{R}}$ denote the quotient set $X/\mathcal{R}$. Denote by $x_{\mathcal{R}}$ the equivalence class corresponding to a given element $x \in X$. Define on $X_{\mathcal{R}}$ the binary relation $\prec_{\mathcal{R}}$ given by $x_{\mathcal{R}} \prec_{\mathcal{R}} y_{\mathcal{R}} \iff x \prec y \iff f(x)+1 < f(y)$ $(x, y \in \mathbb{R})$ and the real-valued map $f_{\mathcal{R}} : X_{\mathcal{R}} \to \mathbb{R}$ given by $f_{\mathcal{R}}(x_{\mathcal{R}}) = f(x)$ $(x \in X)$. It is straightforward to see now that $\prec_{\mathcal{R}}$ is a semiorder on $X_{\mathcal{R}}$ such that $x_{\mathcal{R}} \prec_{\mathcal{R}} y_{\mathcal{R}} \iff f_{\mathcal{R}}(x_{\mathcal{R}}) + 1 < f_{\mathcal{R}}(y_{\mathcal{R}})$ $(x, y \in \mathbb{R})$. Moreover, $f_{\mathcal{R}}$ is a bijection. Let $h_{\mathcal{R}} : X_{\mathcal{R}} \to X_{\mathcal{R}}$ be the map defined by $h_{\mathcal{R}}(x_{\mathcal{R}}) = f_{\mathcal{R}}^{-1}(f_{\mathcal{R}}(x_{\mathcal{R}})+1)$ $(x_{\mathcal{R}} \in X_{\mathcal{R}})$. For every $x_{\mathcal{R}} \in X_{\mathcal{R}}$ we fix an element $x^* \in X$ such that $x^*_{\mathcal{R}} = h_{\mathcal{R}}(x_{\mathcal{R}})$. Now we consider the map $h : X \to X$ given by $h(x) = x^*$ $(x \in X)$. We get $f(h(x)) = f(x^*) = f_{\mathcal{R}}(x^*_{\mathcal{R}}) = f_{\mathcal{R}}(h_{\mathcal{R}}(x_{\mathcal{R}})) = f_{\mathcal{R}}(x_{\mathcal{R}}) + 1 = f(x) + 1$ $(x \in X)$, so that f also satisfies a generalized Abel equation. Again, it is clear that $x \prec y \iff f(x) + 1 < f(y) \iff f(h(x)) < f(y)$ $(x, y \in X)$.

Case 3. If f is injective but it is *not* surjective (i.e: $f(X) \subsetneq \mathbb{R}$) then we enlarge the set X in the following way: for every $\alpha \in \mathbb{R} \setminus f(X)$ we add an extra element x_α to X. Let $\bar{X}$ denote the enlarged set $X \cup \{x_\alpha : \alpha \in \mathbb{R} \setminus f(X)\}$. Consider now the real-valued map $\bar{f} : \bar{X} \to \mathbb{R}$ given, for every $t \in \bar{X}$, by $\bar{f}(t) = f(t)$ if $t \in X$; $\bar{f}(t) = \alpha$ if $t = x_\alpha$ for some $\alpha \in \mathbb{R} \setminus f(X)$. Now it is plain that $\bar{f}$ is a bijection. Finally, define a binary relation $\bar{\prec}$ on $\bar{X}$ by declaring that $s \bar{\prec} t \iff \bar{f}(s) + 1 < \bar{f}(t)$ $(s, t \in \bar{X})$. We may easily check that $\bar{\prec}$ is a semiorder whose restriction to X is $\prec$. Let $\bar{h} : \bar{X} \to \bar{X}$ be defined by $\bar{h}(t) = \bar{f}^{-1}(\bar{f}(t) + 1)$ $(t \in \bar{X})$. It follows that $\bar{f}(\bar{h}(t)) = \bar{f}(t) + 1$ $(t \in \bar{X})$, so that $\bar{f}$ satisfies another generalized Abel equation. Moreover $x \prec y \iff x \bar{\prec} y \iff \bar{f}(x) + 1 < \bar{f}(y) \iff \bar{f}(h(x)) < \bar{f}(y) = f(y)$ for every $x, y \in X \subseteq \bar{X}$.

