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# On the factors of Stern polynomials
(Remarks on the preceding paper of M. Ulas)

By ANDRZEJ SCHINZEL (Warszawa)

**Abstract.** It is proved that the Stern polynomial with a prime index does not have a proper divisor over $\mathbb{Q}$ of degree less than 4.

Stern polynomials $B_n(t)$ have been first introduced in [1] and then studied in [3]. The notation is the same as in [3]. In particular, $B_0(t) = 0$, $B_1(t) = 1$, $B_{2n}(t) = tB_n(t)$ and $B_{2n+1}(t) = B_n(t) + B_{n+1}(t)$. In connection with Conjecture 6.4 of [3] we shall prove the following theorems.

**Theorem 1.** *For all integers $n \geq 0$ we have*

$$B_n(1) \leq n^{3/4}.$$

**Theorem 2.** *For no prime $p$ does the polynomial $B_p$ have a proper divisor over $\mathbb{Q}$ of degree* 1, 2 *or* 3.

**Theorem 3.** *For no prime $p$ is the polynomial $B_p$ a product over $\mathbb{Q}$ of more than one polynomial of degree* 4.

**Corollary.** *Polynomials $B_p$ are irreducible over $\mathbb{Q}$ for all primes $p < 2017$.*

The value of Theorem 1 for applications lies in its explicit form. If one allows estimates of the form $B_n(1) = O(n^\alpha)$, the best value for $\alpha$ is $\left(\log \frac{1+\sqrt{5}}{2}\right)/\log 2 = 0.694\ldots$ (see [4], Corollary 3).

The value of the Corollary lies in the fact that it is obtained without computation. With computation the results are much stronger (see [3], Remark 6.5).

In connection with Conjecture 6.6 of [3] we shall prove

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**Theorem 4.** *For every $k$ there exists an index $n$ with at least $k$ prime factors such that $B_n$ is irreducible over $\mathbb{Q}$.*

The following lemma generalizes Lemmas 2.1 and 2.2 of [3].

**Lemma 1.** *For all non-negative integers $a$, $m$ and $r$ such that $2^a \ge r \ge 0$ we have*

$$B_{2^a m+r} = B_{2^a - r} B_m + B_r B_{m+1}. \tag{1}$$

Proof by induction on $a$. For $a = 0$ (1) is trivially true for $r = 0$ or 1. Assume that (1) holds for exponent $a-1$ ($a \ge 1$). Then for $r$ even $\le 2^a$

$$B_{2^a m+r} = tB_{2^{a-1}+r/2} = tB_{2^{a-1}-r/2}B_m + tB_{r/2}B_{m+1} = B_{2^a-r}B_m + B_r B_{m+1},$$

for $r$ odd $< 2^a$

$$\begin{aligned} B_{2^a m+r} &= B_{2^{a-1}m+\frac{r-1}{2}} + B_{2^{a-1}m+\frac{r+1}{2}} \\ &= B_{2^{a-1}-\frac{r-1}{2}}B_m + B_{\frac{r-1}{2}}B_{m+1} + B_{2^{a-1}-\frac{r+1}{2}}B_m + B_{\frac{r+1}{2}}B_{m+1} \\ &= B_{2^a-r}B_m + B_r B_{m+1}. \end{aligned}$$

Proof of Theorem 1. We shall show the stronger inequality

$$B_n(1) \le (n-1)^{3/4} \quad \text{for } n \ne 0, 1, 3, 5. \tag{2}$$

We consider $n$ in the interval $[16^{k-1}, 16^k)$ and proceed by induction on $k$. For $k = 1$ we check directly (2) for $n \ne 1, 3, 5$ and also $B_n(1) \le n^{3/4}$ for all $n < 16$. For $k = 2$ we have $n = 16m + r$, when $1 \le m < 16$, $0 \le r < 16$ and from Lemma 1

$$B_n(1) = B_{16-r}(1)B_m(1) + B_r(1)B_{m+1}(1). \tag{3}$$

If $m \ne 2, 4$, then by the inductive assumption

$$B_m(1) \le m^{3/4}, \qquad B_{m+1}(1) \le m^{3/4},$$

hence for $r \ne 0$

$$B_n(1) \le (B_{16-r}(1) + B_n(1))m^{3/4} \le 8m^{3/4} \le 8\left(\frac{n-1}{16}\right)^{3/4} = (n-1)^{3/4}. \tag{4}$$

