Publ. Math. Debrecen
79/1-2 (2011), 7–22
DOI: 10.5486/PMD.2011.4826

Global signed total domination in graphs

By MARYAM ATAPOUR (Tabriz), SEYED MAHMOUD SHEIKHOLESLAMI (Tabriz)
and ABDOLLAH KHODKAR (Carrollton)

Abstract. A function $f : V(G) \to \{-1, 1\}$ defined on the vertices of a graph G is a signed total dominating function (STDF) if the sum of its function values over any open neighborhood is at least one. A STDF f of G is called a global signed total dominating function (GSTDF) if f is also a STDF of the complement $\overline{G}$ of G. The global signed total domination number $\gamma_{gst}(G)$ of G is defined as $\gamma_{gst}(G) = \min\{\sum_{v\in V(G)} f(v) \mid f$ is a GSTDF of $G\}$. In this paper first we find lower and upper bounds for the global signed total domination number of a graph. Then we prove that if T is a tree of order $n \geq 4$ with $\Delta(T) \leq n-2$, then $\gamma_{gst}(T) \leq \gamma_{st}(T)+4$. We characterize all the trees which satisfy the equality. We also characterize all trees T of order $n \geq 4$, $\Delta(T) \leq n-2$ and $\gamma_{gst}(T) = \gamma_{st}(T) + 2$.

1. Introduction

In the whole paper, G is a simple graph without isolated vertices and with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the open neighborhood $N_G(v) = N(v)$ is the set $\{u \in V \mid uv \in E\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$. The *minimum* and *maximum* degrees of G are respectively denoted by δ and Δ. For a vertex v in a rooted tree T, we let $C(v)$ denote the set of children of v and let $D(v)$ denote the set of descendants of v and $D[v] = D(v)\cup\{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A leaf of T is a vertex of degree 1 and a support vertex

---

Mathematics Subject Classification: 05C69, 05C05.
Key words and phrases: signed dominating function, signed total domination number, global signed total dominating function, global signed total domination number.

is a vertex adjacent to a leaf. The set of leaves and the set of support vertices in T are denoted by $L(T)$ and $S(T)$, respectively. We use [7] for terminology and notation which are not defined here.

For a real-valued function $f : V \to \mathbb{R}$ the *weight* of f is $\omega(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V)$. For a vertex v in V, we denote $f(N(v))$ by $f[v]$. Let $f : V \to \{-1, 1\}$ be a function which assigns to each vertex of G an element of the set $\{-1, 1\}$. The function f is said to be a *signed total dominating function* (STDF) of G (see [8]) if $f[v] \geq 1$ for every $v \in V$. Note that G admits a STDF if and only if G has no isolated vertices. The *signed total domination number* of G, denoted by $\gamma_{st}(G)$, is the minimum weight of a signed total dominating function of G. The signed total domination number has been studied by several authors (see for example [1], [2], [4], [6]).

A signed total dominating function f of G is called a *global signed total dominating function* (GSTDF) if f is also a STDF of its complement $\overline{G}$. Note that G admits a GSTDF if and only if G and $\overline{G}$ have no isolated vertices. The global signed total domination number of G, denoted by $\gamma_{gst}(G)$, is the minimum weight of a GSTDF of G. A $\gamma_{st}(G)$-function is a STDF of G with $\omega(f) = \gamma_{st}(G)$. A $\gamma_{gst}(G)$-function is defined similarly. For a (global) signed total dominating function f of G we define $P = P_f = \{v \in V \mid f(v) = 1\}$ and $M = M_f = \{v \in V \mid f(v) = -1\}$. Since every GSTDF of G is a STDF on both G and $\overline{G}$, we have

$$\gamma_{gst}(G) \geq \max\{\gamma_{st}(G), \gamma_{st}(\overline{G})\}. \tag{1}$$

Our purpose in this paper is to initiate the study of the global signed total domination numbers in graphs. First we find lower and upper bounds for the global signed total domination number of a graph. Then we prove that if T is a tree of order $n \geq 4$ with $\Delta \leq n - 2$, then $\gamma_{gst}(T) \leq \gamma_{st}(T) + 4$. We characterize all trees which satisfy the equality. Note that the condition $\Delta \leq n - 2$ guaranties that $\overline{T}$, the complement of T, has no isolated vertices. We also characterize all trees T of order $n \geq 4$, $\Delta(T) \leq n - 2$ and $\gamma_{gst}(T) = \gamma_{st}(T) + 2$. Finally, we calculate $\gamma_{gst}(G)$ for complete bipartite graphs G.

2. Preliminary and bounds

We make use of the following results.

Theorem A ([4]). *If G is a graph of order n with minimum degree $\delta \geq 2$ and maximum degree Δ, then*

$$\gamma_{st}(G) \geq \left(\frac{\lceil \frac{\delta-1}{2} \rceil - \lfloor \frac{\Delta-1}{2} \rfloor + 1}{\lceil \frac{\delta-1}{2} \rceil + \lfloor \frac{\Delta-1}{2} \rfloor + 1} \right) n.$$

An immediate consequence of Theorem A now follows.

Corollary 1. *For every graph G of order $n \geq 13$ with $2 \leq \delta \leq \Delta \leq 3$, $\gamma_{gst}(G) = \gamma_{st}(G)$.*

Proof. Let f be a $\gamma_{st}(G)$-function and $v \in V$. We show that f is also a STDF of $\overline{G}$. By Theorem A, we have $\gamma_{st}(G) \geq n/3$ and so $\gamma_{st}(G) \geq 5$. Since $\Delta \leq 3$, $f(N_G[v]) \leq 4$. It follows that

$$f(N_{\overline{G}}(v)) \geq \gamma_{st}(G) - 4 \geq 1.$$

Now the result follows by (1). □

Theorem B ([4]). *If T is a tree of order $n \geq 2$, then $\gamma_{st}(T) \geq 2$ with equality if and only if every vertex $v \in V(T) \setminus L(T)$ has odd degree and is adjacent to at least $\frac{\deg(v)-1}{2}$ leaves. Moreover, if $\gamma_{st}(T) = 2$ and f is a $\gamma_{st}(T)$-function, then $\sum_{x \in N(v)} f(x) = 1$ for each $v \in V(T)$ and also $M_f \subseteq L(T)$.*

Theorem C ([8]). *For every graph G of order $n \geq 2$ and $\delta(G) \geq 1$, $\gamma_{st}(G) \equiv n \pmod 2$.*

Theorem D ([8]). *For $n \geq 3$, $\gamma_{st}(C_n) = n$.*

We conclude this section with some propositions on $\gamma_{gst}(G)$.

