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The D numbers and the central factorial numbers

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Abstract. In this paper the author establishes some formulas for the D numbers $D_{2n}^{(k)}$ and $d_{2n}^{(k)}$. Several identities involving the D numbers, the Bernoulli numbers and the central factorial numbers are also presented.

1. Introduction

The Bernoulli polynomials $B_n^{(k)}(x)$ of order k, for any integer k, may be defined by (see [1], [3], [6], [10])

$$\left(\frac{t}{e^t-1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \quad |t| < 2\pi. \tag{1.1}$$

The numbers $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of order k, $B_n^{(1)} = B_n$ are the ordinary Bernoulli numbers (see [3], [5]). By (1.1), we can get (see [10, p. 145])

$$\frac{d}{dx}B_n^{(k)}(x) = nB_{n-1}^{(k)}(x), \tag{1.2}$$

$$B_n^{(k+1)}(x) = \frac{k-n}{k}B_n^{(k)}(x) + (x-k)\frac{n}{k}B_{n-1}^{(k)}(x) \tag{1.3}$$

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and

$$B_n^{(k+1)}(x+1) = \frac{nx}{k} B_{n-1}^{(k)}(x) - \frac{n-k}{k} B_n^{(k)}(x), \tag{1.4}$$

where $n \in \mathbb{N}$, $\mathbb{N}$ being the set of positive integers.

The numbers $B_n^{(n)}$ are called the Nörlund numbers (see [2]–[4], [10]). A generating function for the Nörlund numbers $B_n^{(n)}$ is (see [10, p. 150])

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!}. \tag{1.5}$$

The D numbers $D_{2n}^{(k)}$ may be defined by (see [8]–[11])

$$(t \csc t)^k = \sum_{n=0}^{\infty} (-1)^n D_{2n}^{(k)} \frac{t^{2n}}{(2n)!}, \quad |t| < \pi. \tag{1.6}$$

By (1.1), (1.6), and note that $\csc t = \frac{2i}{e^{it}-e^{-it}}$ (where $i^2 = -1$), we can get

$$\sum_{n=0}^{\infty} (-1)^n D_{2n}^{(k)} \frac{t^{2n}}{(2n)!} = \left(\frac{2it}{e^{it}-e^{-it}}\right)^k = \left(\frac{2it}{e^{2it}-1}\right)^k e^{kit} = \sum_{n=0}^{\infty} B_n^{(k)}\left(\frac{k}{2}\right) \frac{(2it)^n}{n!}.$$

Therefore,

$$D_{2n}^{(k)} = 4^n B_{2n}^{(k)}\left(\frac{k}{2}\right). \tag{1.7}$$

Taking $k = 1, 2$ in (1.7), and noting that $B_{2n}^{(1)}(\frac{1}{2}) = (2^{1-2n}-1)B_{2n}$, $B_{2n}^{(2)}(1) = (1-2n)B_{2n}$ (see [10, p. 22, p. 145]), we have

$$D_{2n}^{(1)} = (2-2^{2n})B_{2n} \quad \text{and} \quad D_{2n}^{(2)} = 4^n(1-2n)B_{2n}. \tag{1.8}$$

The D numbers $D_{2n}^{(k)}$ satisfy the recurrence relation (see [8])

$$D_{2n}^{(k)} = \frac{(2n-k+2)(2n-k+1)}{(k-2)(k-1)} D_{2n}^{(k-2)} - \frac{2n(2n-1)(k-2)}{k-1} D_{2n-2}^{(k-2)}. \tag{1.9}$$

By (1.9), we can immediately deduce the following (see [10,p. 147]):

$$D_{2n}^{(2n+1)} = \frac{(-1)^n(2n)!}{4^n}\binom{2n}{n}, \quad D_{2n}^{(2n+2)} = \frac{(-1)^n 4^n}{2n+1}(n!)^2 \tag{1.10}$$

and

$$D_{2n}^{(2n+3)} = \frac{(-1)^n(2n)!}{2\cdot 4^{2n}}\binom{2n+2}{n+1}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\frac{1}{(2n+1)^2}\right). \tag{1.11}$$

