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An approach to operator Dunkl–Williams inequalities

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Abstract. We prove several operator versions of the Dunkl–Williams inequality and discuss the case when equality holds in a new approach. We mainly present some necessary and sufficient conditions for the case where equality holds. We also deduce some norm inequalities and get a generalization of the Dunkl–Williams inequality in the framework of Hilbert C^*-modules.

1. Introduction

In 1964, Dunkl and Williams [6] showed that for any nonzero elements x and y in a normed space $(\mathcal{X}, \|.\|)$,

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{4\|x-y\|}{\|x\|+\|y\|}. \tag{1.1}$$

In the same paper, the authors proved that the constant 4 can be replaced by 2 if $\mathcal{X}$ is an inner product space. This inequality has some applications in the study of the geometry of Banach spaces. Kirk and Smiley [11] showed that inequality (1.1) with 2 instead of 4 characterizes the inner product spaces.

Inequality (1.1), which estimates the angular distance between the vectors x and y has many interesting refinements which have obtained over the years, see [3] and references therein. Maligranda [13] presented the following refinement of (1.1)

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{\|x-y\| + \big|\|x\|-\|y\|\big|}{\max\{\|x\|,\|y\|\}}. \tag{1.2}$$

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A reverse of inequality (1.1) was given by MERECER [15] as follows

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geq \frac{\|x-y\| - \big| \|x\| - \|y\| \big|}{\min\{\|x\|, \|y\|\}}. \tag{1.3}$$

Some extensions of (1.2) and (1.3) for finitely many elements of a normed space were obtained in [10], [17], see also [14]. The main result of [17] was generalized by DRAGOMIR in [5].

The operator valued versions of Dunkl–Williams are related to some inequalities for Hilbert space operators. In [19], PEČARIĆ and RAJIĆ showed the following inequality, which is a weaker version than (1.2),

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\big(2\|x-y\|^2 + 2(\|x\| - \|y\|)^2\big)^{\frac{1}{2}}}{\max\{\|x\|, \|y\|\}} \tag{1.4}$$

and introduced an operator version of (1.4) by estimating $|A|A|^{-1} - B|B|^{-1}|$, where A and B are Hilbert space operators such that $|A|$ and $|B|$ are invertible (see Corollary 3.3 below). They also investigated the Dunkl–Williams equality in pre-Hilbert C^*-modules in [18].

In [13], MALIGRANDA considered the p-angular distance ($p \in \mathbb{R}$), as a generalization of the concept of angular distance (when $p = 0$), between nonzero elements x and y in a normed space $(\mathcal{X}, \|.\|)$ as $\alpha_p[x,y] := \left\| \frac{x}{\|x\|^{1-p}} - \frac{y}{\|y\|^{1-p}} \right\|$; see also [4].

Very recently DADIPOUR, FUJII and MOSLEHIAN [2] introduced several operator versions of the Dunkl–Williams inequality with respect to the p-angular distance (see Theorem 3.2 and Theorem 3.4 below) as a generalization of both the main result of PEČARIĆ and RAJIĆ [19] and that of SAITO and TOMINAGA [23].

In this paper, we prove several operator versions of the Dunkl–Williams inequality and discuss the case when equality holds in a new approach. We mainly present some necessary and sufficient conditions for the case where equality holds. We also deduce some norm inequalities and get a generalization of the Dunkl–Williams inequality in the framework of Hilbert C^*-modules.

2. Preliminaries

Let $\mathbb{B}(\mathscr{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ equipped with the operator norm $\|\cdot\| = \|\cdot\|_\infty$.

A C^*-algebra $\mathscr{A}$ is a closed $*$-subalgebra of $\mathbb{B}(\mathscr{H})$. A self-adjoint operator A is positive if $\langle Ax, x\rangle \geq 0$, for all $x \in \mathscr{H}$. Then we write $A \geq 0$. If $A, B \in \mathscr{A}$

are self-adjoint operators such that $B - A \geq 0$, then we say that $A \leq B$. For each operator $A \in \mathscr{A}$ there corresponds the absolute value $|A| = (A^*A)^{\frac{1}{2}}$ of A, which is the positive square root of the positive operator A^*A, where A^* stands for the adjoint of A.