Proposition 2. *Let G be a graph of order n such that G and $\overline{G}$ have no isolated vertices. Then $\gamma_{gst}(G) \equiv n \pmod 2$.*

Proof. Let f be a $\gamma_{gst}(G)$-function . Obviously, $n = |P_f| + |M_f|$ and $\gamma_{gst}(G) = |P_f| - |M_f|$. Therefore, $n - \gamma_{gst}(G) = 2|M_f|$ and the result follows. □

By Theorem C and Proposition 2 we have:

Proposition 3. *Let G be a graph such that G and $\overline{G}$ have no isolated vertices. Then $\gamma_{gst}(G) \equiv \gamma_{st}(G) \pmod 2$.*

The next proposition shows that the global signed total domination number of a graph is a positive integer.

Proposition 4. *Let G be a graph such that G and $\overline{G}$ have no isolated vertices. Then*

$$\gamma_{gst}(G) \geq \max\{3, \gamma_{st}(G), \gamma_{st}(\overline{G})\}.$$

Furthermore, this bound is sharp.

Proof. By assumption $n \geq 4$ and by (1), $\gamma_{gst}(G) \geq \max\{\gamma_{st}(G), \gamma_{st}(\overline{G})\}$. Thus, it suffices to prove $\gamma_{gst}(G) \geq 3$. Let f be a $\gamma_{gst}(G)$-function. Obviously, $P \neq \emptyset$. Assume $x \in P$. Then

$$|N_G(x) \cap P| \geq |N_G(x) \cap M| + 1 \tag{2}$$

and

$$|N_{\overline{G}}(x) \cap P| \geq |N_{\overline{G}}(x) \cap M| + 1. \tag{3}$$

By (2) and (3),

$$|N_G(x) \cap P| + |N_{\overline{G}}(x) \cap P| \geq |N_G(x) \cap M| + |N_{\overline{G}}(x) \cap M| + 2.$$

Since $x \in P$, it follows that $|P| \geq |M| + 3$ and so $\gamma_{gst}(G) = |P| - |M| \geq 3$.

In order to prove the sharpness, assume $k \geq 1$ and let G be a graph with vertex set

$$V(G) = \{u_i, v_i, x_j, y_j, z_j \mid 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 2k + 1\}$$

and edge set

$$E(G) = \{u_i u_{i+1}, u_i v_i, u_{i+1} v_i \mid 1 \leq i \leq 3\} \cup \{u_1 x_j, u_2 y_j, u_3 z_j \mid 1 \leq j \leq 2k + 1\},$$

where $u_4 = u_1$. Define $f : V(G) \to \{-1, 1\}$ by $f(v_i) = f(x_j) = f(y_j) = f(z_j) = -1$ for $1 \leq i \leq 3$, $1 \leq j \leq k$ and $f(x) = 1$ otherwise. It is easy to see that f is a GSTDF of G with $\omega(f) = 3$. Thus $\gamma_{gst}(G) = 3$ and the proof is complete. □

The next theorem shows that the difference $\gamma_{gst}(G) - \max\{\gamma_{st}(G), \gamma_{st}(\overline{G})\}$ can be arbitrarily large.

Theorem 5. *For every positive integer k, there exists a connected graph G such that $\overline{G}$ is connected and*

$$\gamma_{gst}(G) - \max\{\gamma_{st}(G), \gamma_{st}(\overline{G})\} \geq 2k + 1.$$

Proof. Let G be the graph with vertex set $V(G) = \{u_i, v_i \mid 0 \leq i \leq 4k - 1\}$ and edge set $E(G) = \{v_i v_j \mid 0 \leq i \neq j \leq 4k - 1\} \cup \{u_i v_i, u_i v_{i+1}, \ldots, u_i v_{i+2k-1} \mid 0 \leq i \leq 4k - 1\}$, where the indices are taken modulo $4k$. Obviously, $G \simeq \overline{G}$ and so $\gamma_{st}(G) = \gamma_{st}(\overline{G})$. Define $f : V(G) \to \{-1, 1\}$ by $f(v_i) = 1$ if $i \in \{0, 1, \ldots, 3k\}$ and $f(x) = -1$ otherwise. It is easy to see that f is a STDF of G which implies that $\gamma_{st}(G) \le \omega(f) = 2 - 2k$. Therefore $\max\{\gamma_{st}(G), \gamma_{st}(\overline{G})\} \le 2 - 2k$. By Proposition 4, $\gamma_{gst}(G) - \max\{\gamma_{st}(G), \gamma_{st}(\overline{G})\} \ge 2k + 1$ and the proof is complete. $\square$

Theorem 6. *For every graph G of order n,*

$$\gamma_{gst}(G) \le n - 2\min\left\{\left\lfloor\frac{\delta(G)-1}{2}\right\rfloor, \left\lfloor\frac{\delta(\overline{G})-1}{2}\right\rfloor\right\}.$$

Proof. Let, without loss of generality,

$$\theta = \left\lfloor\frac{\delta(G)-1}{2}\right\rfloor = \min\left\{\left\lfloor\frac{\delta(G)-1}{2}\right\rfloor, \left\lfloor\frac{\delta(\overline{G})-1}{2}\right\rfloor\right\}.$$

Suppose that $v_1, \ldots, v_\theta$ are distinct vertices of G. Define $f : V(G) \to \{-1, 1\}$ by $f(v_i) = -1$ for $i = 1, \ldots, \theta$ and $f(x) = 1$ if $x \notin \{v_1, \ldots, v_\theta\}$. It is easy to see that f is a GSTDF of G and $\omega(f) = n - 2\theta$. Now the result follows. $\square$

3. Trees

In this section we study the global signed total domination numbers in trees. We note that a tree T of order $n \ge 4$ admits a GSTDF if and only if $\Delta(T) \le n-2$. The condition $\Delta(T) \le n-2$ guaranties that $\overline{T}$ has no isolated vertices. Recall that, for every pair u, v of distinct vertices in V, the *distance* $d(u, v)$ is the minimum length of a $(u$-$v)$-*path*. We begin with the following lemma.

