Publ. Math. Debrecen
44 / 3-4 (1994), 335–357

Construction of periodic solutions of differential equations with impulse effect

By N. I. RONTO (Kiev–Miskolc) and Á. TUZSON (Miskolc)

Abstract. In this paper we establish both the convergency and the error estimation of the trigonometric collocation method for forming a periodic solution of a system of differential equations of the form

$$\boldsymbol{dx}/dt = \boldsymbol{f}(t,\boldsymbol{x}) = \begin{cases} \boldsymbol{f}_1(1,\boldsymbol{x})\,, & t \in [0,\tau)\,, \\ \boldsymbol{f}_2(t,\boldsymbol{x})\,, & t \in (\tau, T]\,, \end{cases}$$

where $\boldsymbol{f}(t,\boldsymbol{x})$ is a piecewise continuous function and

$$\begin{gathered} \boldsymbol{x}(0) = \boldsymbol{x}(T)\,, \\ \boldsymbol{S}_1\,\boldsymbol{x}(\tau+0) + \boldsymbol{S}_2\,\boldsymbol{x}(\tau-0) = \boldsymbol{g}\,. \end{gathered}$$

In order to do this the Green function is also constructed for the linear boundary value problem with one impulse effect

$$\boldsymbol{dx}/dt = \begin{cases} \boldsymbol{A}_1(t)\,\boldsymbol{x}\,, & t \in [a,\tau) \\ \boldsymbol{A}_2(t)\,\boldsymbol{x}\,, & t \in (\tau, b]\,, \end{cases}$$

$$\begin{gathered} \boldsymbol{B}_1\,\boldsymbol{x}(a) + \boldsymbol{B}_2\,\boldsymbol{x}(b) = \boldsymbol{0}\,, \\ \boldsymbol{S}_1\,\boldsymbol{x}(\tau+0) + \boldsymbol{S}_2\,\boldsymbol{x}(\tau-0) = \boldsymbol{0}\,, \end{gathered}$$

with piecewise continuous coefficients.

1. Introduction

The algebraic and trigonometric collocation methods together with other numerical methods can effectively be applied for forming approximate solutions of broad classes of two-point, multipoint, periodic, linear

Supported by project "Impulse" 1/182.

and nonlinear boundary value problems of ordinary differential equations with retarded argument (see [1], [2]).

The study of the extension of the collocation method in case of impulse effect problems is of theoretical and practical interest. Though the theory of differential equations with impulse effect developed intensively in the last decade, most of the papers are devoted to qualitative problems as existence, asymptotical behaviour and stability, and only a few of them to the construction of the solutions (see [3], [4], [5], [6]).

In this paper for the sake of simplicity the T-periodic boundary value problem with one impulse effect is studied. We shall show the application of the trigonometric collocation method to T-periodic systems with impulse effect. Assuming the existence of a periodic solution the solvability of the equations determining the coefficients of the approximate solution is established.We shall also give the speed of convergence of the approximate solution.

We note that in case of a finite number of impulse effects similar results are valid.

We are going to solve the system

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t,\boldsymbol{x}) = \begin{cases} \boldsymbol{f}_1(t,\boldsymbol{x}), & t \in [0,\tau), \\ \boldsymbol{f}_2(t,\boldsymbol{x}), & t \in (\tau, T\,], \end{cases} \tag{1}$$

with piecewise continuous right-hand side under the conditions

$$\boldsymbol{S}_1\boldsymbol{x}(\tau+0) + \boldsymbol{S}_2\boldsymbol{x}(\tau-0) = \boldsymbol{g},\ \det \boldsymbol{S}_1, \boldsymbol{S}_2 \neq 0 \tag{2}$$

$$\boldsymbol{x}(0) = \boldsymbol{x}(T)\,. \tag{3}$$

This boundary value problem is a reduction of the problem of T-periodic solutions of (1)–(2) in $t \in \mathbb{R}$ with impulse effect acting at each $t = \tau + kT$ $(k = 0 \pm 1, \pm 2, \dots)$.

Let the vector functions $\boldsymbol{f}_i(t,\boldsymbol{x}) = (f_{i1}(t,\boldsymbol{x}), \dots, f_{in}(t,\boldsymbol{x}))$ and the *Jacobian*-matrices $\boldsymbol{A}_i(t,\boldsymbol{x}) = \left(\dfrac{\partial \boldsymbol{f}_i(t,\boldsymbol{x})}{\partial \boldsymbol{x}}\right)$, $(i = 1, 2\,)$ be defined and continuous in the domains

$$\begin{aligned} \boldsymbol{f}_1 &: [0,\tau] \times \boldsymbol{D} \to \mathbb{R}^n\,, \\ \boldsymbol{f}_2 &: [\tau, T] \times \boldsymbol{D} \to \mathbb{R}^n\,, \end{aligned} \tag{4}$$

where $\boldsymbol{D}$ is a bounded and closed set in the n–dimensional *Euclidean* space $\mathbb{R}^n$ and the right-hand side of (1) is a periodic function in t of period T

$$\boldsymbol{f}_1(0,\boldsymbol{x}) = \boldsymbol{f}_2(T,\boldsymbol{x})\,.$$

We denote by $C_\tau[0,T] = C_\tau$ the space of piecewise continuous vector-functions

$$\boldsymbol{x}(t) = \begin{cases} \boldsymbol{x}_1(t)\,, & t \in [0,\tau)\,, \\ \boldsymbol{x}_2(t)\,, & t \in (\tau, T\,]\,, \end{cases} \tag{5}$$

in $[0,T]$ having a point of discontinuity of the first kind at $t=\tau$

$$\boldsymbol{x}(\tau+0)-\boldsymbol{x}(\tau-0)=\boldsymbol{x}_2(\tau)-\boldsymbol{x}_1(\tau),$$

where

$$\boldsymbol{x}_1(t)=(x_{11}(t),\dots,x_{1n}(t))\in C[0,\tau]\,,$$
$$\boldsymbol{x}_2(t)=(x_{21}(t),\dots,x_{2n}(t))\in C[\tau,T]\,.$$

We define the following uniform norms in $C[0,\tau]\,,C[\tau,T]$

$$|\boldsymbol{x}_1|_{C[0,\tau]}=\max_{i=1,\dots,n}\ \max_{t\in[0,\tau]}|x_{1i}(t)|\,,$$
$$|\boldsymbol{x}_2|_{C[\tau,T]}=\max_{i=1,\dots,n}\ \max_{t\in[\tau,T]}|x_{2i}(t)|\,,$$
$$|\boldsymbol{x}|_{C_\tau[0,T]}=|\boldsymbol{x}|_{C_\tau}=\max(|\boldsymbol{x}_1|_{C[0,\tau]},|\boldsymbol{x}_2|_{C[\tau,T]})\,.$$

Consider the vector functions $\boldsymbol{x}(t)$ in (5) in the space of quadratically summable vector functions $L^2_\tau[0,T]=L^2_\tau$ in the interval $[0,T]$ with the norm

$$|\boldsymbol{x}|_{L^2_\tau[0,T]}=|\boldsymbol{x}|_{L^2_\tau}=|\boldsymbol{x}_1|_{L^2[0,\tau]}+|\boldsymbol{x}_2|_{L^2[\tau,T]}=$$
$$=\left[\int_0^\tau|\boldsymbol{x}_1(t)|^2dt\right]^{\frac12}+\left[\int_\tau^T|\boldsymbol{x}_2(t)|^2dt\right]^{\frac12}.$$

The Green functions will be needed, too, to prove the convergence of the trigonometric collocation method applied to set up the solution of (1)–(3).

