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Pointwise summability of Vilenkin–Fourier series

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Abstract. In this paper we give a characterization of points in which Fejér means of Vilenkin–Fourier series converge.

1. The One-dimensional Vilenkin–Lebesgue points

Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \dots, m_k - 1\}$ the additive group of integers modulo m_k. Define the group G_m as the complete direct product of the groups Z_{m_j}, with the product of the discrete topologies of Z_{m_j}'s. The direct product μ of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$. If the sequence m is bounded, then G_m is called a bounded Vilenkin group. The elements of G_m can be represented by sequences $x := (x_0, x_1, \dots, x_j, \dots)$, $(x_j \in Z_{m_j})$. The group operation $+$ in G_m is given by $x + y = (x_0 + y_0 \mod m_0, \dots, x_k + y_k(\mod m_k, \dots)$, where $x = (x_0, \dots, x_k, \dots)$ and $y = (y_0, \dots, y_k, \dots) \in G_m$. The inverse of $+$ will be denoted by $-$.

It is easy to give a base for the neighborhoods of G_m:

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m | y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$$

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for $x \in G_m,\ n \in \mathbb{N}$. Define $I_n := I_n(0)$ for $n \in \mathbb{N}_+$. Set $e_n := (0, \dots, 0, 1, 0, \dots) \in G_m$ the nth coordinate of which is 1 and the rest are zeros ($n \in \mathbf{N}$).

If we define the so-called generalized number system based on m in the following way: $M_0 := 1, M_{k+1} := m_k M_k (k \in \mathbb{N})$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N}_+)$ and only a finite number of n_j's differ from zero. We use the following notation. Let (for $n > 0$) $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$).

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. At first define the complex valued functions $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions in this way

$$r_k(x) := \exp \frac{2\pi \imath x_k}{m_k} \ (\imath^2 = -1,\ x \in G_m,\ k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as follows.

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh–Paley one if $m \equiv 2$.
The Vilenkin system is orthonormal and complete in $L^1(G_m)$ [1].
Now, introduce analogues of the usual definitions of the Fourier analysis. If $f \in L^1(G_m)$ we can establish the following definitions in the usual way:
Fourier coefficients:

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu \qquad (k \in \mathbb{N}),$$

partial sums:

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k \qquad (n \in \mathbb{N}_+,\ S_0 f := 0),$$

Fejér means:

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f \qquad (n \in \mathbb{N}_+),$$

Dirichlet kernels:

$$D_n := \sum_{k=0}^{n-1} \psi_k \qquad (n \in \mathbb{N}_+),$$

Fejér kernels:

$$K_n(x) := \frac{1}{n} \sum_{k=1}^{n} D_k(x).$$

Recall that

$$D_{M_n}(x)=\begin{cases} M_n, & \text{if } x\in I_n,\\ 0, & \text{if } x\in G_m\backslash I_n. \end{cases} \tag{1}$$

It is well known that

$$\sigma_n f(x)=\int_{G_m} f(t)\,K_n(x-t)\,d\mu(t).$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p:=\left(\int_{G_m}|f(x)|^p\,d\mu(x)\right)^{1/p}\quad(0<p<+\infty).$$

The space weak-$L_p(G_m)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_p(G_m)}:=\sup_{\lambda>0}\lambda\mu(|f|>\lambda)^{1/p}<+\infty.$$

The σ-algebra generated by the intervals $\{I_n(x):x\in G_m\}$ will be denoted by $F_n(n\in\mathbb{N})$.

Denote by $f=(f^{(n)},n\in\mathbb{N})$ a martingale with respect to $(F_n,n\in\mathbb{N})$ (for details see, e.g. [12], [15]).

The maximal function of a martingale f is defined by

$$f^*=\sup_{n\in\mathbb{N}}\left|f^{(n)}\right|.$$

In case $f\in L_1(G_m)$, the maximal functions are also be given by

$$f^*(x)=\sup_{n\in\mathbb{N}}\frac{1}{\mu(I_n(x))}\left|\int_{I_n(x)}f(u)\,d\mu(u)\right|.$$

For $0<p<\infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p}:=\|f^*\|_p<\infty.$$

If $f\in L_1(G_m)$ then it is easy to show that the sequence $(S_{M_n}(f):n\in\mathbb{N})$ is a martingale. If f is a martingale, that is $f=(f^{(n)}:n\in\mathbb{N})$, then the Vilenkin–Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i)=\lim_{k\to\infty}\int_{G_m}f^{(k)}(x)\overline{\psi}_i(x)\,d\mu(x).$$

The Vilenkin–Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n \in \mathbb{N})$ obtained from f.

A bounded measurable function l is an $H_p(G_m)$-atom if there exists a Vilenkin interval $I \in F_n$ for any $n \in \mathbb{N}$ such that

$$\begin{cases} \text{a)} & \operatorname{supp}(l) \subset I \\ \text{b)} & \int_I l d\mu = 0 \\ \text{c)} & \|l\|_\infty \leq (\mu(I))^{-1/p} \end{cases} . \tag{2}$$

Weisz [13] proved that the following is true.

Theorem W1. *Suppose that the operator T is sublinear and*

$$\int_{G_m \setminus I} |Tl|^p \, d\mu \leq c_p \quad (p_0 < p \leq 1) \tag{3}$$

for every $H_p(G_m)$-atom l. If the operator T is bounded from L_{p_1} to $L_{p_1}(1 < p_1 \leq \infty)$ then

$$\|Tf\|_p \leq c_p \|f\|_{H_p} \quad (f \in H_p)$$

for every $p_0 < p \leq p_1$. Moreover, if $p_0 < 1$ then the operator T is of weak type $(1,1)$, i.e. if $f \in L_1(G_m)$, then

$$\|Tf\|_{\text{weak}-L_1(G_m)} \leq c \|f\|_1 .$$

In the one-dimensional case a point $x \in (-\infty, \infty)$ is called a Lebesgue point of a function f if

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| d\mu(t) = 0.$$

It is known that a.e. point $x \in (-\infty, \infty)$ is a Lebesgue point of $f \in L_1$ and that the Fejér means of the trigonometric Fourier series of $f \in L_1$ converge to f at each Lebesgue point (see Butzer and Nessel [2]).