Lemma 7. *Let T be a tree of order $n \ge 4$ with $\Delta(T) \le n-2$. If f is a signed total dominating function of T, then $\sum_{u\in N_{\overline{T}}(v)} f(u) \ge 0$ for every $v \in V(T)$.*

Proof. Let T be rooted at a vertex v. If v is a leaf with support vertex w, then $f(w) = 1$ and by Theorem B,

$$\sum_{u\in N_{\overline{T}}(v)} f(u) = \sum_{x\in V(T)} f(x) - f(v) - f(w) \ge 1 - f(v) \ge 0.$$

Now assume that v is not a leaf and $N(v) = \{v_1, v_2, \ldots, v_t\}$. We consider two cases.

Case 1. $f(v) = -1$. Then v is not a support vertex and f, restricted to T_{v_i}, is a signed total dominating function of T_{v_i} for each $1 \le i \le t$. By Theorem B, $f(N_{\overline{T}}(v) \cap V(T_{v_i})) = f(V(T_{v_i})) - f(v_i) \ge 2 - f(v_i) \ge 1$. Hence,

$$\sum_{u \in N_{\overline{T}}(v)} f(u) = \sum_{i=1}^{t} f(N_{\overline{T}}(v) \cap V(T_{v_i})) \ge 2.$$

Case 2. $f(v) = 1$. Since $\Delta(T) \le n-2$, v has some neighbors which are not leaves. Without loss of generality, we may assume $v_1, \dots, v_s$ are the neighbors of v with $\deg(v_i) \ge 2$ for $1 \le i \le s \le t$. Let $T_i = T[V(T_{v_i}) \cup \{v\}]$ be the subgraph induced by $V(T_{v_i}) \cup \{v\}$ for each $1 \le i \le s$. We consider two subcases.

Subcase 2.1. $f(v_i) = 1$ for each $1 \le i \le s$. Then f, restricted to T_i, is a signed total dominating function of T_i for each i. By theorem B,

$$f(N_{\overline{T}}(v) \cap V(T_i)) = f(V(T_i)) - f(v_i) - f(v) \ge 0. \tag{4}$$

Hence,

$$\sum_{u \in N_{\overline{T}}(v)} f(u) = \sum_{i=1}^{s} f(N_{\overline{T}}(v) \cap V(T_i)) \ge 0. \tag{5}$$

Subcase 2.2. $f(v_i) = -1$ for some $1 \le i \le s$. Let, without loss of generality, $f(v_1) = -1$. Set $S = V(T) \setminus (L(T) \cup \{v\})$ and $S_i = V(T_i) \setminus L(T_i)$. Then

$$\sum_{u \in N_{\overline{T}}(v)} f(u) = \sum_{i=1}^{s} f(N_{\overline{T}}(v) \cap V(T_i)) = \sum_{i=1}^{s} \sum_{u \in S_i} f(C(u)). \tag{6}$$

Since $f[u] \ge 1$ for each $u \in S$ and $N(u)$ consists of the parent and children of u, it follows that $f(C(u)) \ge 0$. On the other hand, since $f(v_1) = -1$, v_1 is not a support vertex and for each $u \in C(v_1)$ we have $1 \le f[u] = f(v_1) + f(C(u)) = -1 + f(C(u))$. Therefore for each $u \in C(v_1)$, $f(C(u)) \ge 2$ and so $f(N_{\overline{T}}(v) \cap V(T_1)) = \sum_{u \in S_1} f(C(u)) \ge 2$. By (6),

$$\sum_{u \in N_{\overline{T}}(v)} f(u) = \sum_{i=1}^{s} \sum_{u \in S_i} f(C(u)) \ge 2.$$

This completes the proof. $\square$

A closer look at the proof of Lemma 7 shows that:

Corollary 8. *Let T be a tree of order n with $\Delta(T) \le n-2$, $v \in V(T)$ and f a signed total dominating function of T. If $\sum_{u \in N_{\overline{T}}(v)} f(u) = 0$, then either v is* *a leaf or $f(v) = 1$ and f assigns the value 1 to every neighbor of v whose degree is at least two.*

Theorem 9. *Let T be a tree of order $n \geq 4$ with $\Delta(T) \leq n - 2$. Then*

$$\gamma_{gst}(T) \leq \gamma_{st}(T) + 4.$$

PROOF. Let f be a $\gamma_{st}(T)$-function. If $|M_f| \leq 2$, then define $g : V(T) \to \{-1, 1\}$ by $g(x) = 1$ for each $x \in V(T)$. Obviously, g is a GSTDF of T and so

$$\gamma_{gst}(T) \leq \omega(g) \leq \omega(f) + 4 = \gamma_{st}(T) + 4.$$

Now let $|M_f| \geq 3$. First let there exist vertices u and v in M_f such that $d(u, v) \geq 3$. Define $g : V(T) \to \{-1, 1\}$ by

$$g(x) = 1 \text{ if } x \in P_f \cup \{u, v\} \text{ and } g(x) = -1 \text{ otherwise.}$$

We claim that g is a GSTDF of T. Obviously, g is a STDF of T. Let $x \in V(T)$. Since $d(u, v) \geq 3$, $u \in N_{\overline{T}}(x)$ or $v \in N_{\overline{T}}(x)$. By Lemma 7,

$$\sum_{z \in N_{\overline{T}}(x)} g(z) = \sum_{z \in N_{\overline{T}}(x) - \{u,v\}} f(z) - \sum_{z \in N_{\overline{T}}(x) \cap \{u,v\}} f(z) \geq 1 + 1 = 2$$

and so g is a GSTDF of $\overline{T}$. Therefore

$$\gamma_{gst}(T) \leq \omega(g) \leq \omega(f) + 4 = \gamma_{st}(T) + 4.$$

Now let for each pair u, v in M_f, $d(u, v) \leq 2$. Then T has a subgraph isomorphic to a star $K_{1,t}, (t \geq 2)$ with center, say x, and leaves $\{x_1, \ldots, x_t\}$ such that $x_1, \ldots, x_t \in M_f$ and $M_f \subseteq \{x, x_1, \ldots, x_t\}$. Define $g : V(T) \to \{-1, 1\}$ by

$$g(z) = 1 \text{ if } z \in P_f \cup \{x_1, x_2\} \text{ and } g(z) = -1 \text{ otherwise.}$$

We claim that g is a GSTDF of T. Obviously, g is a STDF of T. It suffices to prove that g is a STDF of $\overline{T}$. Let $y \in V(T)$. First let $y = x$. Since $g(z) = 1$ for each $z \in V(T) \setminus N_T(x)$, it follows that $\sum_{z \in N_{\overline{T}}(x)} g(z) \geq 1$.