The numbers $D_{2n}^{(2n)}$ satisfy the recurrence relation (see [8])

$$\sum_{j=0}^{n} \frac{(-1)^j}{4^j(2j+1)} \binom{2j}{j} \frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!} = \frac{(-1)^n}{4^n} \binom{2n}{n}, \tag{1.12}$$

so we find $D_0^{(0)} = 1$, $D_2^{(2)} = -\frac{2}{3}, D_4^{(4)} = \frac{88}{15}, D_6^{(6)} = -\frac{3056}{21}, D_8^{(8)} = \frac{319616}{45}$, $D_{10}^{(10)} = -\frac{18940160}{33}, \ldots$.

By (1.14), we can get a generating function for $D_{2n}^{(2n)}$ (see [8]):

$$\frac{t}{\sqrt{1+t^2}\log(t+\sqrt{1+t^2}\,)} = \sum_{n=0}^{\infty} D_{2n}^{(2n)} \frac{t^{2n}}{(2n)!}, \quad |t| < 1. \tag{1.13}$$

The numbers $D_{2n}^{(2n-1)}$ satisfy the recurrence relation (see [9]):

$$\sum_{j=1}^{n} \binom{2n}{2j} (-1)^{j-1} 4^{j-1} ((j-1)!)^2 D_{2n-2j}^{(2n-1-2j)} = \frac{(-1)^{n-1} 2(2n)!}{4^n} \binom{2n-2}{n-1}, \tag{1.14}$$

so we find $D_2^{(1)} = -\frac{1}{3}$, $D_4^{(3)} = \frac{17}{5}$, $D_6^{(5)} = -\frac{1835}{21}$, $D_8^{(7)} = \frac{195013}{45}$, $D_{10}^{(9)} = -\frac{3887409}{11}, \ldots$.

By (1.16), we can get a generating function for $D_{2n}^{(2n-1)}$ (see [9])

$$\sum_{n=0}^{\infty} D_{2n}^{(2n-1)} \frac{t^{2n}}{(2n)!} = \frac{1}{\sqrt{1+t^2}} \left(\frac{t}{\log(t+\sqrt{1+t^2}\,)} \right)^2. \tag{1.15}$$

These numbers $D_{2n}^{(2n)}$ and $D_{2n}^{(2n-1)}$ have many important applications. For example (see [10, p. 246])

$$\int_0^{\frac{\pi}{2}} \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n D_{2n}^{(2n)}}{(2n+1)!}, \quad \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} dt = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{2^{2n}(2n-1)(n!)^2}, \tag{1.16}$$

and

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{(2n-1)(2n)!}. \tag{1.17}$$

The D numbers d_{2n} of the second kind may be defined by

$$\frac{t}{\log(t+\sqrt{1+t^2}\,)} = \sum_{n=0}^{\infty} d_{2n} t^{2n}. \tag{1.18}$$

By (1.18), we have $d_0 = 1$, $d_2 = \frac{1}{6}$, $d_4 = -\frac{17}{360}$, $d_6 = \frac{367}{15120}$, $d_8 = -\frac{195013}{27216000}$, $d_{10} = \frac{1295803}{252806400}$.

We now turn to the central factorial numbers $t(n,k)$ of the first kind, which are usually defined by (see [7], [12])

$$x\left(x+\frac{n}{2}-1\right)\left(x+\frac{n}{2}-2\right)\dots\left(x+\frac{n}{2}-n+1\right)=\sum_{k=0}^{n}t(n,k)x^k, \tag{1.19}$$

or by means of the following generating function:

$$\left(2\log\left(\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right)\right)^k = k!\sum_{n=k}^{\infty}t(n,k)\frac{x^n}{n!}. \tag{1.20}$$

It follows from (1.19) or (1.20) that

$$t(n,k)=t(n-2,k-2)-\frac{1}{4}(n-2)^2t(n-2,k), \tag{1.21}$$

and that

$$t(n,0)=\delta_{n,0}\quad (n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}),\qquad t(n,n)=1\quad (n\in\mathbb{N}),$$
$$t(n,k)=0\quad (n+k\quad \text{odd})\quad \text{and}\quad t(n,k)=0\quad (k>n \text{ or } k<0),$$

where (and in what follows) $\delta_{m,n}$ denotes the Kronecker symbol.