2. The Green function for systems with impulse effect and piecewise continuous coefficients

Consider a homogeneous two-point boundary value problem with one impulse effect

$$\frac{d\boldsymbol{x}}{dt}=\boldsymbol{A}(t)\boldsymbol{x},\quad t\neq\tau\,,\ t\in[a,b]\,,\ \boldsymbol{x}\in\mathbb{R}^n\,, \tag{6}$$

$$\boldsymbol{S}_1\boldsymbol{x}\,(\tau+0)+\boldsymbol{S}_2\boldsymbol{x}\,(\tau-0)=\boldsymbol{0},,\quad \tau\in(a,b)\,,\ \det\boldsymbol{S}_1,\boldsymbol{S}_2\neq0\,, \tag{7}$$

$$\boldsymbol{B}_1\boldsymbol{x}(a)+\boldsymbol{B}_2\boldsymbol{x}(b)=\boldsymbol{0}\,, \tag{8}$$

where $\boldsymbol{A}(t)$ is a piecewise continuous $n\times n$ matrix in $[a,b]$ the elements of which may have discontinuity of the first kind at $t=\tau$

$$\boldsymbol{A}(t)=\begin{cases}\boldsymbol{A}_1(t)\,, & t\in[a,\tau)\,,\\ \boldsymbol{A}_2(t)\,, & t\in(\tau,b]\,.\end{cases}$$

$\boldsymbol{B}_1, \boldsymbol{B}_2$ are $n \times n$ constant matrices.

Let $\boldsymbol{\Phi}_1$ be the fundamental matrix of the following system of homogeneous differential equations without impulse effect, normalized at $t = a$

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}_1(t)\boldsymbol{x}\,, \quad t \in [a, \tau)\,, \qquad \boldsymbol{\Phi}_1(a) = \boldsymbol{E}\,, \tag{9}$$

and $\boldsymbol{\Phi}_2(t)$ the fundamental matrix of the system

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}_2(t)\boldsymbol{x}\,, \quad t \in (\tau, b]\,. \tag{10}$$

Lemma 1. *Let the homogeneous boundary value problem* (6)–(8) *have only trivial solution in the space* $C_\tau[a, b]$

$$\boldsymbol{x}(t) = \begin{cases} \boldsymbol{x}_1(t) \equiv \boldsymbol{0}\,, & t \in [a, \tau)\,, \\ \boldsymbol{x}_2(t) \equiv \boldsymbol{0}\,, & t \in (\tau, b] \end{cases} \tag{11}$$

(especially the continuous solution $\boldsymbol{x}(t) \equiv \boldsymbol{0}$*). Then the matrices* $\boldsymbol{S}_1, \boldsymbol{S}_2$*,* $\boldsymbol{B}_1, \boldsymbol{B}_2$ *and the fundamental matrices* $\boldsymbol{\Phi}_1(t), \boldsymbol{\Phi}_2(t)$ *of the homogeneous system* (9),(10) *fulfill the inequality*

$$\det \boldsymbol{D} \neq 0\,, \; \boldsymbol{D} = \boldsymbol{B}_1 - \boldsymbol{B}_2\boldsymbol{\Phi}_2(b)\boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\,. \tag{12}$$

Proof. It is known that the solution $\boldsymbol{x}(t)$ of the system (9) passing through the point $\boldsymbol{x} = \boldsymbol{x}(a)$ in case $t = a$ can be written in the form

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}_1(t)\boldsymbol{\Phi}_1(a)\boldsymbol{x}(a) = \boldsymbol{\Phi}_1(t)\boldsymbol{x}(a)\,, \; t \in [a, \tau)\,. \tag{13}$$

From (7) with (13) we obtain that

$$\boldsymbol{x}(\tau + 0) = -\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\boldsymbol{x}(\tau - 0) = -\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\boldsymbol{x}(a)\,. \tag{14}$$

Hence, the solution of (10) under the initial condition (14) is

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}_2(t)\boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{x}(\tau) = -\boldsymbol{\Phi}_2(t)\boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\boldsymbol{x}(a)\,, \; t \in (\tau, b]\,.$$

On the ground of this relation and the boundary condition (8) we obtain

$$[\boldsymbol{B}_1 - \boldsymbol{B}_2\boldsymbol{\Phi}_2(b)\boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)]\boldsymbol{x}(a) = \boldsymbol{0}\,.$$

Since the boundary value problem (6)–(8) has only trivial solution (11), hence $\boldsymbol{x}(a) \equiv \boldsymbol{0}$ and relation (12) is valid. This completes the proof.

Corollary. *If* (12) *holds for the linear boundary value problem with one impulse effect described in* (6)–(8) *then this problem possesses only trivial solution* (11).

Definition. Let a matrix function $\boldsymbol{G}(t,s)$ of two variables t,s be defined for all $t,s\in[a,b]$, $t,s\neq\tau$, $t\neq s$ as

$$\boldsymbol{G}(t,s)=\begin{cases}\boldsymbol{G}_1(t,s)\,, & t\in[a,s)\,,\\ \boldsymbol{G}_2(t,s)\,, & t\in(s,b]\,,\end{cases}\tag{15}$$

where

$$\boldsymbol{G}_1(t,s)=\begin{cases}\boldsymbol{G}_1^{(1)}(t,s)\,, & t\in[a,\tau)\,,\ s\in(t,\tau)\,,\\ \boldsymbol{G}_1^{(2)}(t,s)\,, & t\in[a,\tau)\,,\ s\in(\tau,t)\,,\\ \boldsymbol{G}_1^{(3)}(t,s)\,, & t\in(\tau,b]\,,\ s\in(t,b]\,,\end{cases}\tag{16}$$

$$\boldsymbol{G}_2(t,s)=\begin{cases}\boldsymbol{G}_2^{(1)}(t,s)\,, & t\in[a,\tau)\,,\ s\in[a,t)\,,\\ \boldsymbol{G}_2^{(2)}(t,s)\,, & t\in(\tau,b]\,,\ s\in[a,\tau)\,,\\ \boldsymbol{G}_2^{(3)}(t,s)\,, & t\in(\tau,b]\,,\ s\in(\tau,b]\,.\end{cases}\tag{17}$$

$\boldsymbol{G}(t,s)$ is the Green function of the homogeneous boundary value problem (6)–(8) if it satisfies the following conditions:

1. for all fixed $s\in[a,b]$, $s\neq\tau$ $\boldsymbol{G}(t,s)$ as a function of the variable t satisfies the homogeneous differential equation (6) for all $t\in[a,b]$, $t\neq s\neq\tau$

$$\frac{\boldsymbol{d}\boldsymbol{G}(t,s)}{dt}=\boldsymbol{A}(t)\boldsymbol{G}(t,s)\,,\qquad s\in[a,b]\,,\ t\neq s\neq\tau\,,\tag{18}$$

i.e.

$$\begin{aligned}\frac{\boldsymbol{d}\boldsymbol{G}(t,s)}{dt}&=\boldsymbol{A}_1(t)\boldsymbol{G}(t,s)\,, && s\in[a,b]\,,\\ & && s\neq\tau\,;\ t\in[a,\tau)\,,\ t\neq s\,,\\ \frac{\boldsymbol{d}\boldsymbol{G}(t,s)}{dt}&=\boldsymbol{A}_2(t)\boldsymbol{G}(t,s)\,, && s\in[a,b]\,,\\ & && s\neq\tau\,;\ t\in(\tau,b]\,,\ t\neq s\,;\end{aligned}\tag{19}$$

2. for $t=a$, $t=b$ and for all fixed $s\in[a,b]$, $s\neq\tau$ $\boldsymbol{G}(t,s)$ satisfies the boundary condition (8)

$$\boldsymbol{B}_1\boldsymbol{G}(a,s)+\boldsymbol{B}_2\boldsymbol{G}(b,s)=\boldsymbol{0}\,,\ s\in[a,b]\,,\quad s\neq\tau\,;\tag{20}$$

3. for $t=s\neq\tau$ $\boldsymbol{G}(t,s)$ has discontinuity of the first kind

$$\text{(21)} \qquad \boldsymbol{G}(s+0,s) - \boldsymbol{G}(s-0,s) = \boldsymbol{E}\,,\ s \in (a,b)\,, \quad s \neq \tau\,;$$

4. for $t = \tau$ and for all fixed $s \in [a,b]\,,\ s \neq \tau\ \ \boldsymbol{G}(t,s)$ satisfies the impulse effect condition (7)

$$\text{(22)} \qquad \boldsymbol{S}_1\boldsymbol{G}(\tau+0,s) + \boldsymbol{S}_2\boldsymbol{G}(\tau-0,s) = \boldsymbol{0}\,, \quad s \in [a,b]\,,\ s \neq \tau\,.$$

It comes from (16) and (17) that the Green function in (15) constructed to (6)–(8) is a piecewise continuous function represented by six functions, as it is shown in the *Figure* below.

