Publ. Math. Debrecen
79/1-2 (2011), 1–6
DOI: 10.5486/PMD.2011.4818

On additive countably continuous functions

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Abstract. We construct an example of additive Darboux function $f : \mathbb{R} \to \mathbb{R}$ which is strongly countably continuous and discontinuous. We show also that if an additive function f is covered by countable family of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, then it can be also covered by countably many linear functions. Finally we remark that every finitely continuous and additive function is continuous.

Let us establish some of terminology to be used. By $\mathbb{R}$ and $\mathbb{Q}$ we denote the sets of all reals and rationals, respectively. Let, moreover, $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$. For $A \subset \mathbb{R}$ and $x \in \mathbb{R}$, define $A + x = \{a + x : a \in A\}$ and $xA = \{xa : a \in A\}$. The symbol $|A|$ stands for the cardinality of a set A. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$.

We will consider $\mathbb{R}$ as a linear space over the field $\mathbb{Q}$. For $A \subset \mathbb{R}$, $\mathrm{LIN}\,(A)$ denotes the linear subspace of $\mathbb{R}$ generated by A. Any basis of $\mathbb{R}$ over $\mathbb{Q}$ will be referred as a Hamel basis. Recall that every function defined on a Hamel basis has the unique extension to additive function defined on whole $\mathbb{R}$. (See e.g. [MK] for more details.)

A function $f : \mathbb{R} \to \mathbb{R}$ is:

- *additive* ($f \in \mathrm{Add}$) if $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$;
- *Darboux* ($f \in \mathrm{D}$) if f maps intervals onto intervals;
- *countably continuous* ($f \in \mathrm{CC}$), if there is a decomposition $(A_n)_{n<\omega}$ of $\mathbb{R}$ such that $f \restriction A_n$ is continuous for every $n < \omega$;

Mathematics Subject Classification: Primary: 26A15; Secondary: 26A30.
Key words and phrases: additive function, Darboux function, strongly countably continuous function, almost continuity, Hamel basis.

- *strongly countably continuous* ($f \in \mathrm{SCC}$), if there is a sequence $(f_n)_{n<\omega}$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f \subset \bigcup_{n<\omega} f_n$ [GH], cf. [GF].
- *Sierpiński-Zygmund function* ($f \in \mathrm{SZ}$) if the restriction $f\restriction A$ is discontinuous for each $A \subset \mathbb{R}$ of size $\mathfrak{c}$ [SZ].

Let F be a family of partial functions from $\mathbb{R}$ to $\mathbb{R}$. We say that $f : \mathbb{R} \to \mathbb{R}$ is *countably* F ($f \in \mathrm{CF}$), if there exists a sequence $\langle f_n \rangle_n \subset \mathrm{F}$ such that $f \subset \bigcup_{n<\omega} f_n$. The class of continuous functions $f : \mathbb{R} \to \mathbb{R}$ is denoted by C. By L we denote the class of all linear functions, i.e., all functions $f : A \to \mathbb{R}$, $A \subset \mathbb{R}$, of the form $f(x) = ax + b$, where a, b are fixed. (Then a will be called *the direction of f*.) Note that $\mathrm{CL} \subset \mathrm{SCC} \subset \mathrm{CC}$.

Obviously no SZ function is CC. It is not hard to construct an example of $f \in \mathrm{Add} \cap \mathrm{SZ}$ (see e.g. [NR]), thus there are $f \in \mathrm{Add} \setminus \mathrm{CC}$. On the other hand, Z. Grande and A. Fatz-Grupka constructed a function $f \in \mathrm{Add} \cap \mathrm{SCC} \setminus \mathrm{C}$ with uncountable image [GF, Example 2]. This result has been strengthened recently by G. Horbaczewska. She gives an example of $f \in \mathrm{Add} \cap \mathrm{SCC} \setminus \mathrm{C}$ with an image which intersects every uncountable Borel subset of $\mathbb{R}$ [GH, Example 2]. We will show that such a function can maps every interval onto whole $\mathbb{R}$. (This means, in particular, that f is Darboux.)

Proposition 1. *There exists* $f \in \mathrm{Add} \cap \mathrm{D} \cap \mathrm{SCC} \setminus \mathrm{C}$.