By (1.21), we have

$$t(2n+1,1)=\frac{(-1)^n1^2\cdot 3^2\dots(2n-1)^2}{4^n},$$
$$t(2n+2,2)=(-1)^n(n!)^2\ (n\in\mathbb{N}_0) \tag{1.22}$$

and

$$t(2n+2,4)=(-1)^{n+1}(n!)^2\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}\right)\quad (n\in\mathbb{N}). \tag{1.23}$$

By (1.9) and (1.21), we have

$$D_{2n-2k}^{(2n+1)}=\frac{4^{n-k}}{\binom{2n}{2k}}t(2n+1,2k+1)\quad (n\ge k\ge 0). \tag{1.24}$$

The central factorial numbers $T(n,k)$ of the second kind can be defined by (see [7], [12])

$$x^n=\sum_{k=0}^{n}T(n,k)x\left(x+\frac{k}{2}-1\right)\left(x+\frac{k}{2}-2\right)\dots\left(x+\frac{k}{2}-k+1\right) \tag{1.25}$$

or by the generating function

$$\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^k = k! \sum_{n=k}^{\infty} T(n,k) \frac{x^n}{n!}. \tag{1.26}$$

It follows from (1.25) or (1.26) that

$$T(n,k) = T(n-2,k-2) + \frac{1}{4}k^2 T(n-2,k), \tag{1.27}$$

and that

$$T(n,0) = \delta_{n,0} \quad (n \in \mathbb{N}_0), \qquad T(n,n) = 1 \quad (n \in \mathbb{N}),$$

$$T(n,k) = 0 \quad (n+k \quad \text{odd}), \quad \text{and} \quad T(n,k) = 0 \quad (k > n \text{ or } k < 0).$$

The main purpose of this paper is to prove several expressions formulas for the D numbers $D_{2n}^{(k)}$ and $d_{2n}^{(k)}$. We also obtain some identities involving the D numbers, the Bernoulli numbers and the central factorial numbers.

2. Main results

Theorem 1. *Let $n, k \in \mathbb{N}$. Then*

(i) $$D_{2n}^{(2k)} = 2k \binom{2n}{2k} \sum_{j=0}^{k-1} \frac{4^j t(2k, 2k-2j)}{(2n-2j)(2n-2j-1)} D_{2n-2j}^{(2)} \quad (n \geq k); \tag{2.1}$$

(ii) $$D_{2n}^{(2k+1)} = (2k+1) \binom{2n}{2k+1} \times \sum_{j=0}^{k} \frac{4^j t(2k+1, 2k+1-2j)}{2n-2j} D_{2n-2j}^{(1)} \quad (n \geq k+1). \tag{2.2}$$

By (2.1), (2.2) and (1.8), we may immediately deduce the following Corollary 1:

Corrolary 1. *Let $n, k \in \mathbb{N}$. Then*

(i) $$D_{2n}^{(2k)} = -2k \binom{2n}{2k} \sum_{j=0}^{k-1} \frac{4^n t(2k, 2k-2j)}{2n-2j} B_{2n-2j} \quad (n \geq k); \tag{2.3}$$

(ii) $$D_{2n}^{(2k+1)} = (2k+1) \binom{2n}{2k+1} \times \sum_{j=0}^{k} \frac{4^j (2 - 2^{2n-2j}) t(2k+1, 2k+1-2j)}{2n-2j} B_{2n-2j} \quad (n \geq k+1). \tag{2.4}$$