Weisz introduced the one-dimensional Walsh–Lebesgue point in [12]: $x \in G_2$ is a Walsh–Lebesgue point of $f \in L_1(G_2)$, if

$$\lim_{n \to \infty} \sum_{k=0}^{n} 2^k \int_{I_n(x+e_k)} |f(t) - f(x)| dt = 0,$$

where G_2 is a Walsh group. Weisz proved in [12] that a.e. point $x \in G_2$ is a Walsh–Lebesgue point of an integrable function f. Moreover, the Fejér means of the Walsh–Fourier series of $f \in L_1(G_2)$ converge to f at each Walsh–Lebesgue point.

Pál and Simon [11] proved that the one-dimensional Fejér means of the Vilenkin–Fourier series on bounded Vilenkin group of an integrable function converge a.e. to the function.

In this paper we will characterize the set of convergence of Vilenkin–Fejér means. We introduce first the operator

$$W_A f(x) := \sum_{s=0}^{A-1} M_s \sum_{r_s=1}^{m_s-1} \int_{I_A(x-r_s e_s)} |f(t) - f(x)| d\mu(t).$$

A point $x \in G_m$ is a Vilenkin–Lebesgue point of $f \in L_1(G_m)$, if

$$\lim_{A\to\infty} W_A f(x) = 0.$$

Denote

$$\begin{aligned} V_A f(x) &:= \sum_{s=0}^{A} M_s \sum_{r_s=1}^{m_s-1} \int_{I_A(x-r_s e_s)} f(t)\, d\mu(t) \\ &= \sum_{s=0}^{A} \frac{M_s}{M_A} \sum_{r_s=1}^{m_s-1} \int_{G_m} f(t)\, D_{M_A}(x - r_s e_s - t)\, d\mu(t) \end{aligned}$$

It is easy to see that $W_A f(x) \to 0$ as $A \to \infty$ if and only if

$$\lim_{A\to\infty} V_A(|f - f(x)|)(x) = 0.$$

Let

$$Vf := \sup_A |V_A f|.$$

For one-dimensional Vilenkin–Fejér means we prove that the following is true

Theorem 1. *Let $f \in L_1(G_m)$. Then*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x)$$

for all Vilenkin–Lebesgue points of f.

Theorem 2. *Let $p > 0$. Then*

$$\|Vf\|_p \le c_p \|f\|_p \quad (f \in H_p(G_m))$$

and

$$\sup_\lambda \lambda\mu\{Vf > \lambda\} \le c\|f\|_1 .$$

It is easy to show that $\lim_{A\to\infty} W_A f(x) = 0$ for every Vilenkin polynomials and $x \in G_m$. Since the Vilenkin polynomials are dense in $L_1(G_m)$, Theorem 2 and the usual density argument (see Marcinkiewicz and Zygmund [9]) imply

Corollary 1. *Let $f \in L_1(G_m)$. Then*

$$\lim_{A\to\infty} W_A f(x) = 0 \quad \text{for a.e. } x \in G_m,$$

thus a.e. point is a Vilenkin–Lebesgue point of f.

Corollary 2. *Let $f \in L_1(G_m)$, where G_m is a bounded Vilenkin group. Then*

$$\sigma_n f(x) \to f(x) \quad \text{for a.e. } x \in G_m.$$

We note that for the unbounded Vilenkin group and for special indices Gát [6] proved that $\sigma_{M_n} f(x) \to f(x)$ a.e. for $f \in L(G_m)$ (see also [4]). On the other hand, for full indices and unbounded Vilenkin group Gát [4] showed that $\sigma_n f(x) \to f(x)$ a.e. for $f \in L_p(G_m)$ $(1 < p \leq \infty)$.

Proof of Theorem 1. Since [11], [1]

$$|K_{M_A}(x)| \leq c \sum_{s=0}^{A} M_s \sum_{r_s=1}^{m_s-1} 1_{I_A(r_s e_s)}(x) \tag{4}$$

and

$$n|K_n(x)| \leq c \sum_{A=0}^{|n|} M_A |K_{M_A}(x)| \tag{5}$$

we can write

$$\begin{aligned} |\sigma_n f(x) - f(x)| &\leq \frac{c}{n} \sum_{A=0}^{|n|} M_A \int_{G_m} |f(t) - f(x)| \, |K_{M_A}(x-t)| \, d\mu(t) \\ &\leq \frac{c}{n} \sum_{A=0}^{|n|} M_A \sum_{s=0}^{A} M_s \sum_{r_s=1}^{m_s-1} \int_{I_A(x - r_s e_s)} |f(t) - f(x)| \, d\mu(t) \\ &= \frac{c}{n} \sum_{A=0}^{|n|} M_A W_A f(x) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Theorem 1 is proved. □

Proof of Theorem 2. By Theorem W1, the proof of Theorem 2 will be complete if we show that the operator V satisfies (3) and is bounded from $L_\infty(G_m)$ to $L_\infty(G_m)$.

The boundedness follows from the inequality

$$\|Vf\|_\infty \le c\|f\|_\infty \sup_A \frac{1}{M_A}\sum_{s=0}^{A-1} M_s \le c\,\|f\|_\infty\,.$$

To verify (3), let l be an arbitrary atom with support $I_a(z)$. It is easy to see that $V_A(a) = 0$ if $A < a$. Therefore we can suppose that $A \ge a$. Since the dyadic addition is a measure preserving group operation, we may assume that $z = 0$.