Lemma 2. *Let the homogeneous two-point boundary value problem with impulse effect* (6)–(8) *have only zero solution* (11). *Then the Green function* $\boldsymbol{G}(t,s)$ *to* (6)–(8), *given by* (15)–(17) *and fulfilling the conditions* (18)–(22), *is uniquely defined.*

PROOF. The function having the form (15)–(17) and satisfying the conditions (18)–(22) can be directly constructed. Since $\boldsymbol{G}(t,s)$ in t satisfies (18) , (19) for all fixed $s \in [a,b]$, $t \neq s \neq \tau$ in each subdomain of $[a,b]\times[a,b]$, the functions in (16) , (17) necessarily have the following representation

$$
\begin{aligned}
&\boldsymbol{G}_1^{(1)}(t,s) = \boldsymbol{\Phi}_1(t)\boldsymbol{C}_1^{(1)}(s)\,,\ t \in [a,\tau)\,, \quad s \in (t,\tau)\,,\\
&\boldsymbol{G}_1^{(2)}(t,s) = \boldsymbol{\Phi}_1(t)\boldsymbol{C}_1^{(2)}(s)\,,\ t \in [a,\tau)\,, \quad s \in (\tau,b]\,,\\
&\boldsymbol{G}_1^{(3)}(t,s) = \boldsymbol{\Phi}_2(t)\boldsymbol{C}_1^{(3)}(s)\,,\ t \in (\tau,b]\,, \quad s \in (t,b]\,,\\
&\boldsymbol{G}_2^{(1)}(t,s) = \boldsymbol{\Phi}_1(t)\boldsymbol{C}_2^{(1)}(s)\,,\ t \in [a,\tau)\,, \quad s \in [a,t)\,,\\
&\boldsymbol{G}_2^{(2)}(t,s) = \boldsymbol{\Phi}_2(t)\boldsymbol{C}_2^{(2)}(s)\,,\ t \in (\tau,b]\,, \quad s \in [a,\tau)\,,\\
&\boldsymbol{G}_2^{(3)}(t,s) = \boldsymbol{\Phi}_2(t)\boldsymbol{C}_2^{(3)}(s)\,,\ t \in (\tau,b]\,, \quad s \in (\tau,b]\,,
\end{aligned}
\tag{23}
$$

where $\boldsymbol{C}_i^{(j)}(s)$ ($i=1,2$, $j=1,2,3$) are still unknown matrices depending on s. To find the matrices $\boldsymbol{C}_i^{(j)}$ substitute (23) into (20)–(22)

$$
\begin{aligned}
\boldsymbol{B}_1\boldsymbol{\Phi}_1(a)\boldsymbol{C}_1^{(1)}(s) + \boldsymbol{B}_2\boldsymbol{\Phi}_2(b)\boldsymbol{C}_2^{(2)}(s) &= \boldsymbol{0}\,,\\
\boldsymbol{B}_1\boldsymbol{\Phi}_1(a)\boldsymbol{C}_1^{(2)}(s) + \boldsymbol{B}_2\boldsymbol{\Phi}_2(b)\boldsymbol{C}_2^{(3)}(s) &= \boldsymbol{0}\,,\\
\boldsymbol{\Phi}_1(s)\boldsymbol{C}_2^{(1)}(s) - \boldsymbol{\Phi}_1(s)\boldsymbol{C}_1^{(1)}\ (s) &= \boldsymbol{E}\,,\\
\boldsymbol{\Phi}_2(s)\boldsymbol{C}_2^{(3)}(s) - \boldsymbol{\Phi}_2(s)\boldsymbol{C}_1^{(3)}(s) &= \boldsymbol{E}\,,\\
\boldsymbol{S}_1\boldsymbol{\Phi}_2(\tau)\boldsymbol{C}_2^{(2)}(s) + \boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\boldsymbol{C}_2^{(1)}(s) &= \boldsymbol{0}\,,\\
\boldsymbol{S}_1\boldsymbol{\Phi}_2(\tau)\boldsymbol{C}_1^{(3)}(s) + \boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\boldsymbol{C}_1^{(2)}(s) &= \boldsymbol{0}\,.
\end{aligned}
\tag{24}
$$

From (24) and the condition (12) of Lemma 1 one obtains

$$
\begin{aligned}
&\boldsymbol{C}_1^{(1)}(s) = (\boldsymbol{D}^{-1}\boldsymbol{B}_1 - \boldsymbol{E})\boldsymbol{\Phi}_1^{-1}(s)\,,\\
&\boldsymbol{C}_1^{(2)}(s) = -\boldsymbol{D}^{-1}\boldsymbol{B}_2\boldsymbol{\Phi}_2(b)\boldsymbol{\Phi}_2^{-1}(s)\,,\\
&\boldsymbol{C}_1^{(3)}(s) = \boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\boldsymbol{D}^{-1}\boldsymbol{B}_2\boldsymbol{\Phi}_2(b)\boldsymbol{\Phi}_2^{-1}(s)\,,\\
&\boldsymbol{C}_2^{(1)}(s) = \boldsymbol{D}^{-1}\boldsymbol{B}_1\boldsymbol{\Phi}_1^{-1}(s)\,,\\
&\boldsymbol{C}_2^{(2)}(s) = -\boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\boldsymbol{D}^{-1}\boldsymbol{B}_1\boldsymbol{\Phi}_1^{-1}(s)\,,\\
&\boldsymbol{C}_2^{(3)}(s) = [\boldsymbol{E} + \boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\boldsymbol{D}^{-1}\boldsymbol{B}_2\boldsymbol{\Phi}_2(b)]\boldsymbol{\Phi}_2^{-1}(s)\,.
\end{aligned}
\tag{25}
$$

Substituting (25) into (23), the Green function of the boundary value problem with impulse effect, described in (6)–(8), is expressed in terms of the fundamental matrices of the system (9), (10) without impulse effect. This completes the proof.

Now, instead of (6), consider an inhomogeneous system of differential equations with impulse effect and piecewise continuous coefficients

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{g}(t)\,, \quad t \neq \tau\,, \tag{26}$$

where $\boldsymbol{g}(t) \in C_\tau[a,b]$, i.e.