Theorem 2. *Let $n, k \in \mathbb{N}$. Then*

$$\text{(i)}\quad D_{2n}^{(2k)} = 4^n \sum_{j=0}^{n} \frac{(2k)!(2j)!}{(2k+2j)!} t(2k+2j, 2k) T(2n, 2j); \tag{2.5}$$

$$\text{(ii)}\quad D_{2n}^{(2k+1)} = 4^n \sum_{j=0}^{n} \frac{(2k+1)!(2j)!}{(2k+1+2j)!} t(2k+1+2j, 2k+1) T(2n, 2j). \tag{2.6}$$

By (2.3), (2.4), (2.5) and (2.6), we may immediately deduce the following Corollary 2:

Corrolary 2. *Let $n, k \in \mathbb{N}$. Then*

$$\text{(i)}\quad -\binom{2n}{2k} \sum_{j=0}^{k-1} \frac{t(2k, 2k-2j)}{2n-2j} B_{2n-2j}$$

$$= \sum_{j=0}^{n} \frac{(2k-1)!(2j)!}{(2k+2j)!} t(2k+2j, 2k) T(2n, 2j) \quad (n \geq k); \tag{2.7}$$

$$\text{(ii)}\quad \binom{2n}{2k+1} \sum_{j=0}^{k} \frac{(2^{2j+1-2n}-1) t(2k+1, 2k+1-2j)}{2n-2j} B_{2n-2j}$$

$$= \sum_{j=0}^{n} \frac{(2k)!(2j)!}{(2k+1+2j)!} t(2k+1+2j, 2k+1) T(2n, 2j) \quad (n \geq k+1). \tag{2.8}$$

Remark 1. Setting $k = 1$ in (2.7), and noting that $t(2j+2, 2) = (-1)^j (j!)^2$, we obtain

$$B_{2n} = \frac{1}{1-2n} \sum_{j=0}^{n} \frac{(-1)^j (j!)^2}{(j+1)(2j+1)} T(2n, 2j).$$

Remark 2. Setting $k = 2$ in (2.7), and noting that $t(2j+4, 4) = (-1)^j ((j+1)!)^2 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(j+1)^2}\right)$, we obtain

$$(1-n) B_{2n} + n B_{2n-2} = \frac{72}{(2n-1)(2n-3)}$$

$$\times \sum_{j=0}^{n} \frac{(-1)^j ((j+1)!)^2 (2j)!}{(4+2j)!} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(j+1)^2}\right) T(2n, 2j).$$

Theorem 3. *Let $n, k \in \mathbb{N}_0$. Then*

$$\sum_{n=0}^{\infty} D_{2n}^{(2n-k)} \frac{t^{2n}}{(2n)!} = \frac{1}{\sqrt{1+t^2}} \left(\frac{t}{\log(t + \sqrt{1+t^2}\,)} \right)^{k+1} \quad (|t| < 1). \tag{2.9}$$

Theorem 4. *Let $n, k \in \mathbb{N}_0$ and*

$$d_{2n}^{(k)} := \sum_{\substack{v_1,\dots,v_k \in \mathbb{N}_0 \\ v_1+\cdots+v_k=n}} d_{2v_1} d_{2v_2} \dots d_{2v_k}. \tag{2.10}$$

Then

$$(2n)! d_{2n}^{(k)} = \frac{k}{k-2n} D_{2n}^{(2n-k)}. \tag{2.11}$$

Theorem 5. *Let $n, k \in \mathbb{N}_0$. Then*

$$(2n)! d_{2n}^{(k)} = 4^n \sum_{j=0}^{n} \frac{k!(2j)!}{(k+2j)!} T(k+2j, k) t(2n, 2j). \tag{2.12}$$

3. Proofs of Theorems 1–5

Proof of Theorem 1. (i) We prove (2.1) by using mathematical induction.