From (1), (4) and (5) we can write

$$\begin{aligned}|V_A l\,(x)| &\le c\sum_{s=0}^{A} \frac{M_s}{m_A}\sum_{r_s=1}^{m_s-1}\int_{G_m} |l\,(t)|\,|D_{M_A}\,(x - r_s e_s - t)\,|d\mu\,(t)\\ &\le cM_a^{1/p}\sum_{s=0}^{a-1}\frac{M_s}{M_A}\sum_{r_s=1}^{m_s-1}\int_{I_a}|D_{M_A}\,(x - r_s e_s - t)\,|d\mu\,(t)\\ &\le \frac{cM_a^{1/p}}{M_a}\sum_{s=0}^{a-1} M_s\sum_{r_s=1}^{m_s-1} 1_{I_a(r_s e_s)}\,(x)\,,\quad x\in \overline{I}_a.\end{aligned}$$

Hence

$$\int_{\overline{I}_a} (Vl\,(x))^p\,dx \le \frac{cM_a}{M_a^p}\frac{1}{M_a}\sum_{s=0}^{a-1} M_s^p \le c_p < \infty.$$

Theorem 2 is proved. □

2. The two-dimensional Vilenkin–Lebesgue points

The rectangular partial sums of the double Vilenkin–Fourier series are defined as follows:

$$S_{M,N}(f;x^1,x^2) := \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} \widehat{f}(i,j)\psi_i\left(x^1\right)\psi_j\left(x^2\right),$$

where the number

$$\widehat{f}\,(i,j) = \int_{G_m\times G_m} f\left(x^1,x^2\right)\overline{\psi}_i\left(x^1\right)\overline{\psi}_j\left(x^2\right)d\mu\left(x^1,x^2\right)$$

is said to be the (i,j)th Vilenkin–Fourier coefficient of the function f.

The norm (or quasinorm) of the space $L_p(G_m \times G_m)$ is defined by ($\boldsymbol{\mu}$ is the product measue $\mu \times \mu$)

$$\|f\|_p := \left(\int_{G_m \times G_m} \left| f\left(x^1, x^2\right) \right|^p d\boldsymbol{\mu}\left(x^1, x^2\right) \right)^{1/p} \quad (0 < p < +\infty).$$

The space weak-$L_p(G_m \times G_m)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_p(G_m \times G_m)} := \sup_{\lambda > 0} \lambda \boldsymbol{\mu} \left(|f| > \lambda\right)^{1/p} < +\infty.$$

Let

$$I_{n,k}\left(x^1, x^2\right) := I_n\left(x^1\right) \times I_k\left(x^2\right).$$

The σ-algebra generated by the rectangles $\{I_{n,k}(x^1, x^2) : (x^1, x^2) \in G_m \times G_m\}$ will be denoted by $F_{n,k} (n, k \in \mathbb{N})$.

Denote by $f = (f^{(n,k)}, n, k \in \mathbb{N})$ a martingale with respect to $(F_{n,k}, n, k \in \mathbb{N})$ (for details see, e.g. [12], [15]).

The maximal function of a martingale f is defined by

$$f^* = \sup_{n,k \in \mathbb{N}} \left| f^{(n,k)} \right|.$$

In case $f \in L_1(G_m \times G_m)$, the maximal function is also be given by

$$f^*\left(x^1, x^2\right) = \sup_{n,k \in \mathbb{N}} \frac{1}{\boldsymbol{\mu}\left(I_{n,k}(x^1, x^2)\right)} \left| \int_{I_{n,k}(x^1, x^2)} f\left(u^1, u^2\right) d\boldsymbol{\mu}\left(u^1, u^2\right) \right|$$

$$\left(x^1, x^2\right) \in G_m \times G_m.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m \times G_m)$ consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

A function $l \in L_2(G_m \times G_m)$ is an $H_p(G_m \times G_m)$-atom if there exists a Vilenkin rectangle $R \in F_{a,b}$ for any $a, b \in \mathbb{N}$ such that

$$\begin{cases} \text{a)} \quad \operatorname{supp}(l) \subset R \\ \text{b)} \quad \|l\|_2 \le (\boldsymbol{\mu}(R))^{1/2 - 1/p} \\ \text{c)} \quad \displaystyle\int_{G_m} l\left(x^1, x^2\right) d\mu\left(x^i\right) = 0, \; i = 1, 2. \end{cases} \tag{6}$$

We say that a measurable function f is in $L\log^{+}L(G_m\times G_m)$ if

$$\int_{G_m\times G_m}\left|f\left(x^1,x^2\right)\right|\log^{+}\left|f\left(x^1,x^2\right)\right|d\mu\left(x^1,x^2\right)<\infty,$$

where $\log^{+}u:=1_{\{u>1\}}\log u$.

For a two-dimensional integrable function f we need to introduce the hybrid maximal function

$$f^{\#}\left(x^1,x^2\right):=\sup_{n\in\mathbb{N}}\frac{1}{\mu\left(I_n\left(x^1\right)\right)}\left|\int_{I_n(x^1)}f\left(t,x^2\right)d\mu\left(t\right)\right|.$$

We say that a two-dimensional function $f\in L_1\left(G_m\times G_m\right)$ is in the space $H_1^{\#}$ if

$$\|f\|_{H_1^{\#}}:=\left\|f^{\#}\right\|_1<\infty.$$

Recall that $L\log^{+}L\left(G_m\times G_m\right)\subset H_1^{\#}\left(G_m\times G_m\right)$, more exactly,

$$\|f\|_{H_1^{\#}}\leq c+c\left\||f|\log^{+}|f|\right\|_1\quad\left(f\in L\log^{+}L\right).$$

However, for a non-negative function $f\in L_1\left(G_m\times G_m\right)$, the conditions $f\in L\log^{+}L\left(G_m\times G_m\right)$ and $f\in H_1^{\#}\left(G_m\times G_m\right)$ are equivalent.