By a unitarily invariant norm we mean a norm defined on a two-sided ideal $\mathcal{C}_{|||\cdot|||}$ of $\mathbb{B}(\mathscr{H})$ enjoying the invariant property $|||UAV||| = |||A|||$ for all $A \in \mathcal{C}_{|||\cdot|||}$ and all unitary operators $U, V \in \mathbb{B}(\mathscr{H})$. It is known that

$$|||ABC||| \leq \|A\| \, |||B||| \, \|C\| \qquad (B \in \mathcal{C}_{|||\cdot|||},\ A, C \in \mathbb{B}(\mathscr{H})).$$

Some well-known examples of unitarily invariant norms are the Schatten ℓ-norms $\|B\|_\ell := (\operatorname{tr}|B|^\ell)^{1/\ell}$ for $1 \leq \ell < \infty$ and the Ky-Fan norms; cf. [1], [24]. A unitarily invariant norm $\|\cdot\|_Q$ is called a Q-norm if there is a unitarily invariant norm $\|\cdot\|_{\widehat{Q}}$ such that $\|B^*B\|_{\widehat{Q}} = \|B\|_Q^2$ $(B \in \mathcal{C}_Q)$. It is known that $\|\cdot\|_\ell$ is a Q-norm if and only if $2 \leq \ell \leq \infty$.

The notion of Hilbert C^*-module is a generalization of that of Hilbert space in which the field of scalars $\mathbb{C}$ is replaced by a C^*-algebra. Any Hilbert space can be regarded as a Hilbert $\mathbb{C}$-module and any C^*-algebra $\mathscr{A}$ is a Hilbert C^*-module over itself via $\langle a, b\rangle = a^*b$ $(a, b \in \mathscr{A})$. A mapping T on a Hilbert C^*-module $\mathscr{X}$ is adjointable if there is a mapping $T^* : \mathscr{X} \to \mathscr{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ $(x, y \in \mathscr{X})$. The space $\mathcal{L}(\mathscr{X})$ of all adjointable mappings T on $\mathscr{X}$ is a C^*-algebra. For every $x \in \mathscr{X}$ the absolute value of x is defined as the unique positive square root of $\langle x, x\rangle \in \mathscr{A}$, that is, $|x| = \langle x, x\rangle^{\frac{1}{2}}$.

A Hilbert C^*-module $\mathscr{X}$ over a C^*-algebra $\mathscr{A}$ can be embedded into a certain C^*-algebra $\Lambda(\mathscr{X})$. To see this, let us denote by $\mathscr{F} = \mathscr{X} \oplus \mathscr{A}$, the direct sum of Hilbert $\mathscr{A}$-modules $\mathscr{X}$ and $\mathscr{A}$ equipped with the $\mathscr{A}$-inner product

$$\big\langle (x_1, a_1), (x_2, a_2)\big\rangle = \langle x_1, x_2\rangle + a_1^*a_2.$$

Let us identify each $x \in \mathscr{X}$ with the mapping $\mathscr{A} \to \mathscr{X}$, $a \mapsto xa$. The adjoint of this map is $x^*(y) = \langle x, y\rangle$. Set

$$\Lambda(\mathscr{X}) = \left\{ \begin{bmatrix} T & x \\ y^* & a \end{bmatrix} : a \in \mathscr{A},\ x, y \in \mathscr{X},\ T \in \mathcal{L}(\mathscr{X}) \right\}.$$

$\Lambda(\mathscr{X})$ is a C^*-subalgebra of $\mathcal{L}(\mathscr{F})$, called *the linking algebra* of $\mathscr{X}$. Then

$$\mathscr{X} \simeq \begin{bmatrix} 0 & \mathscr{X} \\ 0 & 0 \end{bmatrix}, \mathscr{A} \simeq \begin{bmatrix} 0 & 0 \\ 0 & \mathscr{A} \end{bmatrix}, \mathcal{L}(\mathscr{X}) \simeq \begin{bmatrix} \mathcal{L}(\mathscr{X}) & 0 \\ 0 & 0 \end{bmatrix}.$$

Furthermore, $\langle x, y\rangle$ of $\mathscr{X}$ becomes the product x^*y in $\Lambda(\mathscr{X})$ and the module multiplication $\mathscr{X} \times \mathscr{A} \to \mathscr{X}$ becomes a part of the internal multiplication of $\Lambda(\mathscr{X})$.