1. When $k = 1, 2$, (2.1) is true by (1.9).

2. Suppose (2.1) is true for some natural number k. By the superposition of (1.9) and (1.21), we have

$$\begin{aligned}
D_{2n}^{(2k+2)} &= \frac{(2n-2k)(2n-2k-1)}{(2k)(2k+1)} D_{2n}^{(2k)} - \frac{2n(2n-1)(2k)}{2k+1} D_{2n-2}^{(2k)} \\
&= \frac{(2n-2k)(2n-2k-1)}{(2k)(2k+1)} 2k \binom{2n}{2k} \sum_{j=0}^{k-1} \frac{4^j t(2k, 2k-2j)}{(2n-2j)(2n-2j-1)} D_{2n-2j}^{(2)} \\
&\quad - \frac{2n(2n-1)(2k)}{2k+1} 2k \binom{2n-2}{2k} \sum_{j=0}^{k-1} \frac{4^j t(2k, 2k-2j)}{(2n-2j-2)(2n-2j-3)} D_{2n-2j-2}^{(2)} \\
&= (2k+2) \binom{2n}{2k+2} \sum_{j=0}^{k-1} \frac{4^j t(2k, 2k-2j)}{(2n-2j)(2n-2j-1)} D_{2n-2j}^{(2)} \\
&\quad - (2k+2) \binom{2n}{2k+2} (2k)^2 \sum_{j=1}^{k} \frac{4^{j-1} t(2k, 2k+2-2j)}{(2n-2j)(2n-2j-1)} D_{2n-2j}^{(2)} \\
&= (2k+2) \binom{2n}{2k+2} \sum_{j=0}^{k} \frac{4^j \left(t(2k, 2k-2j) - \frac{1}{4}(2k)^2 t(2k, 2k+2-2j) \right)}{(2n-2j)(2n-2j-1)} D_{2n-2j}^{(2)} \\
&= (2k+2) \binom{2n}{2k+2} \sum_{j=0}^{k} \frac{4^j t(2k+2, 2k+2-2j)}{(2n-2j)(2n-2j-1)} D_{2n-2j}^{(2)},
\end{aligned} \tag{3.1}$$

and (3.1) shows that (2.1) is also true for the natural number $k+1$. From 1 and 2, we know that (2.1) is true.

(ii) Use the same method as (i). This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. By (1.1), (1.20), (1.26) and noting the identity

$$t = 2\log\left(\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{2} + \sqrt{1+\left(\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{2}\right)^2}\right),$$

we have

$$\sum_{n=0}^{\infty} B_n^{(k)}\left(\frac{k}{2}\right)\frac{t^n}{n!} = \left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^k = \left(\frac{2\log\left(\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{2} + \sqrt{1+\left(\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{2}\right)^2}\right)}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^k$$

$$= k!\sum_{j=k}^{\infty}\frac{t(j,k)}{j!}(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^{j-k} = k!\sum_{j=k}^{\infty}\frac{t(j,k)}{(2j)!}(j-k)!\sum_{n=j-k}^{\infty}T(n,j-k)\frac{t^n}{n!}$$

$$= k!\sum_{j=0}^{\infty}\frac{t(k+j,k)}{(k+j)!}j!\sum_{n=j}^{\infty}T(n,j)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{k!j!}{(k+j)!}t(k+j,k)T(n,j)\frac{t^n}{n!}. \tag{3.2}$$

Comparing the coefficient $\frac{t^n}{n!}$ on both sides of (3.2), we have

$$B_n^{(k)}\left(\frac{k}{2}\right) = \sum_{j=0}^{n}\frac{k!j!}{(k+j)!}t(k+j,k)T(n,j). \tag{3.3}$$

By (3.3), (1.7), and noting that $t(n,k)=T(n,k)=0$ ($n+k$ odd), we immediately obtain Theorem 2. This completes the proof of Theorem 2. $\square$

Proof of Theorem 3. Note the identity (see [10, p. 203])

$$B_{2n}^{(k)}\left(x+\frac{k}{2}\right) = \sum_{j=0}^{n}\binom{2n}{2j}\frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}}x^2(x^2-1^2)(x^2-2^2)\dots(x^2-(j-1)^2), \tag{3.4}$$

we have

$$\frac{B_{2n}^{(k)}\left(x+\frac{k}{2}\right) - B_{2n}^{(k)}\left(\frac{k}{2}\right)}{x^2}$$

$$= \sum_{j=1}^{n}\binom{2n}{2j}\frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}}(x^2-1^2)(x^2-2^2)\dots(x^2-(j-1)^2). \tag{3.5}$$