For $r = 0$, $m$ arbitrary

$$B_n(1) = B_m(1) \le m^{3/4} < (n-1)^{3/4}. \tag{5}$$

For $m = 2$, $r \neq 0$

$$B_n(1) = B_{16-r}(1)B_2(1) + B_r(1)B_3(1) = B_{16-r}(1) + 2B_r(1) \leq 13 < 32^{3/4}.$$

Finally, for $m = 4$, $r \neq 0$

$$B_n(1) = B_{16-r}(1)B_4(1) + B_r(1)B_r(1) = B_{16-r}(1) + 3B_5(1) \leq 18 < 64^{3/4}.$$

For $k > 2$ we have $n = 16m + r$, where $m \in [16^{k-2}, 16^{k-1})$, $0 \leq r < 16$ and, from Lemma 1, (3) holds. By the inductive assumption

$$B_m(1) \leq (m-1)^{3/4}, \qquad B_{m+1}(1) \leq m^{3/4}$$

hence (4) or (5) holds for $r \neq 0$ and $r = 0$, respectively. □

It follows that $B_n(1) \leq (n-1)^{3/4}$.

**Lemma 2.** *If $f$ is a polynomial of degree $n$, then its leading coefficient is $\frac{1}{n!}\Delta^n f(a)$ for arbitrary $a$, where $\Delta f(a) = f(a+1) - f(a)$.*

Proof. See [2], Satz 23. □

**Lemma 3.** $B_n(2) = n$, $B_n(0) = 2\{\frac{n}{2}\}$, $B_n(-1) = 3\{\frac{n}{3} + \frac{1}{2}\} - \frac{3}{2}$.

Proof. See [3], Theorem 5.1. □

**Lemma 4.** *Every divisor of $B_n(t)$ over $\mathbb{R}$ $(n > 0)$ with the leading coefficient $l$ satisfies $lf(0) \geq 0$ and $f(a)f(0) \geq 0$ for all $a > 0$.*

Proof. If $f(a)f(0) < 0$, then by the Darboux property of $f$, $f$ has a zero in the interval $(0, a)$, which is also a zero of $B_n$, a contradiction, since $B_n$ has non-negative coefficients, not all 0. If $lf(0) < 0$, then for sufficiently large $a$: $f(a)f(0) < 0$, which has been shown impossible. □

Proof of Theorem 2. If $B_p$ ($p$ an odd prime) has over $\mathbb{Q}$ a proper divisor $f$ of degree 1, $f$ could be normalized, by Lemmas 3 and 4, to the form $lx + 1$, $l > 0$. By Lemma 2, $l = \Delta f(1) = f(2) - f(1)$ and by Lemmas 3, 4 and, by Theorem 1, $f(2) = 1$ or $p$, $1 \leq f(1) \leq B_p(1) \leq p^{3/4}$. Thus $l \geq p - p^{3/4}$, hence for $p \geq 5$, $B_p(2) > 4(p - p^{3/4}) > p$, a contradiction with Lemma 3. □

If $B_p$ had over $\mathbb{Q}$ a proper divisor $f$ of degree 2, $f$ could be normalized to the form $lx^2 + mx + 1$, $l > 0$. From Lemmas 2, 3, 4 and by Theorem 1

$$l = \frac{1}{2}\Delta^2 f(0) = \frac{1}{2}\left(f(2) - 2f(1) + f(0)\right) \geq \frac{1}{2}(p - 2B_p(1) + 1) \geq \frac{1}{2}(p - 2p^{3/4} + 1).$$

Let $e(n)$ be degree of $B_n$, as in [2] and [3]. We have (see [1], Corollary 13)

$$e(2n) = e(n) + 1, \quad e(4n+1) = e(n) + 1, \quad e(4n+3) = e(n+1) + 1.$$

Hence for $e(p) \geq 4$, $p \geq 31$ and $B_p(2) > 16l \geq 8(p - 2p^{3/4} + 1) > p$, contrary to Lemma 3.

If $B_p$ had over $\mathbb{Q}$ a proper divisor $f$ of degree 3, $f$ could be normalized to the form $lx^3 + mx^2 + nx + 1$, where $l > 0$.

From Lemmas 2, 3, 4 and by Theorem 1

$$l = \frac{1}{6}\Delta^3 f(-1) = \frac{1}{6}\left(f(2) - 3f(1) + 3f(0) - f(-1)\right) \geq \frac{1}{6}(p - 3p^{3/4} + 2).$$

Hence for $e(p) \geq 6$, $p \geq 127$

$$B_p(2) > 64l \geq \frac{32}{3}(p - 3p^{3/4} + 2) > p,$$

contrary to Lemma 3.