For each Vilenkin rectangle $R\in F_{a,b}$ let $R^r(r\in N)$ be the Vilenkin rectangle from $F_{a-r,b-r}$ for which $R\subset R^r$. Let

$$R^r:=I_{a-r}\times I_{b-r}.$$

Theorem W2 (Weisz [14]**).** *Suppose that the operator T is sublinear and $p_0<p\leq 1$. Furthermore, assume that there exist $\delta>0$ such that for every $H_p(G_m\times G_m)$-atom a supported on the rectangle R and for every $r\geq 1$ one has*

$$\int_{(G_m\times G_m)\setminus R^r}|Tl|^p d\mu\leq c_p 2^{-\delta r},\tag{7}$$

where c_p is a constant depending only on p. If T is bounded from $L_2(G_m\times G_m)$ to $L_2(G_m\times G_m)$, then

$$\|Tf\|_p\leq c_p\|f\|_{H_p}\quad\left(f\in H_p\left(G_m\times G_m\right),\ p_0<p\leq 2\right).$$

Moreover, T is of weak type $(H_1^{\#}(G_m\times G_m),L_1(G_m\times G_m))$ i.e. if $f\in H_1^{\#}(G_m\times G_m)$ then

$$\sup_{\lambda}\lambda\mu\left(|Tf|>\lambda\right)\leq c\|f\|_{H_1^{\#}}.$$

If $f \in L_1(G_m \times G_m)$ then it is easy to show that the sequence $(S_{M_n,M_k}(f) : n,k \in \mathbb{N})$ is a martingale. If f is a martingale, that is $f = (f^{(n,k)} : n,k \in \mathbb{N})$, then the Vilenkin–Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i,j) = \lim_{\min(k,l)\to\infty} \int_{G_m \times G_m} f^{(k,l)}\left(x^1,x^2\right) \overline{\psi}_i\left(x^1\right) \overline{\psi}_j\left(x^2\right) d\mu\left(x^1,x^2\right).$$

The Vilenkin–Fourier coefficients of $f \in L_1(G_m \times G_m)$ are the same as those of the martingale $(S_{M_n,M_k}(f) : n,k \in \mathbb{N})$ obtained from f.

For $n,k \in \mathbb{N}_+$ and a martingale f the Fejér mean of order (n,k) of the double Vilenkin–Fourier series of f is given by

$$\sigma_{n,k} f(x^1,x^2) = \frac{1}{nk} \sum_{i=1}^{n} \sum_{j=1}^{k} S_{i,j} f(x^1,x^2).$$

For a martingale f the unrestricted maximal operators of the Fejér means are defined by

$$\sigma^* f\left(x^1,x^2\right) = \sup_{n,k\in\mathbb{N}} |\sigma_{n,k} f(x^1,x^2)|.$$

It is well known that

$$\sigma_{n,k} f(x^1,x^2) = \int_{G_m \times G_m} f\left(t^1,t^2\right) K_n\left(x^1 - t^1\right) K_k\left(x^2 - t^2\right) d\mu\left(t^1,t^2\right).$$

In 1992 Móricz, Schipp and Wade [10] proved with respect to the Walsh–Paley system that

$$\sigma_{n,k} f \to f$$

a.e. for each $f \in L\log^+ L(G_2 \times G_2)$, when $\min(n,k) \to \infty$. In 2000 Gát proved [5] that the theorem of Móricz, Schipp and Wade above can not be improved. Namely, let $\delta : [0,+\infty) \to [0,+\infty)$ be a measurable function with property $\lim_{t\to\infty} \delta(t) = 0$. Gát proved [5] the existence of a function $f \in L_1(G_2 \times G_2)$ such that $f \in L\log^+ L\delta(L)$, and $\sigma_{n,k} f$ does not converge to f a.e. as $\min\{n,k\} \to \infty$. That is, the maximal convergence space for the $(C,1)$ means of two-dimensional partial sums is $L\log^+ L(G_2 \times G_2)$, and not $L_1(G_2 \times G_2)$.

In [13] Weisz proved with respect to the Vilenkin system that

$$\sigma_{n,k} f \to f$$

a.e. for each $f \in L\log^+ L(G_m \times G_m)$, when $\min(n,k) \to \infty$. Feichtinger and Weisz [3] investigate two-dimensional Lebesgue points for trigonometric Fourier series (see also Weisz [16])

In this section we will characterize the set of convergence of two-dimensional Vilenkin–Fejér means. We introduce first the operators

$$W_{A,B}f(x^1,x^2) := \sum_{i=0}^{A}\sum_{j=0}^{B} M_i M_j \sum_{r_i^1=1}^{m_i-1}\sum_{r_j^2=1}^{m_j-1} \times \int_{I_A(x-r_i^1 e_i)\times I_B(x^2-r_j^2 e_j)} |f(t^1,t^2)-f(x^1,x^2)|d\mu(t^1,t^2),$$

$$H_B^{(2)}f(x^1,x^2) := \sum_{j=0}^{B} M_j \sum_{r_j^2=1}^{m_j-1} \int_{G_m\times I_B(x^2-r_j^2 e_j)} |f(t^1,t^2)-f(x^1,x^2)|d\mu(t^1,t^2)$$

and

$$H_A^{(1)}f(x^1,x^2) := \sum_{i=0}^{A} M_i \sum_{r_i^1=1}^{m_i-1} \int_{I_A\left(x^1-r_i^1 e_i\right)\times G_m} \left|f\left(t^1,t^2\right)-f\left(x^1,x^2\right)\right|d\mu\left(t^1,t^2\right).$$