Therefore,

$$\lim_{x\to 0}\frac{B_{2n}^{(k)}\left(x+\frac{k}{2}\right)-B_{2n}^{(k)}\left(\frac{k}{2}\right)}{x^2}=\sum_{j=1}^{n}\binom{2n}{2j}\frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}}(-1)^{j-1}((j-1)!)^2. \tag{3.6}$$

By (3.6) and (1.2), we have

$$\lim_{x\to 0}\frac{2n(2n-1)B_{2n-2}^{(k)}\left(x+\frac{k}{2}\right)}{2}=\sum_{j=1}^{n}\binom{2n}{2j}\frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}}(-1)^{j-1}((j-1)!)^2, \tag{3.7}$$

i.e.,

$$n(2n-1)B_{2n-2}^{(k)}\left(\frac{k}{2}\right)=\sum_{j=1}^{n}\binom{2n}{2j}\frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}}(-1)^{j-1}((j-1)!)^2. \tag{3.8}$$

By (3.8) and (1.7), we have

$$n(2n-1)D_{2n-2}^{(k)}=\sum_{j=1}^{n}\binom{2n}{2j}(-1)^{j-1}4^{j-1}((j-1)!)^2D_{2n-2j}^{(k-2j)}. \tag{3.9}$$

Thus

$$\frac{1}{2}\sum_{n=1}^{\infty}D_{2n-2}^{(2n-k)}\frac{t^{2n}}{(2n-2)!}=\sum_{n=1}^{\infty}\sum_{j=1}^{n}\binom{2n}{2j}(-1)^{j-1}4^{j-1}((j-1)!)^2D_{2n-2j}^{(2n-k-2j)}\frac{t^{2n}}{(2n)!}.$$

i.e.

$$\begin{aligned}&\frac{t^2}{2}\sum_{n=0}^{\infty}D_{2n}^{(2n-(k-2))}\frac{t^{2n}}{(2n)!}\\ &=\sum_{n=1}^{\infty}(-1)^{n-1}4^{n-1}((n-1)!)^2\frac{t^{2n}}{(2n)!}\sum_{n=1}^{\infty}D_{2n}^{(2n-k)}\frac{t^{2n}}{(2n)!}.\end{aligned} \tag{3.10}$$

Note the identity (see [6])

$$\sum_{n=0}^{\infty}(-1)^n4^n(n!)^2\frac{t^{2n}}{(2n)!}=\frac{1}{1+t^2}\left(1-\frac{t}{\sqrt{1+t^2}}\log\left(t+\sqrt{1+t^2}\right)\right), \tag{3.11}$$

where $|t|<1$. We have

$$\sum_{n=0}^{\infty}(-1)^n4^n(n!)^2\frac{t^{2n+2}}{(2n+2)!}=\frac{1}{2}\left(\log\left(t+\sqrt{1+t^2}\right)\right)^2, \tag{3.12}$$

i.e.

$$\sum_{n=1}^{\infty}(-1)^{n-1}4^{n-1}((n-1)!)^2\frac{t^{2n}}{(2n)!}=\frac{1}{2}\left(\log\left(t+\sqrt{1+t^2}\,\right)\right)^2. \tag{3.13}$$

Thus, by (3.10) and (3.13), we have

$$\sum_{n=1}^{\infty}D_{2n}^{(2n-k)}\frac{t^{2n}}{(2n)!}=\left(\frac{t}{\log(t+\sqrt{1+t^2}\,)}\right)^2\sum_{n=0}^{\infty}D_{2n}^{(2n-(k-2))}\frac{t^{2n}}{(2n)!}. \tag{3.14}$$