Now let $y \neq x$. Then either $yx_1 \notin E(T)$ or $yx_2 \notin E(T)$. Therefore $yx_1 \in E(\overline{T})$ or $yx_2 \in E(\overline{T})$. Then

$$\sum_{u \in N_{\overline{T}}(y)} g(u) = \sum_{u \in (N_{\overline{T}}(y) - \{x_1, x_2\})} f(u) - \sum_{u \in (N_{\overline{T}}(y) \cap \{x_1, x_2\})} f(u)$$

$$= \underbrace{\sum_{u \in N_{\overline{T}}(y)} f(u)}_{\geq 0 \text{ by Lemma 7}} - 2 \sum_{u \in (N_{\overline{T}}(y) \cap \{x_1, x_2\})} f(u) \geq 2.$$

Therefore g is a GSTDF of T and

$$\gamma_{gst}(T) \leq \omega(g) \leq \omega(f) + 4 = \gamma_{st}(T) + 4,$$

as desired. □

In what follows, we characterize all trees T which achieve the bound in Theorem 9.

Lemma 10. *Let T be a tree of order $n \geq 4$ with $\Delta(T) \leq n-2$. If $\gamma_{st}(T) = 2$, then $\gamma_{gst}(T) = \gamma_{st}(T) + 4$.*

Proof. Let $\gamma_{st}(T) = 2$ and let f be a $\gamma_{st}(T)$-function. By Theorem B,

$$M_f \subseteq L(T), \tag{7}$$

and each vertex $v \in V(T) \setminus L(T)$ is a support vertex. Assume that g is a GSTDF of T such that $|M_g \cap M_f|$ is maximum. Since g is a STDF of T and each vertex $v \in V(T) \setminus L(T)$ is a support vertex, $g(v) = 1$. Hence,

$$M_g \subseteq L(T). \tag{8}$$

We claim that $M_g \subseteq M_f$. Suppose to the contrary that $M_g \not\subseteq M_f$ and $u \in M_g - M_f$. Let v be the support vertex of u. Since $\omega(f) = \gamma_{st}(G) = 2$ and g is a STDF of T, we have $g[v] \geq 1 = f[v]$ and hence $|M_f \cap L(v)| \geq |M_g \cap L(v)|$, where $L(v)$ is the set of leaves adjacent to v. Let $w \in (M_f \cap L(v)) \setminus (M_g \cap L(v) - \{u\})$. Then the function $h : V(G) \to \{-1, 1\}$ defined by

$$h(u) = 1, h(w) = -1 \text{ and } h(x) = g(x) \text{ if } x \in V(G) - \{u, w\}$$

is obviously a GSTDF of T such that $|M_g \cap M_f| < |M_h \cap M_f|$ which is a contradiction. Thus $M_g \subseteq M_f$. Since $\gamma_{st}(T) = 2$ and by Proposition 4, $\gamma_{gst}(T) \geq 3$, we have $|M_f| \geq |M_g| + 1$. Assume $|M_f| = |M_g| + 1$. Let $M_f \setminus M_g = \{u\}$ and let v be the support vertex of u. Then by Theorem B,

$$2 = \gamma_{st}(T) = \sum_{x \in N_T(v)} f(x) + \sum_{x \in N_{\overline{T}}(v)} g(x) + g(v) = 2 + \sum_{x \in N_{\overline{T}}(v)} g(x),$$

which implies that $\sum_{x \in N_{\overline{T}}(v)} g(x) = 0$, a contradiction. Therefore $|M_f| \geq |M_g| + 2$ and so $\gamma_{gst}(T) \geq \gamma_{st}(T) + 4$. Now the result follows by Theorem 9. □

Lemma 11. *Let T be a tree of order $n \geq 4$ with $\Delta(T) \leq n-2$. If $\gamma_{gst}(T) = \gamma_{st}(T) + 4$, then $\gamma_{st}(T) = 2$.*

Proof. Let $\gamma_{gst}(T) = \gamma_{st}(T) + 4$ and let f be a $\gamma_{st}(T)$-function. By assumption, f is not a GSTDF of T. By Lemma 7, there exists a vertex $v \in V(T)$ such that $\sum_{u \in N_{\overline{T}(v)}} f(u) = 0$. By Corollary 8, either v is a leaf or $f(v) = 1$ and f assigns the value 1 to every neighbor of v whose degree is at least two. First assume v is a leaf and w is its support vertex. Since $\gamma_{st}(T) \geq 2$, by Theorem B, and $\sum_{u \in N_{\overline{T}(v)}} f(u) = 0$, we have $f(v) = f(w) = 1$. This implies that

$$\gamma_{st}(T) = \sum_{u \in N_{\overline{T}(v)}} f(u) + f(v) + f(w) = 2.$$

Now suppose that v is not a leaf. Let $N_T(v)=\{v_1, v_2, \ldots, v_t\}$ and $\deg(v_i)\geq 2$ for $1 \leq i \leq s \leq t$. By Corollary 8, $f(v) = f(v_i) = 1$ for $1 \leq i \leq s$. Let $T_i = T[V(T_{v_i}) \cup \{v\}]$ be the subgraph induced by $V(T_{v_i}) \cup \{v\}$ for each $1 \leq i \leq s$. Clearly, f restricted to T_i is a STDF of T_i for each $1 \leq i \leq s$. Since $\sum_{u \in N_{\overline{T}(v)}} f(u) = 0$, by (4) and (5),

$$f(N_{\overline{T}}(v) \cap V(T_{v_i})) = 0 \tag{9}$$

for $1 \leq i \leq s$. Hence, $\gamma_{st}(T_i) = 2$ for $1 \leq i \leq s$. We claim that $f[v] = 1$, which implies that

$$\gamma_{st}(T) = f(v) + f[v] + \sum_{i=1}^{s} f(N_{\overline{T}}(v) \cap V(T_{v_i})) = 2.$$