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}_1(t)\boldsymbol{x} + \boldsymbol{g}_1(t)\,, \quad t \in [a,\tau)\,, \tag{27}$$

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}_2(t)\boldsymbol{x} + \boldsymbol{g}_2(t)\,, \quad t \in (\tau,b]\,. \tag{28}$$

We shall study the above equations completed with either homogeneous conditions (7), (8) or the following inhomogeneous impulse effect and two-point boundary condition

$$\boldsymbol{S}_1\boldsymbol{x}(\tau+0) + \boldsymbol{S}_2\boldsymbol{x}(\tau-0) = \boldsymbol{q}\,, \quad \boldsymbol{q} \in \mathbb{R}^n \tag{29}$$

$$\boldsymbol{B}_1\boldsymbol{x}(a) + \boldsymbol{B}_2\boldsymbol{x}(b) = \boldsymbol{d}\,, \quad \boldsymbol{d} \in \mathbb{R}^n\,. \tag{30}$$

Lemma 3. *Let the boundary value problem with impulse effect* (6)–(8) *have only the* (11) *zero solution. Then for any piecewise continuous* $\boldsymbol{g}(t) \in C_\tau[a,b]$

1. *There exists a unique solution* $\boldsymbol{x}(t)$ *of the semi-homogeneous boundary value problem with one impulse effect* (26), (7), (8) *in* $C_\tau[a,b]$, *that can be given as*

$$\boldsymbol{x}(t) = \int_a^b \boldsymbol{G}(t,s)\boldsymbol{g}(s)ds\,, \tag{31}$$

in detail

$$\boldsymbol{x}_1(t) = \boldsymbol{\Phi}_1(t)\Big[\int_a^t \boldsymbol{C}_2^{(1)}(s)\boldsymbol{g}_1(s)ds + \int_t^\tau \boldsymbol{C}_1^{(1)}(s)\boldsymbol{g}_1(s)ds + \\ + \int_\tau^b \boldsymbol{C}_1^{(2)}(s)\boldsymbol{g}_2(s)ds\Big]\,, \quad t \in [a,\tau)\,, \tag{32}$$

$$(33)\quad \boldsymbol{x}_2(t)=\boldsymbol{\Phi}_2(t)\Big[\int_a^\tau \boldsymbol{C}_2^{(2)}(s)\boldsymbol{g}_1(s)ds+\int_\tau^t \boldsymbol{C}_2^{(3)}(s)\boldsymbol{g}_2(s)ds+ \\ +\int_t^b \boldsymbol{C}_1^{(3)}(s)\boldsymbol{g}_2(s)ds\Big]\,,\quad t\in(\tau,b]\,,$$

where $\boldsymbol{G}(t,s)$ *is the Green function given in* (15)–(17), (23), (25) *and satisfying the* (6)–(8) *homogeneous boundary value problem.*

2. *There exists a unique solution* $\boldsymbol{x}(t)$ *of the* (26), (29), (30) *inhomogeneous boundary value problem with one impulse effect in* $C_\tau[a,b]$

$$(34)\qquad \boldsymbol{x}(t)=\boldsymbol{H}(t)+\int_a^b \boldsymbol{G}(t,s)\boldsymbol{g}(s)ds\,,$$

where $\boldsymbol{H}(t)\in C_\tau[a,b]$ *and*

$$(35)\qquad \boldsymbol{H}(t)=\begin{cases}\boldsymbol{H}_1(t)=\boldsymbol{\Phi}_1(t)\boldsymbol{D}^{-1}[\boldsymbol{d}-\boldsymbol{B}_2\boldsymbol{\Phi}_2(b)\boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{S}_1^{-1}\boldsymbol{g}]\,,\\ \qquad t\in[a,\tau]\,,\\ \boldsymbol{H}_2(t)=\boldsymbol{\Phi}_2(t)\boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{S}_1^{-1}[(\boldsymbol{E}+\\ \qquad +\boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\boldsymbol{D}^{-1}\boldsymbol{B}_2\boldsymbol{\Phi}_2(b)\boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{S}_1^{-1})\boldsymbol{g}-\\ \qquad -\boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\boldsymbol{D}^{-1}\boldsymbol{d}]\,,\qquad t\in(\tau,b]\,.\end{cases}$$

Proof. The functions (32), (33) satisfy the equations (27), (28), respectively, and conditions (7), (8). Since the integrand in (31) is continuously differentiable in the intervals $[a,t]\,,[t,\tau]\,,[\tau,b]$, we get

$$\frac{\boldsymbol{dx}_1(t)}{dt}=\int_a^t\frac{\partial\boldsymbol{G}(t,s)}{\partial t}\boldsymbol{g}_1(s)\,ds+\boldsymbol{G}(t,s)|_{s=t-0}\,\boldsymbol{g}_1(t)+ \\ +\int_t^\tau\frac{\partial\boldsymbol{G}(t,s)}{\partial t}\boldsymbol{g}_1(s)\,ds-\boldsymbol{G}(t,s)|_{s=t+0}\,\boldsymbol{g}_1(t)+\int_\tau^b\frac{\partial\boldsymbol{G}(t,s)}{\partial t}\boldsymbol{g}_1(s)\,ds= \\ =\int_a^b\frac{\partial\boldsymbol{G}(t,s)}{\partial t}\boldsymbol{g}(s)\,ds+\boldsymbol{g}_1(t)\,.$$

Thus for each $t \in [a,\tau)$

$$\frac{d\boldsymbol{x}_1(t)}{dt} - \boldsymbol{A}_1(t)\boldsymbol{x}_1 =$$

$$= \int_a^b \frac{\partial \boldsymbol{G}(t,s)}{\partial t}\boldsymbol{g}(s)\,ds - \boldsymbol{A}_1(t)\int_a^b \boldsymbol{G}(t,s)\boldsymbol{g}(s)\,ds + \boldsymbol{g}_1(t) =$$

$$= \int_a^b \left[\frac{\partial \boldsymbol{G}(t,s)}{\partial t} - \boldsymbol{A}_1(t)\boldsymbol{G}(t,s)\right]\boldsymbol{g}(s)\,ds + \boldsymbol{g}_1(t) = \boldsymbol{g}_1(t)\,.$$

In a similar way one can show that the function in (33) satisfies equation (28) for all $t \in (\tau, b]$. The homogeneous conditions (7), (8) are also fulfilled by the function (31)

$$\boldsymbol{S}_1\int_a^b \boldsymbol{G}(\tau+0,s)\boldsymbol{g}(s)\,ds + \boldsymbol{S}_2\int_a^b \boldsymbol{G}(\tau-0,s)\boldsymbol{g}(s)ds =$$

$$= \int_a^b [\boldsymbol{S}_1\boldsymbol{G}(\tau+0,s) + \boldsymbol{S}_2\boldsymbol{G}(\tau-0,s)]\boldsymbol{g}(s)\,ds = \boldsymbol{0}\,,$$

$$\boldsymbol{B}_1\int_a^b \boldsymbol{G}(a,s)\boldsymbol{g}(s)\,ds + \boldsymbol{B}_2\int_a^b \boldsymbol{G}(b,s)\boldsymbol{g}(s)\,ds =$$

$$= \int_a^b [\boldsymbol{B}_1\boldsymbol{G}(a,s) + \boldsymbol{B}_2\boldsymbol{G}(b,s)]\boldsymbol{g}(s)\,ds = \boldsymbol{0}\,.$$

Since the second term on the right hand side of (34) satisfies the inhomogeneous equation (26) and the homogeneous conditions (7), (8): the functions $\boldsymbol{H}_1(t)$, $\boldsymbol{H}_2(t)$ in (34), as solutions of the homogeneous equations of (9), (10), have to satisfy the inhomogeneous conditions (29), (30). Choosing $\boldsymbol{H}_1(t)\,,\boldsymbol{H}_2(t)$ in the form

$$\boldsymbol{H}_1(t) = \boldsymbol{\Phi}_1(t)\boldsymbol{p}_1\,,\ \boldsymbol{H}_2(t) = \boldsymbol{\Phi}_2(t)\boldsymbol{p}_2\,,$$

the above requirements can be fulfilled by suitably chosen constant vectors $\boldsymbol{p}_1\,,\boldsymbol{p}_2$. Thus

$$\boldsymbol{S}_1\boldsymbol{\Phi}_2(\tau)\boldsymbol{p}_2 + \boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\boldsymbol{p}_1 = \boldsymbol{q}\,,$$

$$\boldsymbol{B}_1\boldsymbol{\Phi}_1(a)\boldsymbol{p}_1 + \boldsymbol{B}_2\boldsymbol{\Phi}_2(b)\boldsymbol{p}_2 = \boldsymbol{d}\,,$$

and we obtain

$$\begin{aligned} \boldsymbol{p}_1 &= \boldsymbol{D}^{-1}(\boldsymbol{d} - \boldsymbol{B}_2\boldsymbol{\Phi}_2(b)\boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{S}_1^{-1}\boldsymbol{q})\,, \\ \boldsymbol{p}_2 &= \boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{S}_1^{-1}[(\boldsymbol{E} + \boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\boldsymbol{D}^{-1}\boldsymbol{B}_2\boldsymbol{\Phi}_2(b)\boldsymbol{\Phi}_2^{-1}(\tau)\boldsymbol{S}_1^{-1})\boldsymbol{q} - \\ &\quad - \boldsymbol{S}_2\boldsymbol{\Phi}_1(\tau)\boldsymbol{D}^{-1}\boldsymbol{d}]\,. \end{aligned}$$

Consequently, $\boldsymbol{H}_1(t)\,,\boldsymbol{H}_2(t)$ have the form given in (35).