Definition 1. Let $f \in L_1(G_m \times G_m)$. A point $(x^1,x^2) \in G_m \times G_m$ is a two-dimensional Vilenkin–Lebesgue point of f, if

$$\lim_{\min(A,B)\to\infty} W_{A,B}f\left(x^1,x^2\right) = 0, \tag{8}$$

$$\sup_A H_A^{(1)}f\left(x^1,x^2\right) < \infty \tag{9}$$

and

$$\sup_B H_B^{(2)}f\left(x^1,x^2\right) < \infty. \tag{10}$$

First, we note that condition (8) does not imply the condition (9) and (10), indeed let

$$f\left(t^1,t^2\right) := \begin{cases} \dfrac{1}{\sqrt{|t^1|}}, & (t^1,t^2)\in \left(G_m\backslash\left(\{(0,0)\}\bigcup I_2(1,1)\right)\right)\times I_2(1,1) \\ \dfrac{1}{\sqrt{|t^2|}}, & (t^1,t^2)\in I_2(1,1)\times\left(G_m\backslash\left(\{(0,0)\}\bigcup I_2(1,1)\right)\right), \\ 0, & \text{otherwise} \end{cases}$$

where

$$|x| := \sum_{k=0}^{\infty}\frac{x_k}{M_{k+1}}, \quad x\in G_m.$$

Then it is easy to show that

$$W_{A,B}f\left(0,0\right)=0.$$

On the other hand,

$$\begin{aligned}H_A^{(1)}f(0,0)&=\sum_{i=0}^{A}M_i\sum_{r_i^1=1}^{m_i-1}\int_{I_A(r_i^1e_i)\times G_m}|f(t^1,t^2)|d\mu(t^1,t^2)\\&=\sum_{i=0}^{A}M_i\sum_{r_i^1=1}^{m_i-1}\int_{I_A(r_i^1e_i)\times I_2(1,1)}\frac{1}{\sqrt{|t^1|}}d\mu(t^1,t^2)\\&\geq\frac{c}{M_A}\sum_{i=0}^{A-1}M_i^{3/2}\to\infty\quad\text{as } A\to\infty.\end{aligned}$$

Analogously, we can prove that

$$H_B^{(2)}f\left(0,0\right)\to\infty\quad\text{as } B\to\infty.$$

Let$|f|\in H_1^{\#}(G_m\times G_m)$. We prove that the conditions (9) and (10) hold for a.e. $(x^1,x^2)\in G_m\times G_m$. Indeed, let

$$F\left(t^2\right):=\int_{G_m}\left|f\left(t^1,t^2\right)\right|d\mu\left(t^1\right).$$

Then we can write

$$\begin{aligned}&\int_{G_m}\left(\sup_n\frac{1}{|I_n(x^2)|}\int_{I_n(x^2)}F(t^2)d\mu(t^2)\right)d\mu(x^2)\\&\leq\int_{G_m\times G_m}\left(\sup_n\frac{1}{|I_n(x^2)|}\int_{I_n(x^2)}|f(t^1,t^2)|d\mu(t^2)\right)d\mu(t^1)d\mu(x^2)\\&=\int_{G_m\times G_m}|f|^{\#}\left(t^1,x^2\right)d\mu\left(t^1,x^2\right)<\infty.\end{aligned}$$

Hence $F\in H_1(G_m)$.

Since

$$\sup_A\sum_{i=0}^{A}M_i\sum_{r_i^1=1}^{m_i-1}\int_{I_A(x^1-r_i^1e_i)}\left(\int_{G_m}|f(t^1,t^2)|d\mu(t^2)\right)d\mu\left(t^1\right)=VF\left(x^1\right).$$

From Theorem 2 we conclude that $VF\in L_1(G_m)$. Hence

$$\sup_A H_A^{(1)}f(x^1,x^2)<\infty\quad\text{a.e. }\left(x^1,x^2\right)\in G_m. \tag{11}$$

Analogously, we can prove that

$$\sup_B H_B^{(2)} f\left(x^1, x^2\right) < \infty \text{ a. e. } \left(x^1, x^2\right) \in G_m. \tag{12}$$

Denote

$$V_{A,B} f\left(x^1, x^2\right)$$

$$:= \sum_{i=0}^{A} \sum_{j=0}^{B} M_i M_j \sum_{r_i^1=1}^{m_i-1} \sum_{r_j^2=1}^{m_j-1} \int_{I_A(x^1-r_i^1 e_i)\times I_B\left(x^2-r_j^2 e_j\right)} f\left(t^1, t^2\right) d\mu\left(t^1, t^2\right).$$

It is easy to see that $W_{A,B} f(x^1, x^2) \to 0$ as $\min(A, B) \to \infty$ if and only if

$$\lim_{\min(A,B)\to\infty} V_{A,B}\left(|f - f(x^1, x^2)|\right)\left(x^1, x^2\right) = 0.$$

Let

$$Vf := \sup_{A,B} |V_{A,B} f|.$$

For the two-dimensional Vilenkin–Fejér means we prove that the following is true

Theorem 3. *Let $f \in L_1(G_m \times G_m)$. Then*

$$\lim_{\min(n,k)\to\infty} \sigma_{n,k} f\left(x^1, x^2\right) = f\left(x^1, x^2\right)$$

for all two-dimensional Vilenkin–Lebesgue points of f.