Let, to the contrary, $f[v] \geq 2$. Since $\gamma_{st}(T_1) = 2$ and the order of T_1 is at least 3, by Theorem B, f must assign the value -1 to a leaf in T_1. Let $z \in V(T_1)$ be a leaf in T such that $f(z) = -1$. Define $g : V(T) \to \{-1, 1\}$ by

$$g(z) = 1 \text{ and } g(x) = f(x) \text{ otherwise.}$$

We claim that g is a GSTDF of T. Obviously, g is a STDF of T. It suffices to prove that g is a STDF of $\overline{T}$. Let $y \in V(T)$. If $y = v$, then by (9)

$$\sum_{x \in N_{\overline{T}}(v)} g(x) = \underbrace{g(N_{\overline{T}}(v) \cap V(T_{v_1}))}_{\geq 2} + \sum_{i=2}^{s} f(N_{\overline{T}}(v) \cap V(T_{v_i})) \geq 2.$$

Let $y \neq v$. First assume $y \in \{v_{s+1}, \ldots, v_t\}$ if $t > s$. Then

$$\sum_{x \in N_{\overline{T}}(y)} g(x) = \underbrace{\sum_{x \in N(v)-\{y\}} g(x)}_{\geq 1 \text{ since } f[v] \geq 2} + \sum_{i=1}^{s} g(N_{\overline{T}}(y) \cap V(T_{v_i})) \geq 1.$$

Now let $y \in \{v_1, \ldots, v_s\}$. By the definition of g, $g(V(T_1)) = 4$ and $g(V(T_i)) = 2$ for $2 \leq i \leq s$ and we also have $g[v_i] = 1$ for $i = 2, \ldots, s$ and $g[v_1] = 3$ when v_1 is adjacent to z and $g[v_1] = 1$ otherwise. Since $g(y) = 1$, $g(V(T_1)) = 4$ and $g(V(T_i)) = 2$ for $2 \leq i \leq s$, it follows that

$$\begin{aligned}
g(N_{\overline{T}}(y) \cap (V(T_{v_1}) - \{v_1\})) &= 2 && \text{if } y = v_1 \quad \text{and } yz \notin E(T),\\
g(N_{\overline{T}}(y) \cap (V(T_{v_1}) - \{v_1\})) &= 0 && \text{if } y = v_1 \quad \text{and } yz \in E(T),\\
g(N_{\overline{T}}(y) \cap (V(T_{v_1}) - \{v_1\})) &= 2 && \text{if } y \in \{v_2, v_3, \ldots, v_s\}, \quad \text{and}\\
g(N_{\overline{T}}(y) \cap (V(T_{v_i}) - \{v_i\})) &= 0 && \text{if } 2 \leq i \leq s.
\end{aligned}$$

Therefore

$$\sum_{x \in N_{\overline{T}}(y)} g(x) = \underbrace{\sum_{x \in N(v)-\{y\}} g(x)}_{\geq 1 \text{ since } f[v] \geq 2} + \sum_{i=1}^{s} g(N_{\overline{T}}(y) \cap (V(T_{v_i}) \setminus \{v_i\})) \geq 1.$$

Finally, if $y \notin N_T[v]$, then as above we can see that $\sum_{x \in N_{\overline{T}}(y)} g(x) \geq 1$. Therefore g is a STDF of $\overline{T}$. This implies that $\gamma_{gst}(T) \leq \gamma_{st}(T) + 2$, a contradiction. This completes the proof. $\square$

By Lemmas 10 and 11, for a tree T of order $n \geq 4$ and $\Delta(T) \leq n - 2$, $\gamma_{gst}(T) = \gamma_{st}(T) + 4$ if and only if $\gamma_{st}(T) = 2$. Hence, by Theorem B we can state the following characterization.

Theorem 12. *Let T be a tree of order $n \geq 4$ with $\Delta(T) \leq n - 2$. Then $\gamma_{gst}(T) = \gamma_{st}(T) + 4$ if and only if every vertex $v \in V(T) \setminus L(T)$ has odd degree and is adjacent to at least $\frac{\deg(v)-1}{2}$ leaves.*

4. Trees T with $\gamma_{gst}(T) = \gamma_{st}(T) + 2$

In this section first we characterize the trees T with $\gamma_{st}(T) = 3$. Then we characterize the trees T for which $\gamma_{gst}(T) = \gamma_{st}(T) + 2$. Note that by Proposition 3, there is no tree T with $\gamma_{gst}(T) = \gamma_{st}(T) + 3$. Throughout this section $\ell(v)$

denotes the number of pendant edges at vertex v. We begin with the following observation.

Observation 13. *Let f be a STDF of G and let $v \in V(G)$. If $\deg(v)$ is even, then $f[v] \geq 2$.*

Let $\mathcal{T}_1$ be the collection of all trees of order $n \geq 2$ with exactly one vertex of even degree and $\ell(v) \geq \lfloor (\deg(v)-1)/2 \rfloor$ for every vertex v.

Theorem 14. *For any tree T, $\gamma_{st}(T) = 3$ if and only if $T \in \mathcal{T}_1$. Also, if $T \in \mathcal{T}_1$ and f is a $\gamma_{st}(T)$-function, then $f[v] = 1$ if v is an odd vertex and $f[v] = 2$ when v is an even vertex, for every vertex $v \in V(T)$.*

Proof. Let f be a $\gamma_{st}(T)$-function. We claim that the only vertices v for which $f(v) = -1$ are leaves. Let, to the contrary, v be a non leaf vertex of T and $f(v) = -1$. Then v is not a support vertex. Suppose that $N(v) = \{v_1, \ldots, v_t\}$. Root T at v. It is clear that f, restricted to T_{v_i}, is a STDF of T_{v_i} for each $1 \leq i \leq t$. By Theorem B, we have $f(V(T_{v_i})) \geq 2$ for each $1 \leq i \leq t$. Since

$$3 = \gamma_{st}(T) = \omega(f) = \sum_{i=1}^{t} f(V(T_{v_i})) - 1,$$

we must have $t = 2$ and $\gamma_{st}(T_{v_1}) = \gamma_{st}(T_{v_2}) = 2$. By Theorem B, $f(N_{T_{v_1}}(v_1)) = 1$ which implies that $f(N_T(v_1)) = 0$, a contradiction. Therefore, the only vertices v for which $f(v) = -1$ are leaves.