By (3.14), and note that

$$\lim_{t\to 0}\frac{t}{\log(t+\sqrt{1+t^2}\,)}=1 \quad\text{and}\quad D_0^{(-k)}=1, \tag{3.15}$$

we have

$$\sum_{n=0}^{\infty}D_{2n}^{(2n-k)}\frac{t^{2n}}{(2n)!}=\left(\frac{t}{\log(t+\sqrt{1+t^2}\,)}\right)^2\sum_{n=0}^{\infty}D_{2n}^{(2n-(k-2))}\frac{t^{2n}}{(2n)!}. \tag{3.16}$$

(3.16), (1.13) and (1.15), we immediately obtain (2.9). This completes the proof of Theorem 3. □

Proof of Theorem 4. By Theorem 3, we have

$$\begin{aligned}\frac{d}{dt}\sum_{n=0}^{\infty}D_{2n}^{(2n-k+2)}\frac{t^{2n}}{(2n)!}&=\frac{d}{dt}\frac{1}{\sqrt{1+t^2}}\left(\frac{t}{\log(t+\sqrt{1+t^2}\,)}\right)^{k-1}\\&=-\frac{t}{1+t^2}\sum_{n=0}^{\infty}D_{2n}^{(2n-k+2)}\frac{t^{2n}}{(2n)!}+\frac{k-1}{t}\sum_{n=0}^{\infty}D_{2n}^{(2n-k+2)}\frac{t^{2n}}{(2n)!}\\&\quad-\frac{k-1}{t(1+t^2)}\left(\frac{t}{\log(t+\sqrt{1+t^2}\,)}\right)^{k},\end{aligned}$$

i.e.

$$\begin{aligned}(k-1)\left(\frac{t}{\log(t+\sqrt{1+t^2}\,)}\right)^{k}&=(k-1)\sum_{n=0}^{\infty}D_{2n}^{(2n-k+2)}\frac{t^{2n}}{(2n)!}\\+(k-2)t^2\sum_{n=0}^{\infty}D_{2n}^{(2n-k+2)}\frac{t^{2n}}{(2n)!}&-(1+t^2)\sum_{n=1}^{\infty}D_{2n}^{(2n-k+2)}\frac{t^{2n}}{(2n-1)!}.\end{aligned} \tag{3.17}$$

By (3.17), (1.18), (2.9) and (2.10), we have

$$(k-1)(2n)!d_{2n}^{(k)}=(k-1-2n)D_{2n}^{(2n-k+2)}+2n(2n-1)(k-2n)D_{2n-2}^{(2n-k)}. \tag{3.18}$$

By (3.18) and (1.9), we immediately obtain Theorem 4. This completes the proof of Theorem 4. □

Proof of Theorem 5. By (2.10), (1.18), (1.20) and (1.26), we have

$$\begin{aligned}\sum_{n=0}^{\infty} d_{2n}^{(k)} t^{2n} &= \left(\frac{t}{\log(t+\sqrt{1+t^2}\,)}\right)^k = \left(\frac{e^{\log(t+\sqrt{1+t^2}\,)} - e^{-\log(t+\sqrt{1+t^2}\,)}}{2\log(t+\sqrt{1+t^2}\,)}\right)^k \\ &= k!\sum_{j=k}^{\infty}\frac{T(j,k)}{j!}\left(2\log\left(t+\sqrt{1+t^2}\,\right)\right)^{j-k} \\ &= k!\sum_{j=k}^{\infty}\frac{T(j,k)}{j!}(j-k)!\sum_{n=j-k}^{\infty}2^n t(n,j-k)\frac{t^n}{n!} \\ &= k!\sum_{j=0}^{\infty}\frac{T(k+j,k)}{(k+j)!}j!\sum_{n=j}^{\infty}2^n t(n,j)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}2^n\sum_{j=0}^{n}\frac{k!j!}{(k+j)!}T(k+j,k)t(n,j)\frac{t^n}{n!}. \end{aligned} \tag{3.19}$$