Theorem 4. *Let $p > 0$. Then*

$$\|Vf\|_p \le c_p \|f\|_p \quad (f \in H_p(G_m \times G_m))$$

and

$$\sup_\lambda \lambda \mu\{Vf > \lambda\} \le c\|f\|_{H_1^\#}.$$

It is easy to show that $\lim_{\min(A,B)\to\infty} W_{A,B} f(x^1, x^2) = 0$ for every Vilenkin polynomials and $(x^1, x^2) \in G_m \times G_m$. Since the Vilenkin polynomials are dense in $L_1(G_m \times G_m)$, Theorem 4 and the usual density argument (see Marcinkiewicz and Zygmund [9]) imply

Corollary 3. *Let $f \in L\log^+ L(G_m \times G_m)$ ($|f| \in H^\#(G_m \times G_m)$). Then*

$$\lim_{\min(A,B)\to\infty} W_{A,B} f\left(x^1, x^2\right) = 0 \quad \textit{for a.e. } (x^1, x^2) \in G_m \times G_m.$$

Corollary 4. *Let $f \in L\log^+ L(G_m \times G_m)$. Then*

$$\lim_{\min(n,k)\to\infty} \sigma_{n,k} f\left(x^1, x^2\right) = f\left(x^1, x^2\right) \quad \text{a.e. } \left(x^1, x^2\right) \in G_m \times G_m.$$

For special indicies and two-dimensional unbounded Vilenkin group a.e. convergence of Fejer means was investigated by GÁT [8]

PROOF OF THEOREM 3. From (1), (4) and (5) we can write

$$\begin{aligned} |\sigma_{n,k} f(x^1,x^2) - f(x^1,x^2)| &\le \frac{c}{nk} \sum_{A=0}^{|n|} M_A \sum_{B=0}^{|k|} M_B \int_{G_m\times G_m} |f(t^1,t^2) - f(x^1,x^2)| \\ &\times \left|K_{M_A}\left(x^1 - t^1\right)\right| \left|K_{M_B}\left(x^2 - t^2\right)\right| d\mu\left(t^1,t^2\right) \\ &\le \frac{c}{nk} \sum_{A=0}^{|n|} M_A \sum_{B=0}^{|k|} M_B \sum_{i=0}^{A} \sum_{j=0}^{B} M_i M_j \sum_{r_i^1=1}^{m_i-1} \sum_{r_j^2=1}^{m_j-1} \\ &\times \int_{I_A\left(x-r_i^1 e_i\right)\times I_B\left(x^2-r_j^2 e_j\right)} \left|f\left(t^1,t^2\right) - f\left(x^1,x^2\right)\right| \mu\left(t^1,t^2\right) \\ &= \frac{c}{nk} \sum_{A=0}^{|n|} M_A \sum_{B=0}^{|k|} M_B W_{A,B} f\left(x^1,x^2\right). \end{aligned} \tag{13}$$

Let $\alpha(n) = [(1/4)\log_M n]$, where $M := \sup\{m_i : i \ge 0\}$. Then from (13) we obtain

$$\begin{aligned} &\left|\sigma_{n,k} f\left(x^1,x^2\right) - f\left(x^1,x^2\right)\right| \\ &\quad \le \frac{c}{nk} \left(\sum_{A=0}^{\alpha(n)-1} \sum_{B=0}^{\alpha(k)-1} + \sum_{A=0}^{\alpha(n)-1} \sum_{B=\alpha(k)}^{|k|} + \sum_{A=\alpha(n)}^{|n|} \sum_{B=0}^{\alpha(k)-1} + \sum_{A=\alpha(n)}^{|n|} \sum_{B=\alpha(k)}^{|k|} \right) \\ &\quad\quad \times M_A M_B W_{A,B} f\left(x^1,x^2\right) = \sum_{i=1}^{4} I_i. \end{aligned} \tag{14}$$

Since

$$W_{A,B} f\left(x^1,x^2\right) < c\left(x^1,x^2\right) M_A M_B$$

for I_1 we can write

$$\begin{aligned} I_1 &\le \frac{c}{nk} \sum_{A=0}^{\alpha(n)-1} \sum_{B=0}^{\alpha(k)-1} M_A^2 M_B^2 \le \frac{c M_{\alpha(n)}^2 M_{\alpha(k)}^2}{nk} \\ &\le \frac{c M^{2(\alpha(n)+\alpha(k))}}{nk} \le \frac{c}{(nk)^{1/2}} \to 0 \text{ as } \min(n,k) \to \infty. \end{aligned} \tag{15}$$

Since

$$\begin{aligned}
&W_{A,B} f\left(x^1, x^2\right) \\
&\quad \leq c M_A \sum_{j=0}^{B-1} M_j \sum_{r_j^2=1}^{m_j-1} \int_{G \times I_B\left(x^2 - r_j^2 e_j\right)} \left|f\left(t^1, t^2\right) - f\left(x^1, x^2\right)\right| d\mu\left(t^1, t^2\right) \\
&\quad \leq c M_A H_B^{(2)} f\left(x^1, x^2\right).
\end{aligned}$$

From (10) we have

$$\begin{aligned}
I_2 &\leq \frac{c}{nk} \sum_{A=0}^{\alpha(n)-1} \sum_{B=\alpha(k)}^{|k|} M_A M_B W_{A,B} f\left(x^1, x^2\right) \\
&\leq \frac{c}{nk} \sum_{A=0}^{\alpha(n)-1} M_A^2 \sum_{B=\alpha(k)}^{|k|} M_B H_B^{(2)} f\left(x^1, x^2\right) \\
&\leq \frac{c M_{\alpha(n)}^2}{n} \leq \frac{c}{\sqrt{n}} \to 0 \text{ as } n \to \infty. \qquad (16)
\end{aligned}$$

Analogously, we can prove that

$$I_3 \to 0 \text{ as } m \to \infty. \qquad (17)$$

From (8) it is easy to show that

$$\frac{c}{nk} \sum_{A=\alpha(n)}^{|n|} \sum_{B=\alpha(m)}^{|m|} M_A M_B W_{A,B} f\left(x^1, x^2\right) \to 0 \text{ as } \min(n,k) \to \infty. \qquad (18)$$

Combining (14)–(18) we conclude the proof of Theorem 3. □

Proof of Theorem 4. By Theorem W2, the proof of Theorem 4 will be complete if we show that the operator V satisfies (7) and is bounded from $L_2(G_m \times G_m)$ to $L_2(G_m \times G_m)$.