Throughout this paper $\mathscr{A}$ denotes a C^*-algebra. We refer the reader to [16] for undefined notions on the theory of C*-algebras and to [12], [22] for more information on Hilbert C^*-modules.

3. Dunkl–Williams inequality and related equalities for operators

We start this section with a useful operator version of the Bohr inequality due to Hirzallah [9]. There are other generalizations of Hirzallah's result in the literature, see [20], [21] [7].

Theorem 3.1 ([9]). *Let $A, B \in \mathbb{B}(\mathscr{H})$ and $\frac{1}{r}+\frac{1}{s}=1$ $(r>1)$. Then*

$$|A-B|^2 \le r|A|^2 + s|B|^2. \tag{3.1}$$

Moreover, equality holds if and only if $(1-r)A = B$.

The following theorem is an operator version of the Dunkl–Williams inequality with respect to the p-angular distance which has been proved in [2]. Now we prove the theorem in a new fashion by using the operator Bohr inequality (3.1).

Theorem 3.2. *Let $A, B \in \mathscr{A}$ such that $|A|$ and $|B|$ are invertible, $\frac{1}{r}+\frac{1}{s}=1$ $(r>1)$ and $p \in \mathbb{R}$. Then*

$$\begin{aligned}|A|A|^{p-1} - B|B|^{p-1}|^2 \\ \le |A|^{p-1}[r|A-B|^2 + s(|A|^{1-p}|B|^p - |B|)(|B|^p|A|^{1-p} - |B|)]|A|^{p-1}.\end{aligned} \tag{3.2}$$

Moreover equality holds if and only if

$$(r-1)(A-B)|A|^{p-1} = B(|A|^{p-1} - |B|^{p-1}).$$

Proof.

$$\begin{aligned}
&|A|A|^{p-1} - B|B|^{p-1}|^2 = |A|A|^{p-1} - B|A|^{p-1} - B|B|^{p-1} + B|A|^{p-1}|^2 \\
&= |(A-B)|A|^{p-1} - B(|B|^{p-1} - |A|^{p-1})|^2 \\
&\le r|(A-B)|A|^{p-1}|^2 + s|B(|B|^{p-1} - |A|^{p-1})|^2 \qquad \text{by (3.1)} \\
&= r|A|^{p-1}|A-B|^2|A|^{p-1} + s(|B|^{p-1} - |A|^{p-1})|B|^2(|B|^{p-1} - |A|^{p-1}) \\
&= r|A|^{p-1}|A-B|^2|A|^{p-1} + s|A|^{p-1}(|A|^{1-p}|B|^p - |B|)(|B|^p|A|^{1-p} - |B|)|A|^{p-1} \\
&= |A|^{p-1}[r|A-B|^2 + s(|A|^{1-p}|B|^p - |B|)(|B|^p|A|^{1-p} - |B|)]|A|^{p-1}.
\end{aligned}$$

In addition, equality holds if and only if

$$(r-1)(A-B)|A|^{p-1} = B(|A|^{p-1} - |B|^{p-1}). \qquad \square$$

A special case of Theorem 3.2, where $p = 0$ gives rise to the main result of Pečarić and Rajić [19, Theorem 2.1].

Corollary 3.3. *Let $A, B \in \mathscr{A}$ such that $|A|$ and $|B|$ are invertible, $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$. Then*

$$|A|A|^{-1} - B|B|^{-1}|^2 \le |A|^{-1}[r|A-B|^2 + s(|A| - |B|)^2]|A|^{-1}. \tag{3.3}$$

Further, equality holds if and only if

$$(r-1)(A-B)|A|^{-1} = B(|A|^{-1} - |B|^{-1}).$$

Dadipour et al. [2] proved that the assumption of the invertibility of $|A|$ and $|B|$ in Theorem 3.2, in the case where $0 < p \le 1$, is not required.