Comparing the coefficient $\frac{t^{2n}}{(2n)!}$ on both sides of (3.19), we have

$$(2n)!d_{2n}^{(k)} = 4^n\sum_{j=0}^{n}\frac{k!(2j)!}{(k+2j)!}T(k+2j,k)t(2n,2j). \tag{3.20}$$

This completes the proof of Theorem 5. □

Remark 3. By (1.19) and noting that $t(n,k)=0(n+k \text{ odd})$, we have

$$x^2(x^2-1^2)\dots(x^2-(n-1)^2) = \sum_{k=0}^{n} t(2n,2k)x^{2k}.$$

Therefore,

$$\int_0^{\frac{1}{2}} x^2(x^2-1^2)\dots(x^2-(n-1)^2)dx = \sum_{k=0}^{n}\frac{1}{(2k+1)2^{2k+1}}t(2n,2k), \tag{3.21}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(2k+1)2^{2k+1}}t(2n,2k)\frac{x^{2n}}{(2n)!} &= \sum_{k=0}^{\infty}\frac{1}{(2k+1)2^{2k+1}}\sum_{n=k}^{\infty}t(2n,2k)\frac{x^{2n}}{(2n)!} \\ &= \sum_{k=0}^{\infty}\frac{1}{(2k+1)!2^{2k+1}}\left(2\log\left(\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right)\right)^{2k} \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\log\left(\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right)\right)^{2k}\end{aligned}$$

$$= \frac{1}{4\log\left(\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right)}\left(\sum_{k=0}^{\infty}\frac{1-(-1)^k}{k!}\left(\log\left(\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right)\right)^k\right)$$

$$= \frac{1}{4\log\left(\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right)}\left(e^{\log\left(\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right)}-e^{-\log\left(\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right)}\right)$$

$$= \frac{x}{4\log\left(\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right)} = \sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}d_{2n}x^{2n}.$$

Hence,

$$\sum_{k=0}^{n}\frac{1}{(2k+1)2^{2k+1}}t(2n,2k) = \frac{(2n)!}{2^{2n+1}}d_{2n}. \tag{3.22}$$

By (3.21) and (3.22), we have

$$\int_0^{\frac{1}{2}} x^2(x^2-1^2)\dots(x^2-(n-1)^2)dx = \frac{(2n)!}{2^{2n+1}}d_{2n}. \tag{3.23}$$

By (3.23) and noting that (see [10, p.211])

$$\frac{\sin\pi x}{\pi x} = 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}x^2(x^2-1^2)\dots(x^2-(n-1)^2), \tag{3.24}$$

we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}dx &= \pi\int_0^{\frac{1}{2}}\frac{\sin\pi x}{\pi x}dx \\ &= \frac{\pi}{2}+\pi\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\int_0^{\frac{1}{2}}x^2(x^2-1^2)\dots(x^2-(n-1)^2)dx \\ &= \frac{\pi}{2}+\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}d_{2n} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}d_{2n}.\end{aligned} \tag{3.25}$$

By (3.25) and (2.11), we immediately obtain (see (1.16))

$$\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}dx = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}D_{2n}^{(2n-1)}}{2^{2n}(2n-1)(n!)^2}. \tag{3.26}$$

On the other hand, by (1.13) we have

$$\sum_{n=0}^{\infty}(-1)^nD_{2n}^{(2n)}\frac{t^{2n}}{(2n)!} = \frac{it}{\sqrt{1-t^2}\log(it+\sqrt{1-t^2}\,)}$$

therefore (see (1.16)),

$$\sum_{n=0}^{\infty}\frac{(-1)^n D_{2n}^{(2n)}}{(2n+1)!}=\int_0^1\frac{it}{\sqrt{1-t^2}\log(it+\sqrt{1-t^2}\,)}dt$$
$$=\int_0^{\frac{\pi}{2}}\frac{1}{\cos x}\frac{i\sin x}{\log(i\sin x+\cos x)}d\sin x=\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}dx, \qquad (3.27)$$

where $i^2=-1$.

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