Since $V_{A,B} f = V_A(V_B f)$ and

$$\left\| \sup_A |V_A f| \right\|_p \leq c_p \|f\|_p \quad (1 < p \leq \infty)$$

iterating the one-dimensional result we get easily that

$$\left\| \sup_{A,B} |V_{A,B} f| \right\|_p \leq c_p \|f\|_p \quad (1 < p \leq \infty).$$

To verify (7), let a be an arbitrary atom with support $I_a(z') \times I_b(z'')$. It is easy to see that $V_{A,B}(a) = 0$ if $A < a$ or $B < b$. Therefore we can suppose that $A \geq a$ and $B \geq b$. Since the dyadic addition is a measure preserving group operation, we may assume that $z' = z'' = 0$.

Step 1: Integrating over $I_{a-r} \times \overline{I}_{b-r}$. Since

$$I_{a-r} \times \overline{I}_{b-r} = \left((I_{a-r} \backslash I_a) \times \overline{I}_{b-r}\right) \cup \left(I_a \times \overline{I}_{b-r}\right)$$

we will consider two cases:

a) Let $(x^1, x^2) \in I_a \times \overline{I}_{b-r}$. Then we can write

$$V_{A,B} l\left(x^1, x^2\right) = \sum_{i=0}^{A} \frac{M_i}{M_A} \sum_{j=0}^{B} \frac{M_j}{M_B} \sum_{r_i^1=1}^{m_i-1} \sum_{r_j^2=1}^{m_j-1}$$
$$\times \int_{I_a \times I_b} l\left(t^1, t^2\right) D_{M_A}\left(x^1 - r_i^1 e_i - t^1\right) D_{M_B}\left(x^2 - r_j^2 e_i - t^2\right) d\mu\left(t^1, t^2\right)$$
$$= \sum_{j=0}^{b-r} M_j \sum_{r_j^2=1}^{m_j-1} 1_{I_b\left(r_j^2 e_j\right)}\left(x^2\right) \int_{I_b} \left(\sum_{i=0}^{A} \frac{M_i}{M_A} \sum_{r_i^1=1}^{m_i-1} \right.$$
$$\left. \times \int_{I_a} l\left(t^1, t^2\right) D_{M_A}\left(x^1 - r_i^1 e_i - t^1\right) d\mu\left(t^1\right) \right) d\mu\left(t^2\right)$$
$$= \sum_{j=0}^{b-r} M_j \sum_{r_j^2=1}^{m_j-1} 1_{I_b\left(r_j^2 e_j\right)}\left(x^2\right) \int_{I_b} V_A l\left(x^1, t^2\right) d\mu\left(t^2\right).$$

Hence

$$V l\left(x^1, x^2\right) \leq \sum_{j=0}^{b-r} M_j \sum_{r_j^2=1}^{m_j-1} 1_{I_b\left(r_j^2 e_j\right)}\left(x^2\right) \int_{I_b} \sup_A V_A l\left(x^1, t^2\right) d\mu\left(t^2\right)$$
$$\leq \frac{1}{M_b^{1/2}} \sum_{j=0}^{b-r} M_j \sum_{r_j^2=1}^{m_j-1} 1_{I_b\left(r_j^2 e_j\right)}\left(x^2\right) \left(\int_{I_b} \left(\sup_A V_A l\left(x^1, t^2\right) \right)^2 d\mu\left(t^2\right) \right)^{1/2}.$$

Applying the inequality

$$\left(\sum_k a_k \right)^p \leq \sum_k a_k^p \quad (a_k \geq 0,\ 0 < p \leq 1)$$

from Theorem 2 we can write

$$\int_{I_a} \left(V l\left(x^1, x^2\right)\right)^p d\mu\left(x^1\right) \leq \frac{1}{M_b^{p/2}} \sum_{j=0}^{b-r} M_j^p \sum_{r_j^2=1}^{m_j-1} 1_{I_b\left(r_j^2 e_j\right)}\left(x^2\right)$$

$$\times \int_{I_a} \left(\int_{I_b} \left(\sup_A V_A l \left(x^1, t^2\right) \right)^2 d\mu \left(t^2\right) \right)^{p/2} d\mu \left(x^1\right)$$

$$\leq \frac{1}{M_b^{p/2} M_a^{1-p/2}} \sum_{j=0}^{b-r} M_j^p \sum_{r_j^2=1}^{m_j-1} 1_{I_b\left(r_j^2 e_j\right)} \left(x^2\right)$$

$$\times \left(\int_{I_b} \left(\int_{I_a} \left(\sup_A V_A l \left(x^1, t^2\right) \right)^2 d\mu \left(t^2\right) \right) d\mu \left(x^1\right) \right)^{p/2}$$

$$\leq \frac{1}{M_b^{p/2} M_a^{1-p/2}} \sum_{j=0}^{b-r} M_j^p \sum_{r_j^2=1}^{m_j-1} 1_{I_b\left(r_j^2 e_j\right)} \left(x^2\right)$$

$$\times \left(\int_{I_a} \int_{I_b} \left(l \left(x^1, t^2\right) \right)^2 d\mu \left(x^1\right) d\mu \left(t^2\right) \right)^{p/2}$$

$$\leq \frac{1}{M_b^{p/2} M_a^{1-p/2}} \left(\frac{1}{M_a M_b} \right)^{p/2-1} \sum_{j=0}^{b-r} M_j^p \sum_{r_j^2=1}^{m_j-1} 1_{I_b\left(r_j^2 e_j\right)} \left(x^2\right),$$