Case 4. When f is *neither* injective *nor* surjective, first we enlarge the set X to a set $\bar{X}$ obtained by adding an extra element x_α for each $\alpha \in \mathbb{R} \setminus f(X)$. Define $\bar{f}$ and $\bar{\prec}$ as in the previous case, and observe that $\bar{f}$ is now surjective but not yet injective. Consequently, we consider the quotient set $\bar{X}_{\mathcal{R}} = \bar{X}/\mathcal{R}$ through the equivalence $\mathcal{R}$ that $\bar{f}$ defines on $\bar{X}$. As before, in the quotient set $\bar{X}_{\mathcal{R}}$ we may define in the natural way a real-valued function $\bar{f}_{\mathcal{R}}$ and a map $h_{\mathcal{R}} : \bar{X}_{\mathcal{R}} \to \bar{X}_{\mathcal{R}}$ such that $\bar{f}_{\mathcal{R}}(h_{\mathcal{R}}(t)) = \bar{f}_{\mathcal{R}}(t) + 1$ $(t \in \bar{X}_{\mathcal{R}})$. As in case 2, for every $\bar{x}_{\mathcal{R}} \in \bar{X}_{\mathcal{R}}$ we fix an element $\bar{x}^* \in \bar{X}$ such that $\bar{x}^*_{\mathcal{R}} = h_{\mathcal{R}}(\bar{x}_{\mathcal{R}})$. Now we consider the map $h : \bar{X} \to \bar{X}$ given by $h(\bar{x}) = \bar{x}^*$ $(\bar{x} \in \bar{X})$. We get $\bar{f}(h(\bar{x})) = \bar{f}(\bar{x}^*) = \bar{f}_{\mathcal{R}}(\bar{x}^*_{\mathcal{R}}) = \bar{f}_{\mathcal{R}}(h_{\mathcal{R}}(\bar{x}_{\mathcal{R}})) = \bar{f}_{\mathcal{R}}(\bar{x}_{\mathcal{R}}) + 1 = \bar{f}(x) + 1$ $(\bar{x} \in \bar{X})$, so that $\bar{f}$ also satisfies a generalized Abel equation. Once more, it is clear that $x \prec y \iff f(x) + 1 < f(y) \iff \bar{f}(x) + 1 < \bar{f}(y) \iff \bar{f}(h(x)) < \bar{f}(y) = f(y)$ $(x, y \in X)$.

To prove the converse, just observe that if there exists a superset $\bar{X}$ of X and a map $h : \bar{X} \to \bar{X}$ such that there exists a real-valued function $\bar{f} : \bar{X} \to \mathbb{R}$ satisfying that $\bar{f}(h(\bar{x})) = \bar{f}(\bar{x}) + 1$ $(\bar{x} \in \bar{X})$ as well as $x \prec y \iff \bar{f}(x) + 1 < \bar{f}(y) \iff$

$\bar{f}(h(x)) < \bar{f}(y)$ $(x, y \in X)$ then $\prec$ is indeed representable in the sense of Scott and Suppes through the real-valued function f with threshold 1, where f stands for the restriction of the map $\bar{f}$ to the given set X. □

Till this point we have been working with the Scott–Suppes representability of semiorders. In this kind of representations, the *codomain* of the function involved is the real line. However, we can also consider repesentations of semiorders *"à la Scott–Suppes"* taking values on codomains *different from* $\mathbb{R}$. In these contexts, further generalizations of the Abel functional equation may also appear, and extended versions of Theorem 5.1 arise.

To illustrate this idea, we shall consider an example, namely the *"lexicographic plane"*.

Definition 5.1. We define the *lexicographic plane* as $\mathcal{P} = \mathbb{R} \times \mathbb{R}$ endowed with the lexicographic ordering $\prec_{\mathcal{P}}$ given by $(a, b) \prec_{\mathcal{P}} (c, d) \iff [(a < c) \vee (a = c; b < d)]$ $(a, b, c, d \in \mathbb{R})$.

Now we may consider the following new generalization of the Abel equation.

Definition 5.2. Let X be a nonempty set. Let $h : X \to X$ be a map. A map $f : X \to \mathcal{P}$ is said to satisfy the *generalized Abel equation with values on the lexicographic plane* $\mathcal{P}$ if $f(h(x)) = f(x)\bar{+}(0, 1)$ where $\bar{+}$ stands for the coordinatewise sum on the plane $\mathbb{R}^2$.

In this direction, we get the next Theorem 5.2. (The proof is omitted for the sake of brevity, because it uses similar ideas to that of Theorem 5.1.)