The uniqueness of the solutions (31) and (34) follows from the fact that the homogeneous problem (6)–(8) possesses only the trivial solution (see [7]). This completes the proof.

3. Convergence of the trigonometric collocation method applied to systems with impulse effect

In accordance with the trigonometric collocation method (see [1]) the approximate solution of the periodic boundary value problem with an impulse effect (1)–(3) in $C_\tau[0,T]$ is sought in the form of trigonometric polynomials of order m. Thus

$$\boldsymbol{x}_m(t) = \begin{cases} \boldsymbol{x}_{1m}(t) = \boldsymbol{a}_0^{(1)} + \displaystyle\sum_{k=1}^{m}(\boldsymbol{a}_k^{(1)}\cos k\omega t + \boldsymbol{b}_k^{(1)}\sin k\omega t)\,, \\ \qquad\qquad t \in [0,\tau)\,, \\ \boldsymbol{x}_{2m}(t) = \boldsymbol{a}_0^{(2)} + \displaystyle\sum_{k=1}^{m}(\boldsymbol{a}_k^{(2)}\cos k\omega t + \boldsymbol{b}_k^{(2)}\sin k\omega t)\,, \\ \qquad\qquad t \in (\tau,T]\,, \end{cases} \tag{36}$$

where $\omega = \dfrac{2\pi}{T}$ and $\boldsymbol{a}_k^{(i)} = (a_{1k}^{(i)},\dots,a_{nk}^{(i)})$ $(i = 1,2\,,\ k = 0,1,\dots,m\,)$, $\boldsymbol{b}_k^{(i)} = (b_{1k}^{(i)},\dots,b_{nk}^{(i)})$ $(i = 1,2\,,\ k = 1,\dots,m\,)$ are the unknown $(4m+2)$ vector coefficients.

It is possible to reduce the number of unknown coefficients by substituting (36) into the conditions (3) and (2). Consequently, $\boldsymbol{a}_1^{(1)}$, $\boldsymbol{a}_0^{(1)}$ can be expressed as:

$$
\begin{aligned}
\boldsymbol{a}_1^{(1)} &= \tilde{\boldsymbol{a}}_1^{(1)} = (1-\cos\omega\tau)^{-1}[(\boldsymbol{E}+\boldsymbol{S}_2^{-1}\boldsymbol{S}_1)\boldsymbol{a}_0^{(2)}+\\
&\quad+\sum_{k=1}^{m}(\boldsymbol{E}+\boldsymbol{S}_2^{-1}\boldsymbol{S}_1\cos k\omega\tau)\boldsymbol{a}_k^{(2)}-\boldsymbol{S}_2^{-1}\boldsymbol{q}+\\
&\quad+\sum_{k=2}^{m}(-1+\cos k\omega\tau)\boldsymbol{a}_k^{(1)}+\sum_{k=1}^{m}[\boldsymbol{S}_2^{-1}\boldsymbol{S}_1\boldsymbol{b}_k^{(2)}+\boldsymbol{b}_k^{(1)}]\sin k\omega\tau]\,,\\
\boldsymbol{a}_0^{(1)} &= \tilde{\boldsymbol{a}}_0^{(1)} = [\boldsymbol{E}-(1-\cos\omega\tau)^{-1}(\boldsymbol{E}+\boldsymbol{S}_2^{-1}\boldsymbol{S}_1)]\boldsymbol{a}_0^{(2)}+\\
&\quad+\sum_{k=1}^{m}[\boldsymbol{E}-(1-\cos\omega\tau)^{-1}(\boldsymbol{E}+\boldsymbol{S}_2^{-1}\boldsymbol{S}_1\cos k\omega\tau)]\boldsymbol{a}_k^{(2)}-\\
&\quad-\sum_{k=2}^{m}[1+(1-\cos\omega\tau)^{-1}(-1+\cos k\omega\tau)]\boldsymbol{a}_k^{(1)}+\\
&\quad+(1-\cos\omega\tau)^{-1}\boldsymbol{S}_2^{-1}\boldsymbol{q}-(1-\cos\omega\tau)^{-1}\sum_{k=1}^{m}[\boldsymbol{S}_2^{-1}\boldsymbol{S}_1\boldsymbol{b}_k^{(2)}+\\
&\quad+\boldsymbol{E}\boldsymbol{b}_k^{(1)}]\sin k\omega\tau\,.
\end{aligned}
\tag{36'}
$$

Then the approximate solution

$$
\boldsymbol{x}_m(t)=\begin{cases}
\boldsymbol{x}_{1m}(t)=\tilde{\boldsymbol{a}}_0^{(1)}+\tilde{\boldsymbol{a}}_1^{(1)}\cos\omega t+\sum_{k=2}^{m}\boldsymbol{a}_k^{(1)}\cos k\omega t+\\
\qquad+\sum_{k=1}^{m}\boldsymbol{b}_k^{(1)}\sin k\omega t\,,\qquad t\in[0,\tau)\,,\\
\boldsymbol{x}_{2m}(t)=\boldsymbol{a}_0^{(2)}+\sum_{k=1}^{m}(\boldsymbol{a}_k^{(2)}\cos k\omega t+\boldsymbol{b}_k^{(2)}\sin k\omega t)\,,\\
\qquad\qquad t\in(\tau,T]\,,
\end{cases}
\tag{37}
$$

satisfies (3) and (2) and there remain $4m$ unknown vector coefficients (m is a finite number)

$$
\begin{gathered}
\boldsymbol{a}_2^{(1)},\dots,\boldsymbol{a}_m^{(1)}\,;\quad \boldsymbol{b}_1^{(1)},\dots,\boldsymbol{b}_m^{(1)}\,;\\
\boldsymbol{a}_0^{(2)},\,\boldsymbol{a}_1^{(2)},\dots,\boldsymbol{a}_m^{(2)}\,;\quad \boldsymbol{b}_1^{(2)},\dots,\boldsymbol{b}_m^{(2)}\,.
\end{gathered}
\tag{38}
$$

The unknown coefficients in (38) can be defined by the trigonometric collocation method if one requires the approximate solution (37) to fulfill (1) in $4m$ equidistant points in the intervals $[0,\tau)\,,\,(\tau,T]\,$. Thus

$$
\left\{
\begin{aligned}
\frac{d\boldsymbol{x}_{1m}(t_i)}{dt} = &\quad \boldsymbol{f}_1(t_i,\boldsymbol{x}_{1m}(t_i))\,,\\
\frac{d\boldsymbol{x}_{2m}(\tilde{t}_j)}{dt} = &\quad \boldsymbol{f}_2(\tilde{t}_j,\boldsymbol{x}_{2m}(\tilde{t}_j))\,,
\end{aligned}
\right.
\tag{39}
$$

where

$$
\begin{aligned}
t_i =&(2i-1)\frac{\tau}{4m}\,, \quad i=1,2,\dots,2m\,,\\
\tilde{t}_j =&\tau+(2j-1)\frac{T-\tau}{4m}\,, \quad j=1,2,\dots,2m\,.
\end{aligned}
\tag{40}
$$