Theorem 3.4 ([2])**.** *Let $A, B \in \mathscr{A}$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$ and let $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$ and $0 < p \le 1$. Then*

$$|(U|A|^p - V|B|^p)|A|^{1-p}|^2 \le r|A-B|^2 + s||B|^p|A|^{1-p} - |B||^2.$$

The equality holds if and only if $(r-1)(A-B) = V(|B| - |B|^p|A|^{1-p})$.

The next theorem provides some necessary and sufficient conditions for the case of equality in Theorem 3.4. In addition, it is an extension of some results due to Pečarić and Rajić [19, Lemma 2.6 and Proposition 2.7] with respect to the p-angular distance and a refinement of some results due to Dadipour et al. [2, Proposition 2.6] by removing the invertibility assumption of the operators $|A|$ and $|B|$.

Theorem 3.5. *Let $A, B \in \mathscr{A}$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$ and let $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$ and $0 < p \le 1$. If*

$$(r-1)(A-B) = V(|B| - |B|^p|A|^{1-p}), \tag{3.4}$$

then

(i) $(r-1)|A-B|^2 = \frac{1}{r}|A|^{1-p}|B|^{2p}|A|^{1-p} + \frac{1}{s}|A|^2 - |B|^2$.

(ii) $|B| \le \left(\frac{1}{r}|A|^{1-p}|B|^{2p}|A|^{1-p} + \frac{1}{s}|A|^2\right)^{\frac{1}{2}}$.

(iii) $(r-1)|A-B| = ||B|^p|A|^{1-p} - |B||$ *and* $A - B = -VW|A-B|$, *where W is the partial isometry operator that appears in the polar decomposition of $|B|^p|A|^{1-p} - |B|$.*

Moreover (3.4) *and (iii) are equivalent.*

We need the following two lemmas. The first one seems to be familiar. We however prove it for the sake of completeness.

Lemma 3.6. *Let T be a positive operator and $r > 0$. Then* $\ker(T) = \ker(T^r)$.

Proof. Let $Tx = 0$. First we note that $T^s x = 0$ for all $s \geq 1$. It follows from $\|T^{\frac{1}{2^{n+1}}}x\|^2 = \langle T^{\frac{1}{2^{n+1}}}x, T^{\frac{1}{2^{n+1}}}x\rangle = \langle T^{\frac{1}{2^n}}x, x\rangle$ and the induction that $T^{\frac{1}{2^n}}x = 0$ for all $n \in \mathbb{N}$. From the fact $\frac{1}{2^m} < r$ for some $m \in \mathbb{N}$, we get $T^r x = 0$. Thus $\ker(T) \subseteq \ker(T^r)$. Therefore $\ker(T^r) \subseteq \ker(T^r)^{\frac{1}{r}} = \ker(T)$. $\square$

Lemma 3.7. *Let $A, B \in \mathscr{A}$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$ and let $0 < p \leq 1$ be arbitrary. Then*

(i) $|A|^{1-p}|B|^p V^*V|B|^p|A|^{1-p} = |A|^{1-p}|B|^{2p}|A|^{1-p}$.

(ii) $(|A|^{1-p}|B|^p - |B|)V^*V(|B|^p|A|^{1-p} - |B|) =$
$$(|A|^{1-p}|B|^p - |B|)(|B|^p|A|^{1-p} - |B|).$$

Proof. (i) Let $|A|^{1-p}|B|^p = W||A|^{1-p}|B|^p|$ be the polar decomposition of $|A|^{1-p}|B|^p$. We note that $|B|^p|A|^{1-p} = W^*||B|^p|A|^{1-p}|$ is the polar decomposition of $|B|^p|A|^{1-p}$ [8, Section 2.2.2 Theorem 5]. First, we show that $W^*W \leq V^*V$, or equivalently $\ker(V) \subseteq \ker(W)$, since W^*W and V^*V are the projections on the closed subspaces $(\ker(W))^\perp$ and $(\ker(V))^\perp$, respectively.