Proof. Let H be a Hamel basis and H_0 be a subset of H with $|H_0| = \omega$. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function such that $f(H_0) = H_0 \cup \{0\}$ and f is the identity on $H \setminus H_0$. Then there exists an $h_0 \in H_0 \cap f^{-1}(0)$, so the kernel of f is dense in $\mathbb{R}$. Moreover, $H \subset f(\mathbb{R})$, thus f is a surjection and consequently it maps every non-degenerate interval onto $\mathbb{R}$. (See e.g. [MK, Theorem XII.6.1].) Hence f is Darboux and non-continuous.

We will verify that $f \in \mathrm{SCC}$. Let $V = \mathrm{LIN}\,(H_0)$ and $W = \mathrm{LIN}\,(H \setminus H_0)$. Since $|H_0| = \omega$, V is countable, and we have $\mathbb{R} = W + V = \bigcup_{v\in V}(W + v)$ and, moreover, $f(w) = w$ for $w \in W$. Observe that $f\restriction(W + v)$ is continuous for all $v \in V$. Indeed, if $x \in W + v$ then $w = x - v \in W$ and consequently, $f(x) = f(w+v) = f(w)+f(v) = w+f(v) = (x-v)+f(v)$. Let f_v be the linear function defined by $f_v(x) = (x - v) + f(v)$. Then $f\restriction(W + v) \subset f_v$ and therefore, $f \subset \bigcup_{v\in V} f_v$. □

Notice that for every $f \in \mathrm{SCC}$ the graph of f has measure zero (and is meager) on the plane. Thus f constructed in Proposition 1 is an example (in ZFC) of additive discontinuous Darboux function with a small graph. A similar result has been obtained by K. Ciesielski, who proved that CH implies the existence of an additive almost continuous function $f:\mathbb{R}\to\mathbb{R}$ whose graph has Lebesgue measure zero [KC, Corollary 2.2]. (An analogous example is constructed under CPA in [CP].) A function $f:\mathbb{R}\to\mathbb{R}$ is almost continuous in the sense of Stallings if each open subset of $\mathbb{R}^2$ containing the graph of f contains also a continuous function from $\mathbb{R}$ to $\mathbb{R}$ [JS]. Recall that every almost continuous function is Darboux. See e.g. [TN]. Ciesielski's example is not countably continuous. Thus the following problem seems to be interesting.

Problem 1. *Does there exist an additive almost continuous function $f\in \mathrm{CC}\setminus \mathrm{C}$?*

Notice that the function f constructed in Proposition 1 as well as examples in [GF] and [GH] are in fact countably linear. This remark leads to a natural question: does there exist an additive function $f\in \mathrm{SCC}\setminus \mathrm{CL}$? To answer this query we start with the following fact.

Theorem 2. *Suppose $\varphi:\mathbb{R}\to\mathbb{R}$ is continuous, $H\subset\mathbb{R}$ is non-meager and $f\in \mathrm{Add}$ is such that $f(x)=\varphi(x)$ for $x\in H$. If $f\in\mathrm{CC}$, then f is linear on some non-meager set $B\subset H$.*

Proof. Let $\mathbb{R}=\bigcup_{n<\omega}A_n$ and $f_n=f\restriction A_n$ be continuous for $n<\omega$. Let $\{I_n:\ n<\omega\}$ be a sequence of all open intervals with rational end-points. For each $h\in H$ there exist $n_h,m_h<\omega$ for which the set $(H-h)\cap A_{n_h}$ is nowhere meager in I_{m_h}. Hence there are $n_0,m_0<\omega$ for which the set H_0 of all $h\in H$ with $\langle n_h,m_h\rangle=\langle n_0,m_0\rangle$ is non-meager and consequently nowhere meager in some interval (a,b). Fix $x\in I_{m_0}\cap A_{n_0}$ and $y,y'\in(a,b)\cap H_0$. Then there exist sequences $\langle x_n\rangle_n$ in $H\cap(A_{n_0}+y)$ and $\langle x'_n\rangle_n$ in $H\cap(A_{n_0}+y')$ such that $x=\lim_n(x_n-y)=\lim_n(x'_n-y')$. Since f_{n_0} is continuous at x, $\lim_n f_{n_0}(x_n-y)=\lim_n f_{n_0}(x'_n-y')$. On the other hand, $f_{n_0}(x_n-y)=f(x_n-y)=f(x_n)-f(y)=\varphi(x_n)-\varphi(y)\to_n\varphi(x+y)-\varphi(y)$. Similarly, $f_{n_0}(x'_n-y')\to_n\varphi(x+y')-\varphi(y')$. Therefore for every $x\in I_{m_0}\cap A_{n_0}$ and all $y,y'\in(a,b)\cap H_0$ we have the equality