Let us consider a homogeneous periodic boundary value problem with one impulse effect

$$
\begin{aligned}
&\frac{d\boldsymbol{x}}{dt}+\boldsymbol{Q}\boldsymbol{x}=\boldsymbol{0}\,, \quad t\neq\tau\,,\, t\in[0,T]\,,\\
&\boldsymbol{S}_1\boldsymbol{x}(\tau+0)+\boldsymbol{S}_2\boldsymbol{x}(\tau-0)=\boldsymbol{0}\,,\\
&\boldsymbol{x}(0)-\boldsymbol{x}(T)=\boldsymbol{0}\,,
\end{aligned}
\tag{41}
$$

where $\boldsymbol{x}(t)$ is a function of the form (5) in $C_\tau[0,T]$ and $\boldsymbol{Q}$ is a certain constant $n\times n$ matrix chosen in such a manner that (41) has only the trivial solution (11). In this case the matrix $\boldsymbol{D}$ in (12) takes the form

$$
\boldsymbol{D}=\boldsymbol{E}-\mathrm{e}^{(\tau-T)\boldsymbol{Q}}\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\mathrm{e}^{-\tau\boldsymbol{Q}}\,,
\tag{42}
$$

and, because of the choice of $\boldsymbol{Q}$, $\det\boldsymbol{D}\neq 0$ is fulfilled, too. Thus there exists a uniquely determined Green function $\boldsymbol{G}(t,s)$ to the problem (41). Putting into use (15)–(17), (23), (25), (42) we obtain $\boldsymbol{\Phi}_1(t)=\boldsymbol{\Phi}_2(t)=\mathrm{e}^{-t\boldsymbol{Q}}$, $\boldsymbol{B}_1=\boldsymbol{E}$, $\boldsymbol{B}_2=-\boldsymbol{E}$ and

$$
\boldsymbol{G}_1(t,s)=\begin{cases}
\mathrm{e}^{-t\boldsymbol{Q}}(\boldsymbol{D}^{-1}-\boldsymbol{E})\mathrm{e}^{s\boldsymbol{Q}}\,, & t\in[0,\tau)\,,\, s\in(t,\tau)\,,\\
\mathrm{e}^{-t\boldsymbol{Q}}\boldsymbol{D}^{-1}\mathrm{e}^{(s-T)\boldsymbol{Q}}\,, & t\in[0,\tau)\,,\, s\in(\tau,T]\,,\\
-\mathrm{e}^{-t\boldsymbol{Q}}\mathrm{e}^{\tau\boldsymbol{Q}}\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\mathrm{e}^{-\tau\boldsymbol{Q}}\boldsymbol{D}^{-1}\mathrm{e}^{(s-T)\boldsymbol{Q}}\,, & t\in(\tau,T]\,,\, s\in(t,T]\,,
\end{cases}
$$

(43)

$$
\boldsymbol{G}_2(t,s)=\begin{cases}
\mathrm{e}^{-t\boldsymbol{Q}}\boldsymbol{D}^{-1}\mathrm{e}^{s\boldsymbol{Q}}\,, \quad t\in[0,\tau)\,,\, s\in[0,t)\,,\\
-\,\mathrm{e}^{(\tau-t)\boldsymbol{Q}}\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\mathrm{e}^{-\tau\boldsymbol{Q}}\boldsymbol{D}^{-1}\mathrm{e}^{s\boldsymbol{Q}}\,,\\
\qquad t\in(\tau,T]\,,\ s\in[0,\tau)\,,\\
\mathrm{e}^{-t\boldsymbol{Q}}[\boldsymbol{E}-\mathrm{e}^{\tau\boldsymbol{Q}}\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\mathrm{e}^{-\tau\boldsymbol{Q}}\boldsymbol{D}^{-1}\mathrm{e}^{-T\boldsymbol{Q}}]\mathrm{e}^{s\boldsymbol{Q}}\,,\\
\qquad t\in(\tau,T]\,,\ s\in(\tau,T]\,.
\end{cases}
$$

It follows from (32), (34) that for any piecewise continuous function $\boldsymbol{v}(t) \in C_\tau[0,T]$

$$\boldsymbol{v}(t) = \begin{cases} \boldsymbol{v}_1(t)\,, & t \in [0,\tau)\,, \\ \boldsymbol{v}_2(t)\,, & t \in (\tau,T]\,, \end{cases} \tag{44}$$

the solution of the inhomogeneous boundary value problem with one impulse effect

$$\begin{gathered} \frac{d\boldsymbol{x}}{dt} + \boldsymbol{Q}\boldsymbol{x} = \boldsymbol{v}(t)\,, \quad t \neq \tau\,, \\ \boldsymbol{S}_1\boldsymbol{x}(\tau+0) + \boldsymbol{S}_2\boldsymbol{x}(\tau-0) = \boldsymbol{q}\,, \\ \boldsymbol{x}(0) - \boldsymbol{x}(T) = \boldsymbol{0}\,, \end{gathered} \tag{45}$$

in the intervals $t \in [0,\tau)\,,\ t \in (\tau,T]$ takes the form

$$\begin{aligned} \boldsymbol{x}_1(t) &= \int_a^b \tilde{\boldsymbol{G}}_1(t,s)\boldsymbol{v}(s)\,ds + \boldsymbol{H}_1(t)\,, \quad t \in [0,\tau)\,, \\ &\boldsymbol{H}_1(t) = \mathrm{e}^{-t\boldsymbol{Q}}\boldsymbol{D}^{-1}\mathrm{e}^{(\tau-T)\boldsymbol{Q}}\boldsymbol{S}_1^{-1}\boldsymbol{q}\,, \\ \boldsymbol{x}_2(t) &= \int_a^b \tilde{\boldsymbol{G}}_2(t,s)\boldsymbol{v}(s)\,ds + \boldsymbol{H}_2(t)\,, \quad t \in (\tau,T]\,, \\ &\boldsymbol{H}_2(t) = \mathrm{e}^{(\tau-t)\boldsymbol{Q}}\boldsymbol{S}_1^{-1}(\boldsymbol{E} - \boldsymbol{S}_2\mathrm{e}^{-\tau\boldsymbol{Q}}\boldsymbol{D}^{-1}\mathrm{e}^{(\tau-T)\boldsymbol{Q}}\boldsymbol{S}_1^{-1})\boldsymbol{q}\,. \end{aligned} \tag{46}$$