**Lemma 5.** $|B_n(-2)| \leq n$.

Proof by induction on $n$. For $n = 0$ or 1 true, assume that it is true for all subscripts $< n$, then for $n$ even $|B_n(-2)| = 2|B_{\frac{n}{2}}(-2)| \leq n$, for $n$ odd

$$|B_n(-2)| = 2\left|B_{\frac{n-1}{2}}(-2) + B_{\frac{n+1}{2}}(-2)\right| \leq \frac{n-1}{2} + \frac{n+1}{2} = n.$$

Proof of Theorem 3. Suppose that

$$B_p = \prod_{i=1}^{k} f_i,$$

where $k \geq 2$, $f_i \in \mathbb{Z}[x]$ of degree 4 and with the leading coefficient $l_i > 0$. By Lemma 3 for $p > 2$ we have

$$1 = B_p(0) = \prod_{i=1}^{k} f_i(0)$$

and, by Lemma 4, $f_i(0) = 1$. Also

$$p = B_p(2) = \prod_{i=1}^{k} f_i(2)$$

and, by Lemma 4, for a certain $i$, say $i = 1$ we have $f_i(2) = p$, for all $i > 1$ : $f_i(2) = 1$. From Lemma 2 we have

$$l_i = \frac{1}{24}\Delta^4 f(-2) = \frac{1}{24}\left(f_i(2) - 4f_i(1) + 6f_i(0) - 4f_i(-1) + f_i(-2)\right)$$

and, since $l_i \geq 1$, we have for $i > 1$

$$f_i(-2) \geq 24 - 1 + 4f_i(1) - 6f_i(0) + 4f_i(-1) \geq 17.$$

Since, by Lemma 5,

$$p \geq |B_p(-2)| = \prod_{i=1}^{k} |f_i(-2)| \geq 17|f_1(-2)|$$

we obtain

$$l_1 = \frac{1}{24}\left(p - 4f_1(1) + 6f_1(0) - 4f_1(-1) + f_1(-2)\right) \geq \frac{1}{24}\left(\frac{16}{17}p - 4p^{3/4} + 2\right).$$

Hence, for $e_p \geq 8$, $p \geq 769$

$$B_p(2) \geq 256l \geq \frac{32}{3}\left(\frac{16}{17}p - 4p^{3/4} + 2\right) > p,$$

contrary to Lemma 3. □

Proof of Corollary. By Theorems 2 and 3 if $B_p$ is irreducible over $\mathbb{Q}$, then $e_p \geq 9$ and $p \geq 2017$. □

Theorem 4 is an immediate consequence of the following two lemmas.

**Lemma 6.** *$B_{2^n-3}$ is irreducible over $\mathbb{Q}$ for all integers $n \geq 3$.*

Proof. By definition of $B_n$ and Lemma 2.1 of [3]

$$B_{2^n-3} = tB_{2^{n-2}-1} + B_{2^{n-1}-1} = t \cdot \frac{t^{n-2}-1}{t-1} + \frac{t^{n-1}-1}{t-1} = 2\sum_{i=1}^{n-2} t^i + 1,$$

hence $B_{2^n-3}$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion. □

**Lemma 7.** *For every integer $k > 0$ there exists an integer $n$ such that $2^n - 3$ has at least $k$ prime factors.*

Proof by induction on $k$. For $k = 1$ one takes $n = 3$. Assume that the statement is true for $k-1$ $(k \geq 2)$ and that $2^{n_{k-1}} - 3$ has at least $k-1$ prime factors. Let among them be $q_1, \dots, q_{k-1}$ and let $q_i^{\alpha_i} \| 2^{n_{k-1}} - 3$. By Euler's theorem

$$q_i^{\alpha_i} \| 2^{n_{k-1} + \varphi(q_1^{\alpha_1+1} \dots q_{k-1}^{\alpha_{k-1}+1})} - 3.$$

However,

$$q_1^{\alpha_1} \dots q_{k-1}^{\alpha_{k-1}} < 2^{n_{k-1} + \varphi(q_1^{\alpha_1+1} \dots q_{k-1}^{\alpha_{k-1}+1})}.$$

Therefore, we can take

$$n_k = n_{k-1} + \varphi(q_1^{\alpha_1+1} \dots q_{k-1}^{\alpha_{k-1}+1}).$$

*Remark.* Probably for every integer $k > 0$ there exists an integer $n$ such that $2^n - 3$ has exactly $k$ prime factors.

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ANDRZEJ SCHINZEL
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
ŚNIADECKICH 8
00-956 WARSAW
POLAND

*E-mail:* schinzel@impan.pl

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