$$\int_{I_a \times \overline{I}_{b-r}} \left(V l \left(x^1, x^2\right) \right)^p d\mu \left(x^1\right) d\mu \left(x^2\right)$$

$$\leq \frac{1}{M_b^{p-1}} \frac{1}{M_b} \sum_{j=0}^{b-r} M_j^p \leq c_p 2^{-rp}. \tag{19}$$

b) Let $(x^1, x^2) \in (I_{a-r} \backslash I_a) \times \overline{I}_{b-r}$. Then we can write

$$\left| V_{A,B} l \left(x^1, x^2\right) \right| \leq \sum_{i=0}^{a} M_i \sum_{r_i^1=1}^{m_i-1} 1_{I_a\left(r_i^1 e_i\right)} \left(x^1\right)$$

$$\times \sum_{j=0}^{b-r} M_j \sum_{r_j^2=1}^{m_j-1} 1_{I_b\left(r_j^2 e_j\right)} \left(x^2\right) \int_{I_a \times I_b} \left| l \left(t^1, t^2\right) \right| d\mu \left(t^1, t^2\right)$$

$$\leq \sum_{i=0}^{a} M_i \sum_{r_i^1=1}^{m_i-1} 1_{I_a\left(r_i^1 e_i\right)} \left(x^1\right)$$

$$\times \sum_{j=0}^{b-r} M_j \sum_{r_j^2=1}^{m_j-1} 1_{I_b\left(r_j^2 e_j\right)} \left(x^2\right) \left(\frac{1}{M_a M_b} \right)^{1/2} \left(\frac{1}{M_a M_b} \right)^{1/2-1/p}.$$

Consequently,

$$\int_{(I_{a-r}\setminus I_a)\times \overline{I}_{b-r}} \left(Vl\left(x^1,x^2\right)\right)^p d\mu\left(x^1,x^2\right)$$
$$\leq \left(\frac{1}{M_aM_b}\right)^p \sum_{i=0}^{a} M_i^p \sum_{j=0}^{b-r} M_j^p \leq c_p 2^{-rp}. \quad (20)$$

Step 2: Integrating over $\overline{I}_{a-r}\times \overline{I}_{b-r}$. Then we can write

$$\left|V_{A,B}l\left(x^1,x^2\right)\right| \leq \sum_{i=0}^{a-r} M_i \sum_{r_i^1=1}^{m_i-1} 1_{I_a\left(r_i^1 e_i\right)}\left(x^1\right)$$
$$\times \sum_{j=0}^{b-r} M_j \sum_{r_j^2=1}^{m_j-1} 1_{I_b\left(r_j^2 e_j\right)}\left(x^2\right) \int_{I_a\times I_b} \left|l\left(t^1,t^2\right)\right| d\mu\left(t^1,t^2\right),$$
$$\leq \left(\frac{1}{M_aM_b}\right)^{1-1/p} \sum_{i=0}^{a-r} M_i \sum_{r_i^1=1}^{m_i-1} 1_{I_a\left(r_i^1 e_i\right)}\left(x^1\right) \sum_{j=0}^{b-r} M_j \sum_{r_j^2=1}^{m_j-1} 1_{I_b\left(r_j^2 e_j\right)}\left(x^2\right)$$
$$\times \int_{\overline{I}_{a-r}\times \overline{I}_{b-r}} (Vl\left(x^1,x^2\right))^p d\mu\left(x^1,x^2\right)$$
$$\leq \left(\frac{1}{M_aM_b}\right)^p \sum_{i=0}^{a-r} M_i^p \sum_{j=0}^{b-r} M_j^p \leq c_p 2^{-2rp}. \quad (21)$$

Step 3: Integrating over $\overline{I}_{a-r}\times I_{b-r}$. This case is analogous to the step 1 and we obtain that

$$\int_{\overline{I}_{a-r}\times I_{b-r}} \left(Vl\left(x^1,x^2\right)\right)^p d\mu\left(x^1,x^2\right) \leq c_p 2^{-rp}. \quad (22)$$

Combining (19)–(22) we complete the proof of Theorem 4. □

Finally, we note that only condition (8) does not guarantee convergence of $\sigma_{n,k}f$. Indeed, let

$$f_1\left(x^1\right) := \begin{cases} 0, & \text{if } x^1 \in I_2(0,0) \cup \bigcup_{x_0=1}^{m_0-1} \bigcup_{x_1=1}^{m_1-1} \left(I_2\left(x_0,0\right) \cup I_2(0,x_1)\right) \\ 1 & \text{otherwise} \end{cases}$$

and

$$f_2\left(x^2\right) := \begin{cases} \sum\limits_{k=1}^{\infty} \dfrac{\psi_k\left(x\right)}{k}, & x^2 \neq 0 \\ 0, & x^2 = 0. \end{cases}$$

Let

$$\Phi\left(x^1, x^2\right) := f_1\left(x^1\right) f_2\left(x^2\right).$$

Then it is easy to show that

$$W_{A,B}\Phi\left(0,0\right) = 0.$$

On the other hand,

$$\sigma_{n,k}\Phi\left(0,0\right) = \sigma_n\left(f_1,0\right)\sigma_k\left(f_2,0\right).$$

Since

$$\left|S_k\left(f_2,0\right)\right| \to \infty \quad \text{as } k \to \infty$$

we conclude that (see [17])

$$\left|\sigma_k\left(f_2,0\right)\right| \to \infty \quad \text{as } n \to \infty.$$

For every fixed n we choose $k = k(n)$ such that

$$\left|\sigma_n\left(f_1,0\right)\sigma_{k(n)}\left(f_2,0\right)\right| \geq n.$$

Hence

$$\overline{\lim_{n\to\infty}}\left|\sigma_{n,k(n)}\Phi\left(0,0\right)\right| = +\infty.$$

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