$$\varphi(x+y)-\varphi(y)=\varphi(x+y')-\varphi(y') \tag{1}$$

Since A_{n_0} is dense in I_{m_0}, H_0 is dense in (a,b) and φ is continuous, the equation (1) holds for all $x\in I_{m_0}$ and $y,y'\in(a,b)$.

Fix non-empty open intervals $I\subset I_{m_0}$ and $J\subset(a,b)$ such that I, J and $I+J$ are pairwise disjoint. Now, let $\psi:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\psi(x) = \varphi(x)$ for $x \in J \cup (I + J)$ and $\psi(x) = \varphi(x + y) - \varphi(y)$ for $x \in I$, where $y \in J$. (Note that this definition does not depend on y.) Then for all $x \in I$ and $y \in J$ we have the equality $\psi(x + y) = \psi(x) + \psi(y)$. This implies that there exist $g \in \mathrm{Add}$ and $c \in \mathbb{R}$ such that $\varphi(y) = \psi(y) = g(y) + c$ for $y \in J$. (See [MK, Theorem XIII.6.1].) Since φ is continuous, g is continuous on J, hence there is $a \in \mathbb{R}$ such that $g(y) = ay$ for $y \in \mathbb{R}$, and consequently, φ is linear on J. This implies that f is linear on a non-meager set $B = H_0 \cap J$. □

Lemma 3. *Suppose $f \in \mathrm{Add} \cap \mathrm{CC}$ and H_1, H_2 are disjoint non-meager sets. If $\varphi_1 = f\restriction H_1$ and $\varphi_2 = f\restriction H_2$ are linear then they have the same direction.*

Proof. Let $\varphi_i(x) = a_i x + b_i$ for $x \in H_i$, $i = 1, 2$. Let $\{I_n : n < \omega\}$ be a sequence of all open intervals with rational end-points. Let $\mathbb{R} = \bigcup_{n<\omega} A_n$ and $f_n = f\restriction A_n$ be continuous for all $n < \omega$. For every $h \in H_2$ there exist $n_h, m_h < \omega$ such that the set $(H_1 + h) \cap A_{n_h}$ is nowhere meager in I_{m_h}. There exist n_0, m_0 for which the set $H_0 = \{h \in H_2 : \langle n_h, m_h\rangle = \langle n_0, m_0\rangle\}$ is non-meager. Fix $x \in I_{m_0} \cap A_{n_0}$ and $h, h' \in H_0$ with $h \neq h'$. Then there exist two sequences $\langle x_n\rangle_n$, $\langle y_n\rangle_n$ in H_1 such that $x_n + h, y_n + h' \in A_{n_0}$ and $\lim_n(x_n + h) = x = \lim_n(y_n + h')$. Since f_{n_0} is continuous, $\lim_n f_{n_0}(x_n + h) = \lim_n f_{n_0}(y_n + h')$. On the other hand, $\lim_n f_{n_0}(x_n + h) = a_1(x - h) + b_1 + a_2 h + b_2$ and $\lim_n f_{n_0}(y_n + h') = a_1(x - h') + b_1 + a_2 h' + b_2$. Thus $a_1(x - h) + a_2 h = a_1(x - h') + a_2 h'$ and consequently, $a_1 = a_2$. □

Lemma 4. *Let H be a Hamel basis in $\mathbb{R}$ and let $f \in \mathrm{Add}$. If there exists a sequence $\langle f_n\rangle_n$ of linear functions which covers $f\restriction H$ and all f_n have the same direction, then $f \in \mathrm{CL}$.*