Here $\tilde{\boldsymbol{G}}_1(t,s)\,,\ \tilde{\boldsymbol{G}}_2(t,s)$ mean the following rearranged form of $\boldsymbol{G}_1(t,s)$, $\boldsymbol{G}_2(t,s)$:

$$\begin{gathered} \tilde{\boldsymbol{G}}_1(t,s) = \begin{cases} \mathrm{e}^{-t\boldsymbol{Q}}\boldsymbol{D}^{-1}\mathrm{e}^{s\boldsymbol{Q}}\,, & t \in [0,\tau)\,,\ s \in [0,t)\,, \\ \mathrm{e}^{-t\boldsymbol{Q}}(\boldsymbol{D}^{-1} - \boldsymbol{E})\mathrm{e}^{s\boldsymbol{Q}}\,, & t \in [0,\tau)\,,\ s \in (t,\tau)\,, \\ \mathrm{e}^{-t\boldsymbol{Q}}\boldsymbol{D}^{-1}\mathrm{e}^{(s-T)\boldsymbol{Q}}\,, & t \in [0,\tau)\,,\ s \in (\tau,T]\,, \end{cases} \\ \tilde{\boldsymbol{G}}_2(t,s) = \begin{cases} -\,\mathrm{e}^{(\tau-t)\boldsymbol{Q}}\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\mathrm{e}^{-\tau\boldsymbol{Q}}\boldsymbol{D}^{-1}\mathrm{e}^{s\boldsymbol{Q}}\,, \\ \qquad t \in (\tau,T]\,,\ s \in [0,\tau)\,, \\ \mathrm{e}^{-t\boldsymbol{Q}}[\boldsymbol{E} - \mathrm{e}^{\tau\boldsymbol{Q}}\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\mathrm{e}^{-\tau\boldsymbol{Q}}\boldsymbol{D}^{-1}\mathrm{e}^{-T\boldsymbol{Q}}]\mathrm{e}^{s\boldsymbol{Q}}\,, \\ \qquad t \in (\tau,T]\,,\ s \in (\tau,t)\,, \\ -\,\mathrm{e}^{-t\boldsymbol{Q}}\mathrm{e}^{\tau\boldsymbol{Q}}\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\mathrm{e}^{-\tau\boldsymbol{Q}}\boldsymbol{D}^{-1}\mathrm{e}^{(s-T)\boldsymbol{Q}}\,, \\ \qquad t \in (\tau,T]\,,\ s \in (t,T]\,. \end{cases} \end{gathered} \tag{47}$$

Theorem. *Suppose that*

$$x = x^0(t) = \begin{cases} x_1^0(t)\,, & t \in [0,\tau)\,, \\ x_2^0(t)\,, & t \in (\tau, T] \end{cases} \tag{48}$$

is a piecewise continuously differentiable solution of the T–periodic boundary value problem with one impulse effect, described in (1)–(3)*. The functions $f_i(t,x)$ in* (4) *and the Jacobian matrices $A_i(t,x)$ are defined and continuous in*

$$t \in [0,\tau]\,,\; t \in [\tau, T]\,, \quad |x - x^0(t)|_{C_\tau} \le \delta\,, \quad \delta > |H|_{C_\tau}\,. \tag{49}$$

Let the following impulsive variational system of the equation (1) *with respect to $x^0(t)$*

$$\frac{dx}{dt} = F(t, x^0(t))x\,, \quad t \neq \tau\,, \tag{50}$$

where

$$F(t, x^0(t)) = \begin{cases} A_1(t, x^0(t))\,, & t \in [0,\tau)\,, \\ A_2(t, x^0(t))\,, & t \in (\tau, T]\,, \end{cases}$$

possess only the trivial $T-$periodic solution (11) *under condition* (7)*.*

Then:

1. *There exists $\alpha > 0$ such that in the ball*

$$|\dot{x} + Qx - (\dot{x}^0 + Qx^0)|_{L_\tau^2} \le \alpha\,, \tag{51}$$

   the solution $x^0(t)$ of the periodic boundary value problem with one impulse effect is unique;

2. *With sufficiently large m ($m \ge m_0$) the system of equations* (39)*, derived from the trigonometric collocation method, has a solution. This solution gives the coefficients* (38) *determining the unique approximate T–periodic solution $x_m(t)$ in* (37) *valid in the domain* (51)*.*

3. *The sequence of approximations $x_m(t)$ for all $t \neq \tau$ converges uniformly and $\dfrac{dx_m(t)}{dt}$ also converges in the metric of $L_\tau^2[0,T]$ as $m \to \infty$ to $x^0(t)$ and $\dfrac{dx^0(t)}{dt}$, respectively, with*

$$|x_m(t) - x^0(t)|_{C_\tau} \le c_1 E_m(v^0)\,, \tag{52}$$

$$\left| \frac{dx_m(t)}{dt} - \frac{dx^0(t)}{dt} \right|_{L_\tau^2} \le c_2 E_m(v^0)\,, \tag{53}$$

where c_1, c_2 are m–independent constants; $E_m(\boldsymbol{v}^0) = \max(E_m(\boldsymbol{v}_1^0), E_m(\boldsymbol{v}_2^0))$, and $E_m(\boldsymbol{v}_i^0)$ $(i = 1, 2)$ are the best uniform approximations of the periodic continuation of $\boldsymbol{v}_i^0 = \dfrac{\boldsymbol{dx}_i^0(t)}{dt} + \boldsymbol{Q}\boldsymbol{x}_i^0(t)$, $(i = 1, 2)$, among trigonometric polynomials of order not higher than m.

Proof. On the basis of the matrices in (47) we introduce the operators $\tilde{\mathcal{G}}_1$, $\tilde{\mathcal{G}}_2$ acting on the function $\boldsymbol{v}(t)$ of the form (44)

$$\begin{aligned} &\tilde{\mathcal{G}}_1 : \quad \tilde{\mathcal{G}}_1\boldsymbol{v}(t) = \int_0^T \tilde{\boldsymbol{G}}_1(t,s)\boldsymbol{v}(s)\,ds\,, \ t \in [0,\tau)\,, \\ &\tilde{\mathcal{G}}_2 : \quad \tilde{\mathcal{G}}_2\boldsymbol{v}(t) = \int_0^T \tilde{\boldsymbol{G}}_2(t,s)\boldsymbol{v}(s)\,ds\,, \ t \in (\tau,T]\,. \end{aligned} \tag{54}$$

It follows easily from the properties of the matrices $\tilde{\boldsymbol{G}}_1(t,s)$, $\tilde{\boldsymbol{G}}_2(t,s)$ that $\tilde{\mathcal{G}}_1$, $\tilde{\mathcal{G}}_2$ are completely continuous as operators from $L^2_\tau[0,T]$ into $C_\tau[0,T]$. Since $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$ are bounded operators we can choose in the space $L^2_\tau[0,T]$ a ball with centre at

$$\boldsymbol{v}^0(t) = \frac{\boldsymbol{dx}^0}{dt} + \boldsymbol{Q}\boldsymbol{x}^0(t)$$

with radius

$$\| \boldsymbol{v} - \boldsymbol{v}^0 \|_{L^2_\tau} \le \delta_1\,, \tag{55}$$

such that the functions $\boldsymbol{x}_1(t)$, $\boldsymbol{x}_2(t)$ in (46) satisfy (49). Now define an operator $\mathcal{K}$ on the ball (55) as

$$\mathcal{K}\boldsymbol{v}(t) = \begin{cases} \mathcal{K}_1\boldsymbol{v}(t) = \boldsymbol{f}_1(t, \boldsymbol{H}_1(t) + \tilde{\mathcal{G}}_1\boldsymbol{v}(t)) + \boldsymbol{Q}[\tilde{\mathcal{G}}_1\boldsymbol{v}(t) + \boldsymbol{H}_1(t)]\,, \\ \qquad t \in [0,\tau)\,, \\ \mathcal{K}_2\boldsymbol{v}(t) = \boldsymbol{f}_2(t, \boldsymbol{H}_2(t) + \tilde{\mathcal{G}}_2\boldsymbol{v}(t)) + \boldsymbol{Q}[\tilde{\mathcal{G}}_2\boldsymbol{v}(t) + \boldsymbol{H}_2(t)]\,, \\ \qquad t \in (\tau,T]\,, \end{cases} \tag{56}$$

where $\boldsymbol{H}_1(t)$, $\boldsymbol{H}_2(t)$ are the functions defined in (46). Since $\boldsymbol{f}_1(t,\boldsymbol{x})$, $\boldsymbol{f}_2(t,\boldsymbol{x})$ are continuous and $\tilde{\mathcal{G}}_1$, $\tilde{\mathcal{G}}_2$ completely continuous, the operators $\mathcal{K}_1$, $\mathcal{K}_2$ are completely continuous in the ball (55).