Since $\gamma_{st}(T) = 3$, by Theorem B, T has an even vertex or every vertex $v \in V(T) \setminus L(T)$ has an odd degree and at least one of them, say v, is adjacent to at most $\frac{\deg(v)-1}{2} - 1$ leaves. First let $v \in V(T) \setminus L(T)$ have odd degree and be adjacent to at most $\frac{\deg(v)-1}{2} - 1$ leaves. Root the tree T at v. Then $f[v] \geq 3$ and v has at least three children which are not leaves. Let $S = V(G) - L(T) - \{v\}$. Since $f[u] \geq 1$ for each $u \in S$, and since $N(u)$ consists of the parent of u and the set of children of u, it follows that $f(C(u)) \geq 0$ where the set $C(u)$ denotes the set of children of u. Now the sets $C(u)$, $u \in S$, together with the two sets $C(v)$ and $\{v\}$, partition $V(T)$. Thus

$$3 = \gamma_{st}(T) = \omega(f) = f(v) + f[v] + \sum_{u \in S} f(C(u)) \geq 1 + 3 + \sum_{u \in S} 0 \geq 4,$$

which is a contradiction.

Thus T has an even vertex, say v, and $f(v) = 1$ because v is not a leaf. Root the tree T at v. Then $f[v] = f(C(v))$, where $C(v)$ consists of children of v. If every child of v is a leaf, then $3 = \omega(f) = f(v) + f[v]$ which implies that $f[v] = 2$ and so $T \in \mathcal{T}_1$. Hence we may assume that at least one child of v is not a leaf. Let $S = V(T) \setminus (L(T) \cup \{v\})$. Since $f[u] \geq 1$ for each $u \in S$, and since $N(u)$ consists of the parent and children of u, it follows that $f(C(u)) \geq 0$. Now the sets $C(u)$, $u \in S$, together with the two sets $\{v\}$ and $C(v)$ partition $V(T)$. Thus

$$3 = \gamma_{st}(T) = f(v) + f[v] + \sum_{u \in S} f(C(u)). \tag{10}$$

By (10) and Observation 13, $f[v] = 2$ and $f(C(u)) = 0$ for every $u \in S$. It follows that every vertex in S, and therefore every vertex of $V(T) \setminus \{v\}$ has odd degree. Furthermore, $f[v] = 2$ and $f[w] = 1$ for every vertex $w \in V(T) \setminus \{v\}$.

Let $y \in V(T) \setminus L(T)$. Then $y \in S \cup \{v\}$. If $y = v$, then $f[v] = f(C(v)) = 2$, and so $(\deg(v) - 1)/2$ children of v are assigned the value -1 under f. Hence, v is adjacent to at least $(\deg(v) - 1)/2$ leaves. Suppose, then, that $y \neq v$. Since $f(C(y)) = 0$, $|C(y)|/2 = (\deg(y) - 1)/2$ children of y are assigned the value -1 under f. Hence, y is adjacent to at least $(\deg(y) - 1)/2$ leaves. Thus every vertex $y \in V(T) \setminus (L(T) \cup \{v\})$ has odd degree and also every vertex $y \in V(T) \setminus L(T)$ is adjacent to at least $(\deg(y) - 1)/2$ leaves. Thus $T \in \mathcal{T}_1$.

On the other hand, let $T \in \mathcal{T}_1$ and let u be the even vertex of T. Let g be the function defined as follows: for each $v \in V(T) \setminus L(T)$, assign the value -1 to $\lfloor (\deg(v) - 1)/2 \rfloor$ leaves adjacent to v. Assign to all other vertices the value 1. Then, $g[u] = 2$ and $g[v] = 1$ for each vertex $v \in V(T) \setminus \{u\}$, and every vertex that is assigned -1 under g is a leaf. It follows that g is a STDF of T. We now root T at u. Then, $g(C(u)) = g[u] = 2$ and for each $v \in V(T) \setminus (L(T) \cup \{u\})$, $g(C(v)) = 0$. It follows that

$$\gamma_{st}(T) \leq \omega(g) = g(u) + g[u] + \sum_{v \in V(T) \setminus (L(T) \cup \{u\})} g(C(v)) = 3.$$

By Theorem B, $\gamma_{st}(T) = 3$ and the proof is complete. □

Lemma 15. *Let T be a tree of order $n \geq 4$ with $\Delta(T) \leq n-2$. If $\gamma_{st}(T) = 3$, then $\gamma_{gst}(T) = \gamma_{st}(T) + 2$.*

Proof. By Proposition 3, Theorem 9 and Lemma 11, $\gamma_{gst}(T) \leq \gamma_{st}(T) + 2$. Let, to the contrary, $\gamma_{gst}(T) < \gamma_{st}(T) + 2$. Then by Proposition 3, $\gamma_{gst}(T) \leq \gamma_{st}(T)$. It follows that $\gamma_{gst}(T) = \gamma_{st}(T)$ by Proposition 4.

Assume that g is a $\gamma_{gst}(T)$-function. Then g is a $\gamma_{st}(T)$-function. Since $\gamma_{st}(T) = 3$, T has an even vertex, say u. By Theorem 14, $g(u) = 1$ and $g[u] = 2$.

Then

$$3 = \gamma_{gst}(T) = \omega(g) = g(u) + g[u] + \sum_{x\in N_{\overline{T}}(u)} g(x) = 3 + \sum_{x\in N_{\overline{T}}(u)} g(x).$$

It follows that $\sum_{x\in N_{\overline{T}}(u)} g(x) = 0$, a contradiction. This completes the proof. $\square$

Let $\mathcal{T}_2$ be the collection of all trees T of order $n \geq 2$ with $|V(T) \setminus (L(T) \cup S(T))| = 1$, and every vertex $u \in S(T)$ has odd degree and is adjacent to at least $\frac{\deg(u)-1}{2}$ leaves.