Proof. Let $f_n : x \mapsto ax + b_n$ for $n < \omega$. For any n define $H_n = \{x \in H : f(x) = f_n(x)\} \setminus \bigcup_{i<n} H_i$. Then $H = \bigcup_{n<\omega} H_n$ and H_n are pairwise disjoint. Let $T = \bigcup_{n<\omega} (\mathbb{Q}^*)^n$. Notice that $\mathbb{R} = \bigcup_{\langle q_0,\ldots,q_{n-1}\rangle \in T} \sum_{i<n} q_i H \cup \{0\}$. Fix $\langle q_0, \ldots, q_{n-1}\rangle \in T$. Then $\sum_{i<n} q_i H = \sum_{i<n} \bigcup_{j<\omega} H_j = \bigcup_{s\in\omega^n} \sum_{i<n} q_i H_{s(i)}$. It is enough to observe that for any $s \in \omega^n$, f is linear on the set $\sum_{i<n} q_i H_{s(i)}$. In fact, if $x \in \sum_{i<n} q_i H_{s(i)}$ then $x = \sum_{i<n} q_i h_i$, where $h_i \in H_{s(i)}$, hence $f(x) = ax + d$, where $d = \sum_{i<n} q_i b_{s(i)}$. □

Theorem 5. *Every additive strongly countably continuous function is countably linear.*

Proof. Assume that $f \in \mathrm{Add}$ and $\langle f_n\rangle_n$ is a sequence of continuous functions, $f_n : \mathbb{R} \to \mathbb{R}$, such that $f \subset \bigcup_{n<\omega} f_n$. For every $n < \omega$ set $A_n = \{x \in \mathbb{R} : f(x) = f_n(x)\}$. Let N be the set of all $n < \omega$ for which the set A_n is non-meager.

Notice that $A = \bigcup_{n \notin N} A_n$ is meager. Since A_n is non-meager for $n \in N$, Theorem 2 yields that f_n is linear on some non-meager subset of A_n. By Lemma 3, all f_n for $n \in N$ have the same direction. Now, the set $B = \bigcup_{n \in N} A_n$ is residual, so the Piccard Theorem implies easily that B includes a Hamel basis H. ([MK, Theorem II.9.1], see also Theorems IX.3.2 and IX.3.6 in [MK].) By Lemma 4, this shows that $f \in \mathrm{CL}$. □

Corollary 6. *Every additive and countably linear function f can be covered by countably many linear functions with the same direction.*

Finally, recall that $\mathrm{CC} \setminus \mathrm{SCC} \neq \emptyset$. (In fact, every increasing left-hand continuous function with a countable dense set of points of discontinuity is CC but not SCC, see [GF, Example 1], c.f., [GH].)

Problem 2. *Does there exist $f \in \mathrm{Add} \cap \mathrm{CC} \setminus \mathrm{SCC}$?*

(In fact, we guess that every function $f \in \mathrm{Add} \cap \mathrm{CC}$ can be covered by countably many lines, but we are unable to prove this hypothesis.)

A function $f : \mathbb{R} \to \mathbb{R}$ is *finitely continuous* if there is a decomposition of $\mathbb{R}$ onto finitely many parts A_i, $i < n$, with $f \restriction A_i$ continuous for each $i < n$. (See e.g. [MM] or [MP].)

Proposition 7. *Every additive and finitely continuous function $f : \mathbb{R} \to \mathbb{R}$ is continuous.*

Proof. Assume $\mathbb{R} = \bigcup_{i<n} X_i$ and $f \restriction X_i$ is continuous for each $i < n$. Let J_0 be an non-degenerate interval such that f is bounded on $J_0 \cap X_0$. If $J_0 \cap X_1 = \emptyset$, then set $J_1 = J_0$. Otherwise, let J_1 be a non-degenerate subinterval of J_0 such that f is bounded on $J_1 \cap X_1$. Proceeding in the same way we construct a decreasing sequence of non-degenerate intervals J_i, $i < n$, such that f is bounded on each of sets $J_i \cap X_i$. Then, since $J_{n-1} = \bigcup_{i<n} (J_{n-1} \cap X_i)$, f is bounded on J_{n-1}, so f is continuous [MK, Theorem IX.1.2]. □

Acknowledgements. The author is greatly indebted to the Referee for his helpful suggestions concerning a presentation of results, especially for a simplification of the proof of Proposition 1.

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(Received January 18, 2010; revised February 4, 2011)