Theorem 5.2. *Let X be a nonempty set and $\prec$ a semiorder defined on X. Then $\prec$ is representable* "à la Scott–Suppes" *on the lexicographic plane $\mathcal{P}$ if and only if there exists a superset $\bar{X}$ of X and a map $h : \bar{X} \to \bar{X}$ such that the generalized functional equation $\bar{f}(h(\bar{x})) = \bar{f}(\bar{x})\bar{+}(0, 1)$ $(\bar{x} \in \bar{X}; \bar{f} : \bar{X} \to \mathcal{P})$ has a solution and, in addition, $x \prec y \iff \bar{f}(x)\bar{+}(0, 1) \prec_{\mathcal{P}} \bar{f}(y) \iff \bar{f}(h(x)) \prec_{\mathcal{P}} \bar{f}(y)$ for every $x, y \in X \subseteq \bar{X}$.*

Remark 2. Further results that are similar to Theorem 5.1 and Theorem 5.2 can also be obtained if we consider other codomains. Among them another classical possible codomain is the *long line*, defined as the lexicographically ordered set $\mathcal{L} = [0, \omega_1) \times [0, 1)$ where ω_1 stands for the first uncountable ordinal.

Notice also that if $\alpha \in [0, \omega_1)$ the ordinal $\alpha+1$ is defined, so that *a generalized Abel equation with values on the long line $\mathcal{L}$* can be understood as one of the type $f(h(x)) = f(x)\bar{+}(1, 0)$ where X is a nonempty set, $h : X \to X$ is a map,

$f : X \to \mathcal{L}$ is the unknown function, and $\bar{+}$ stands for the coordinatewise sum on $\mathcal{L}$.

Other remarkable possibility could be using as codomain a (non-trivial) *totally ordered group or semigroup* $(G, \oplus)$, whose representability could also be related to some other kind of functional equations (see e.g. [9]). If we fix an element $a \in G$ different of the null element e, we may consider a *generalized Abel equation with values on the group* $(G, \oplus)$ as one of the type $f(h(x)) = f(x) \oplus a$ where X is a nonempty set, $h : X \to X$ is a map, $f : X \to G$ is the unknown function, and $\oplus$ stands for the algebraic group operation on G.

A third interesting alternative codomain could be some special set of *fuzzy numbers* on which semiordered structures could have a natural representation. This new approach has been recently considered in [7].

6. Final discussion

Although Theorem 5.1 furnishes a characterization of the Scott–Suppes representability of semiorders in terms of generalized Abel equations, several *difficulties* appear when trying to put in practice this key result. First, we observe that if X is a nonempty set where a semiorder $\prec$ has been defined, to guarantee the Scott–Suppes representability of $\prec$ in the terms stated in Theorem 5.1, we must use a suitable superset $\bar{X}$. However, Theorem 5.1 gives no furher information, a priori, about which are the characteristics of this suitable superset. In other words, *we cannot guess (a priori) how the key superset $\bar{X}$ looks like.* Second, even if we know all about the superset $\bar{X}$, *we must still have at hand a suitable map $h : \bar{X} \to \bar{X}$ of which, again, we have no information given a priori*, so that it seems difficult to guess how h looks like.

Consequently, we should use alternative characterizations of the Scott–Suppes representability of semiorders that lean on other approaches or criteria. At this point, we have already quoted that in [10] some characterizations of this kind were given. But, unfortunately, the characterizations of the Scott–Suppes representability of semiorders given in [10] also lean on properties of a suitable *extension* of the given semiordered structure $(X, \prec)$, and it does not seem easy yet to guess how such suitable extension looks like.

Incidentally, we must point out that the extensions used in the proof of the main theorem in [10], namely the so-called *Generalized Scott–Suppes Representability Theorem (GSSRT)*, are related to the superset $\bar{X}$ that appears in the statement of Theorem 5.1. Comparing both approaches, the reader may observe that the map h that appears in the statement of Theorem 5.1 acts as the definition of a *successor element* $S(x)$, a key step in the proof of GSSRT in [10].

To conclude our discussion, it is important to point out the necessity of looking for alternative characterizations of the Scott–Suppes representability of a semiorder $\prec$ defined on a (nonempty) set X that are *internal*, that is, they should be *expressed in terms (only) of the given structure* (a semiorder, in this case). Partial results in this direction have been recently published in [1].

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(Received December 2, 2009; revised November 3, 2010)