Lemma 16. *If $T \in \mathcal{T}_2$, then $\gamma_{gst}(T) = \gamma_{st}(T) + 2$.*

Proof. Let $T \in \mathcal{T}_2$ and let $v \in V(T) \setminus (L(T) \cup S(T))$. If $\deg(v) = 2$, then obviously $T \in \mathcal{T}_1$ and by Theorem 14 and Lemma 15, $\gamma_{gst}(T) = \gamma_{st}(T) + 2$. Now let $\deg(v) \geq 3$. By Theorem 12, $\gamma_{gst}(T) \leq \gamma_{st}(T) + 2$. Let, to the contrary, $\gamma_{gst}(T) < \gamma_{st}(T) + 2$. Then by Proposition 3, $\gamma_{gst}(T) \leq \gamma_{st}(T)$. It follows that $\gamma_{gst}(T) = \gamma_{st}(T)$ by Proposition 4. Assume that g is a $\gamma_{gst}(T)$-function. Then g is also a $\gamma_{st}(T)$-function. Suppose that $N(v) = \{v_1, \ldots, v_t\}$. Since $V(T) = L(T) \cup S(T) \cup \{v\}$ and $v \notin S(T)$, we have $v_i \in S(T)$ for each $1 \leq i \leq t$. It follows that $g(v_1) = \cdots = g(v_t) = 1$.

Claim 1. $g(v) = 1$.

Proof of Claim 1. Let, to the contrary, $g(v) = -1$. By assumption, v_i has odd degree and is adjacent to at least $\frac{\deg(v_i)-1}{2}$ leaves in T for each $1 \leq i \leq t$. Since $g[v_i] \geq 1$ for each $1 \leq i \leq t$, g must assign the value 1 to a leaf, say w_i, adjacent to v_i for each i. Define $f : V(T) \to \{-1, 1\}$ by

$$f(v) = 1, f(w_1) = \cdots = f(w_t) = -1 \text{ and } f(x) = g(x) \text{ otherwise.}$$

It is clear that f is a STDF of T with weight less than g, a contradiction. Thus $g(v) = 1$.

Let T be rooted at v and let $T_i = T[V(T_{v_i}) \cup \{v\}]$ for each $1 \leq i \leq t$. Obviously, g restricted to T_i is a STDF of T_i.

Claim 2. $\omega(g|_{T_i}) = 2$ for each $1 \leq i \leq t$.

Proof of Claim 2. Let, to the contrary, $\omega(g|_{T_i}) > 2$ for some i, say $i = 1$. By assumption and Theorem B, $\gamma_{st}(T_1) = 2$. Let h be a $\gamma_{st}(T_1)$-function. Then $h[v_1] = 1$ by Theorem B. Since v_1 is adjacent to at least $\frac{\deg(v_i)-1}{2} + 1$ leaves in T_1, without loss of generality, we may assume $h(v) = 1$. Define $f : V(T) \to \{-1, 1\}$ by

$$f(x) = h(x) \text{ if } x \in V(T_1) \quad \text{and} \quad f(x) = g(x) \text{ otherwise.}$$

Clearly, f is a STDF of T with weight less than g, a contradiction. Thus $\omega(g|_{T_i}) = 2$ for each $1 \le i \le t$.

Hence, $g(N_{\overline{T}}(v) \cap V(T_i)) = 0$ for each $1 \le i \le t$ and so

$$g(N_{\overline{T}}(v)) = \sum_{i=1}^{t} g(N_{\overline{T}}(v) \cap V(T_i)) = 0,$$

which contradicts the fact that g is a $\gamma_{gst}(T)$-function. This completes the proof. $\square$

Lemma 17. *Let T be a tree of order $n \ge 4$ with $\Delta(T) \le n-2$. If $\gamma_{gst}(T) = \gamma_{st}(T) + 2$, then $T \in \mathcal{T}_1 \cup \mathcal{T}_2$.*

PROOF. Let $\gamma_{gst}(T) = \gamma_{st}(T) + 2$ and let f be a $\gamma_{st}(T)$-function. By Theorem B and Lemma 10, $\gamma_{st}(T) \ge 3$. If $\gamma_{st}(T) = 3$, then by Lemma 15, $T \in \mathcal{T}_1$, as desired. Now let $\gamma_{st}(T) > 3$. Then by Theorems B and 14, either T has at least two vertices of even degree or there is a vertex u which is adjacent to at most $\lfloor(\deg(u) - 1)/2\rfloor - 1$ leaves.

Since $\gamma_{gst}(T) = \gamma_{st}(T) + 2$, f is not a GSTDF of T. By Lemma 7, there exists a vertex $v \in V(T)$ such that

$$\sum_{u \in N_{\overline{T}(v)}} f(u) = 0.$$

Let $N_T(v) = \{v_1, \ldots, v_t\}$ and assume $\deg(v_i) \ge 2$ for $1 \le i \le s \le t$.

Claim. $f(v) = 1$ and $v \in V(T) \setminus (L(T) \cup S(T))$.

Proof of Claim. By Corollary 8, either v is a leaf or $f(v) = f(v_i) = 1$ for $1 \le i \le s$. If v is a leaf and w is its support vertex, since $\gamma_{st}(T) \ge 4$, it follows that $\sum_{u \in N_{\overline{T}(v)}} f(u) \ge 2$, a contradiction. Hence, we may assume v is not a leaf. It remains to show that $v \notin S(T)$. Let, to the contrary, v be a support vertex and let v_t be a leaf. Since $\sum_{u \in N_{\overline{T}(v)}} f(u) = 0$, $\sum_{i=1}^{t} f(v_i) = \gamma_{st}(T) - 1 \ge 3$. Define $g : V(G) \to \{-1, 1\}$ by $g(v_t) = -1$ and $g(x) = f(x)$ for each $x \in V(T) \setminus \{v_t\}$. It is clear that g is a STDF of T of weight less than f, a contradiction. Therefore, v is not a support vertex and hence $f(v_i) = 1$ for $1 \le i \le t$. This proves our claim.