For this, let $x \in \ker V = \ker|B|$. Applying Lemma 3.6 we get $|B|^p x = 0$ and so $|A|^{1-p}|B|^p x = 0$. Thus $||A|^{1-p}|B|^p|x = 0$. Hence $\ker(V) \subseteq \ker(||A|^{1-p}|B|^p|) = \ker(W)$. Therefore,

$$\begin{aligned}|A|^{1-p}|B|^{2p}|A|^{1-p} &= ||B|^p|A|^{1-p}|\;||B|^p|A|^{1-p}| = |A|^{1-p}|B|^p W^*W|B|^p|A|^{1-p}\\ &\leq |A|^{1-p}|B|^p V^*V|B|^p|A|^{1-p}.\\ &\qquad\qquad \big(\text{by } ||B|^p|A|^{1-p}| = W|B|^p|A|^{1-p}\big)\end{aligned}$$

Hence $|A|^{1-p}|B|^{2p}|A|^{1-p} \leq |A|^{1-p}|B|^p V^*V|B|^p|A|^{1-p}$ and the reverse of the preceding inequality holds due to $V^*V \leq I$.

(ii) A straightforward calculation and part (i) follows (ii). $\square$

Proof of Theorem 3.5. (i) From (3.4) we have $|(r-1)A - rB|^2 = |-V|B|^p|A|^{1-p}|^2$. Thus

$$(r-1)^2|A|^2 - r(r-1)(A^*B + B^*A) + r^2|B|^2 = |A|^{1-p}|B|^p V^*V|B|^p|A|^{1-p}.$$

Adding $(r-1)|A|^2 - r|B|^2$ to the both sides and applying Lemma 3.7 (i), we get the required equality.

(ii) It follows from (i) and the celebrated Löwner–Heinz inequality.

(iii) Taking the absolute value of both sides of (3.4), we obtain

$$(r-1)|A-B| = \left((|A|^{1-p}|B|^p - |B|)V^*V(|B|^p|A|^{1-p} - |B|)\right)^{\frac{1}{2}}.$$

Hence $(r-1)|A-B| = ||B|^p|A|^{1-p} - |B||$ by Lemma 3.7 (ii). We also have

$$\begin{aligned} A-B &= -\frac{1}{r-1}V(|B|^p|A|^{1-p} - |B|) = -\frac{1}{r-1}VW||B|^p|A|^{1-p} - |B|| \\ &= -\frac{1}{r-1}VW(r-1)|A-B| = -VW|A-B|. \end{aligned}$$

Now let us suppose that two equality conditions of (iii) are fulfilled. We have

$$\begin{aligned} (r-1)(A-B) &= -(r-1)VW|A-B| = -(r-1)VW\frac{1}{r-1}||B|^p|A|^{1-p} - |B|| \\ &= V(|B| - |B|^p|A|^{1-p}). \end{aligned}$$

The last equality holds by utilizing the polar decomposition of $|B|^p|A|^{1-p} - |B|$. □

The next result provides a Q-norm version of the Dunkl–Williams inequality (3.2).

Proposition 3.8. *Let $\|\cdot\|_Q$ be a Q-norm, $A, B \in \mathcal{C}_{\|\cdot\|_Q}$ such that $|A|^{-1}$ and $|B|^{-1}$ exist and are elements of $\mathcal{C}_{\|\cdot\|_Q}$, $\frac{1}{r}+\frac{1}{s}=1$ $(r>1)$ and $p \in \mathbb{R}$. Then*

$$\begin{aligned} &\|A|A|^{p-1} - B|B|^{p-1}\|_Q^2 \\ &\qquad \le \|\,|A|^{-1}\|^{2-2p}\left(r\|\,|A-B|\,\|_Q^2 + s\left\|\,|B|^p|A|^{1-p} - |B|\,\right\|_Q^2\right). \end{aligned} \tag{3.5}$$

Proof. Let $\|\cdot\|_{\widehat{Q}}$ be a unitarily invariant norm corresponding to the Q-norm $\|\cdot\|_Q$ via $\|\,|C|^2\|_{\widehat{Q}} = \|C\|_Q^2$. Using the fact that $0 \le T \le S$ implies that $|||T||| \le |||S|||$ for any unitarily invariant norm (see [1]) and the triangle inequality we get

$$\begin{aligned} &\|\,|A|A|^{p-1} - B|B|^{p-1}|^2\|_{\widehat{Q}} \le \|\,|A|^{p-1}\| \\ &\times\left(r\|\,|A-B|^2\|_{\widehat{Q}} + s\left\|(|B|^p|A|^{1-p} - |B|)^*(|B|^p|A|^{1-p} - |B|)\right\|_{\widehat{Q}}\right)\|\,|A|^{p-1}\|. \end{aligned}$$