Assume T is rooted at v. Let $T_i = T[V(T_{v_i}) \cup \{v\}]$ be the subgraph induced by $V(T_{v_i}) \cup \{v\}$ for each $1 \le i \le t$. Since $\sum_{u \in N_{\overline{T}(v)}} f(u) = 0$, by (4) and (5) we have

$$f(N_{\overline{T}}(v) \cap V(T_{v_i})) = 0 \tag{11}$$

for each $1 \leq i \leq t$. Hence, $\gamma_{st}(T_i) = 2$ for each $1 \leq i \leq t$ by (11). It follows that each vertex $u \in V(T_i) \setminus \{v\}$ has odd degree in T_i. Now since v is a leaf adjacent to v_i in T_i with $f(v) = 1$ and $M_{f_{T_i}} \subseteq L(T_i)$, v_i is adjacent to at least $\lfloor \frac{\deg(v_i)-1}{2} \rfloor + 1$ leaves in T_i for each i. Hence v_i is adjacent to at least $\lfloor \frac{\deg(v_i)-1}{2} \rfloor$ leaves in T. Now let $u \in V(T_i) \setminus \{v, v_i\}$. Again since $\gamma_{st}(T_i) = 2$, u is adjacent to at least $\lfloor \frac{\deg(u)-1}{2} \rfloor$ leaves in T_i and hence in T. This implies that $T \in \mathcal{T}_2$ and the proof is complete. □

By Lemmas 15, 16 and 17, we can state the following characterization.

Theorem 18. *Let T be a tree of order $n \geq 4$ with $\Delta(T) \leq n-2$. Then $\gamma_{gst}(T) = \gamma_{st}(T) + 2$ if and only if $T \in \mathcal{T}_1 \cup \mathcal{T}_2$.*

5. The global signed total domination number of complete bipartite graphs

As the parameter $\gamma_{gst}(G)$ is new, it is important to determine its values for some familiar graphs. In this section we find the exact value of the global signed total domination number for complete bipartite graphs.

Theorem 19. *Let $K_{a,b}$ be a complete bipartite graph with the bipartition classes A, B such that $|A| = a$, $|B| = b$, $2 \leq b \leq a$. Then*

$$\gamma_{gst}(K_{a,b}) = \begin{cases} 4 & \text{if } a, b \text{ are both even} \\ 6 & \text{if } a, b \text{ are both odd} \\ 5 & \text{if } a \text{ and } b \text{ have different parity.} \end{cases}$$

PROOF. Let $A = \{x_1, x_2, \ldots, x_a\}$ and $B = \{y_1, y_2, \ldots, y_b\}$ be the partite sets of $K_{a,b}$. We consider three cases.

Case 1. a and b are both even.
Define $f : V(K_{a,b}) \to \{-1, 1\}$ by $f(x_i) = 1$ for $1 \leq i \leq \frac{a}{2} + 1$, $f(y_j) = 1$ if $1 \leq j \leq \frac{b}{2} + 1$ and $f(x) = -1$ otherwise. Obviously, f is a GSTDF of G and so $\gamma_{gst}(K_{a,b}) \leq \omega(f) = 4$. On the other hand, since $a + b$ is even, $\gamma_{gst}(K_{a,b}) \equiv 0 \pmod 2$ by Proposition 2. Therefore $\gamma_{gst}(K_{a,b}) \geq 4$ by Proposition 4. Thus $\gamma_{gst}(K_{a,b}) = 4$.

Case 2. a and b are both odd.
By (1) , we have $\gamma_{gst}(\overline{K_{a,b}}) \geq \gamma_{st}(\overline{K_{a,b}}) = \gamma_{st}(K_a) + \gamma_{st}(K_b) \geq 6$. Define $f : V(K_{a,b}) \to \{-1, 1\}$ by $f(x_i) = f(y_j) = 1$ for $1 \leq i \leq \lceil \frac{a}{2} \rceil + 1$, $1 \leq j \leq \lceil \frac{b}{2} \rceil + 1$

and $f(x) = -1$, otherwise. It is easy to verify that f is a GSTDF with $\omega(f) = 6$. This implies that $\gamma_{gst}(K_{a,b}) = 6$.

Case 3. a and b have different parity.
Assume a is even and b is odd (the case "a is odd and b is even" is similar). By (1), we have $\gamma_{gst}(\overline{K_{a,b}}) \geq \gamma_{st}(\overline{K_{a,b}}) = \gamma_{st}(K_a) + \gamma_{st}(K_b) \geq 5$. Define $f : V(K_{a,b}) \to \{-1, 1\}$ by $f(x_i) = 1$ for $1 \leq i \leq \frac{a}{2} + 1$, $f(y_j) = 1$ for $1 \leq j \leq \lceil \frac{b}{2} \rceil + 1$ and $f(x) = -1$ otherwise. It is easy to see that f is a GSTDF of G with $\omega(f) = 5$. This completes the proof. $\square$

References

[1] W. Chen and E. Song, Lower bounds on several versions of signed domination number, *Discrete Math.* **308** (2008), 1837–1846.

[2] L. Harris, J. H. Hattingh and M. A. Henning, Total k-subdominating functions on graphs, *Australas. J. Combin.* **35** (2006), 141–154.

[3] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Domination in Graphs: Advanced Topics, *Marcel Dekker*, *New York*, 1998.

[4] M. A. Henning, Signed total domination in graphs, *Discrete Math.* **278** (2004), 109–125.

[5] E. Sampathkumar, The global domination number of a graph, *J. Math. Phy. Sci.* **23** (1989), 377–385.

[6] E. Shan and T. C. E. Cheng, Upper bounds on the upper signed total domination number of graphs, *Discrete Appl. Math.* **157** (2009), 1098–1103.

[7] D. B. West, Introduction to Graph Theory, *Prentice-Hall, Inc*, 2000.

[8] B. Zelinka, Signed total domination number of a graph, *Czechoslovak Math. J.* **51** (2001), 225–229.

MARYAM ATAPOUR
DEPARTMENT OF MATHEMATICS
AZARBAIJAN UNIVERSITY OF TARBIAT MOALLEM
TABRIZ
I.R. IRAN

SEYED MAHMOUD SHEIKHOLESLAMI
DEPARTMENT OF MATHEMATICS
AZARBAIJAN UNIVERSITY OF TARBIAT MOALLEM
TABRIZ
I.R. IRAN

E-mail: s.m.sheikholeslami@azaruniv.edu

ABDOLLAH KHODKAR
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WEST GEORGIA
CARROLLTON, GA 30118
USA

E-mail: akhodkar@westga.edu

(Received February 1, 2010; revised March 1, 2011)