Hence

$$\|A|A|^{p-1} - B|B|^{p-1}\|_Q^2 \le \|\,|A|^{-1}\|^{2-2p}\left(r\|\,|A-B|\,\|_Q^2 + s\left\|\,|B|^p|A|^{1-p} - |B|\,\right\|_Q^2\right).$$

□

It follows from (3.5), when $p = 0$ and the Q-norm is the Schatten norm $\|\cdot\|_\ell$, that

Corollary 3.9. *Let $\ell \geq 2$, $A, B \in \mathcal{C}_{\|\cdot\|_\ell}$, $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$ such that $|A|^{-1}$ and $|B|^{-1}$ exist and are elements of $\mathcal{C}_{\|\cdot\|_\ell}$. Then*

$$\|A|A|^{-1} - B|B|^{-1}\|_\ell^2 \leq \|\,|A|^{-1}\,\|^2 \left(r\|A - B\|_\ell^2 + s\|\,|A| - |B|\,\|_\ell^2\right).$$

4. Extension to Hilbert C^*-modules

In this section we extend our main result to Hilbert C^*-modules. To achieve this we need the following extension of Theorem 3.1.

Theorem 4.1 ([21, Theorem 3.7])**.** *Let $r, s > 1$ be conjugate components. Then*

$$|x - y|^2 + \frac{1}{r-1}|(1-r)x - y|^2 = r|x|^2 + s|y|^2$$

for all x, y in a Hilbert C^*-module $\mathscr{X}$. Moreover,

(i) $\quad |x - y|^2 + |(1-r)x - y|^2 \leq r|x|^2 + s|y|^2 \Leftrightarrow r \leq 2$ *or* $(1-r)x = y$,

(ii) $\quad |x - y|^2 + |(1-r)x - y|^2 \geq r|x|^2 + s|y|^2 \Leftrightarrow r \geq 2$ *or* $(1-r)x = y$.

Furthermore, in (i) *and* (ii) *equality holds on the left hand side of the equivalence if and only if $r = s = 2$ or $(1-r)x = y$.*

Now we use a linking algebra approach.

Theorem 4.2. *Let x, y be elements of a Hilbert C^*-module $\mathscr{X}$ such that $|x|$ and $|y|$ are invertible, $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$ and $p \in \mathbb{R}$. Then*

$$|x|x|^{p-1} - y|y|^{p-1}|^2 \leq |x|^{p-1}[r|x - y|^2 + s(|x|^{1-p}|y|^p - |y|)(|y|^p|x|^{1-p} - |y|)]|x|^{p-1}.$$

Moreover equality holds if and only if

$$(r-1)(x-y)|x|^{p-1} = y(|x|^{p-1} - |y|^{p-1}).$$

Proof. Let us embed $\mathscr{X}$ in its linking algebra $\Lambda(\mathscr{X})$ via $x \mapsto \left[\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right]$. Then (3.2) of Theorem 3.2 yields that

$$\left|\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\left|\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right|^{p-1} - \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\left|\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right|^{p-1}\right|^2$$

$$\leq \left|\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right|^{p-1} \left(r \left|\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right|^2 + s \left(\left|\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right|^{1-p} \left|\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right|^p \right.\right.$$
$$\left.\left. - \left|\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right| \right) \left(\left|\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right|^p \left|\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right|^{1-p} - \left|\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right| \right)\right) \left|\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right|^{p-1}.$$

Using a straightforward computation based on the fact that

$$\left|\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right|^2 = \begin{bmatrix} 0 & 0 \\ x^* & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & |x|^2 \end{bmatrix},$$

we get

$$|x|x|^{p-1} - y|y|^{p-1}|^2 \leq |x|^{p-1}(r|x-y|^2 + s(|x|^{1-p}|y|^p - |y|)(|y|^p|x|^{1-p} - |y|))|x|^{p-1}.$$

Furthermore, by Theorem 4.1 equality holds if and only if $(r-1)(x-y)|x|^{p-1} = y(|x|^{p-1} - |y|^{p-1})$. □

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