Denote by $\mathcal{P}_1$, $\mathcal{P}_2$ the linear continuous operators embedding the spaces $C[0,\tau]$, $C[\tau,T]$ into the spaces $L^2[0,\tau]$, $L^2[\tau,T]$, respectively, and by $\mathcal{P}_{1m}$, $\mathcal{P}_{2m}$ the linear operators juxtaposing each continuous, periodic continued function with its trigonometric interpolation polynomial of order m.

Make the exchange of variables in (1)–(3) and in (39) as follows

$$\frac{d\boldsymbol{x}}{dt} + \boldsymbol{Q}\boldsymbol{x} = \boldsymbol{v}(t)\,,\ t \neq \tau\,,$$
$$\frac{d\boldsymbol{x}_m}{dt} + \boldsymbol{Q}\boldsymbol{x}_m = \boldsymbol{v}_m\,,\ t \neq \tau\,,$$

where

$$\boldsymbol{v}_m(t) = \begin{cases} \boldsymbol{v}_{1m}(t)\,, & t \in [0,\tau)\,, \\ \boldsymbol{v}_{2m}(t)\,, & t \in (\tau, T]\,. \end{cases}$$

Then, on the base of (45), (46), (54), (56), in $L^2_\tau[0,T]$ (1)–(3) is equivalent to the operator equation

$$\boldsymbol{v} = \mathcal{P}\mathcal{K}\boldsymbol{v}\,, \tag{57}$$

and the system of determining equations (39) is reduced to

$$\mathcal{P}_m\boldsymbol{v}_m = \mathcal{P}_m\mathcal{K}\boldsymbol{v}_m\,, \tag{58}$$

where

$$\mathcal{P}\mathcal{K}\boldsymbol{v} = \begin{cases} \mathcal{P}_1\mathcal{K}_1\boldsymbol{v}\,, & t \in [0,\tau)\,, \\ \mathcal{P}_2\mathcal{K}_2\boldsymbol{v}\,, & t \in (\tau, T]\,, \end{cases}$$

$$\mathcal{P}_m\mathcal{K}\boldsymbol{v}_m = \begin{cases} \mathcal{P}_{1m}\mathcal{K}_1\boldsymbol{v}_m\,, & t \in [0,\tau)\,, \\ \mathcal{P}_{2m}\mathcal{K}_2\boldsymbol{v}_m\,, & t \in (\tau, T]\,. \end{cases}$$

Here the solution (48) of the boundary value problem (1)–(3) and the solution $\boldsymbol{v}^0(t)$ of the operator equation (57) are connected as follows:

$$\begin{aligned} \boldsymbol{v}^0_1(t) &= \frac{d\boldsymbol{x}^0_1(t)}{dt} + \boldsymbol{Q}\boldsymbol{x}^0_1(t)\,, \quad t \in [0,\tau)\,, \\ \boldsymbol{v}^0_2(t) &= \frac{d\boldsymbol{x}^0_2(t)}{dt} + \boldsymbol{Q}\boldsymbol{x}^0_2(t)\,, \quad t \in (\tau,T]\,, \\ \boldsymbol{x}^0_1(t) &= \tilde{\mathcal{G}}_1\boldsymbol{v}^0(t) + \boldsymbol{H}_1(t)\,, \quad t \in [0,\tau)\,, \\ \boldsymbol{x}^0_2(t) &= \tilde{\mathcal{G}}_2\boldsymbol{v}^0(t) + \boldsymbol{H}_2(t)\,, \quad t \in (\tau,T]\,. \end{aligned} \tag{59}$$

There exists an analogous connection between the solutions of the operator equation (58) and the approximate solution (37), (39):

$$
\begin{aligned}
\boldsymbol{v}_{1m}(t) &= \frac{d\boldsymbol{x}_{1m}(t)}{dt} + \boldsymbol{Q}\boldsymbol{x}_{1m}(t)\,, && t \in [0,\tau)\,, \\
\boldsymbol{v}_{2m}(t) &= \frac{d\boldsymbol{x}_{2m}(t)}{dt} + \boldsymbol{Q}\boldsymbol{x}_{2m}(t)\,, && t \in (\tau,T]\,, \\
\boldsymbol{x}_{1m}(t) &= \tilde{\mathcal{G}}_1\boldsymbol{v}_m(t) + \boldsymbol{H}_1(t)\,, && t \in [0,\tau)\,, \\
\boldsymbol{x}_{2m}(t) &= \tilde{\mathcal{G}}_2\boldsymbol{v}_m(t) + \boldsymbol{H}_2(t)\,, && t \in (\tau,T]\,.
\end{aligned} \tag{60}
$$

Since $\boldsymbol{v}_{1m}(t), \boldsymbol{v}_{2m}(t)$ are trigonometric polynomials of order m, the operator equation in (58) can be rewritten as

$$
\boldsymbol{v}_m = \mathcal{P}_m\mathcal{K}\boldsymbol{v}_m = \begin{cases} \mathcal{P}_{1m}\mathcal{K}_1\boldsymbol{v}_m\,, & t \in [0,\tau)\,, \\ \mathcal{P}_{2m}\mathcal{K}_2\boldsymbol{v}_m\,, & t \in (\tau,T]\,. \end{cases} \tag{61}
$$

Thus the operator equations (57), (61), that are equivalent to the initial problem (1)–(3) and to the system of determining equations (39), respectively, have been constructed. To complete the proof of our Theorem, i.e. to establish the solvability of (39) and the validity of (52), (53), as the equation (50) fulfills the required conditions, we can use the analogy with the method suggested in [8] and elaborated in [9], where the author obtained the same results in case of periodic systems of ordinary differential equations without impulse effect.

4. Examples

At first let us consider a very simple scalar differential equation

$$
\frac{dx}{dt} = \begin{cases} f_1(t,x) = x^2 - 2\cos t \cdot x - 2\sin t\,, & t \in [0,\pi)\,, \\ f_2(t,x) = x^2 - (2+\sin t)\cdot x + \cos t\,, & t \in (\pi, 2\pi]\,, \end{cases} \tag{62}
$$

$$
x(0) = x(2\pi)\,, \tag{63}
$$

$$
x(\pi+0) - x(\pi-0) = 4\,. \tag{64}
$$

The approximate solution of the above problem is sought in the form of a trigonometric polynomial as described in (37), with $m = 2$ and $\omega = 1$.

Then from (36') and (37)

$$\tilde{a}_0^{(1)} = a_0^{(2)} + a_2^{(2)} - a_2^{(1)} - \frac{1}{2}q\,,$$
$$\tilde{a}_1^{(1)} = a_1^{(2)} + \frac{1}{2}q\,,$$

$$(65)\qquad \begin{cases} x_{12}(t) = (a_0^{(2)} + a_2^{(2)} - a_2^{(1)} - 2) + (a_1^{(2)} + 2)\cos t + \\ \qquad + a_2^{(1)}\cos 2t + b_1^{(1)}\sin t + b_2^{(1)}\sin 2t\,, \quad t \in [0,\pi)\,, \\ x_{22}(t) = a_0^{(2)} + a_1^{(2)}\cos t + b_1^{(2)}\sin t + a_2^{(2)}\cos 2t + \\ \qquad + b_2^{(2)}\sin 2t\,, \quad t \in (\pi, 2\pi]\,. \end{cases}$$

The points (40) are

$$(66)\qquad \begin{aligned} &t_1 = \frac{\pi}{8}\,, \quad t_2 = \frac{3\pi}{8}\,, \quad t_3 = \frac{5\pi}{8}\,, \quad t_4 = \frac{7\pi}{8}\,, \\ &\tilde{t}_1 = \frac{9\pi}{8}\,, \quad \tilde{t}_2 = \frac{11\pi}{8}\,, \quad \tilde{t}_3 = \frac{13\pi}{8}\,, \quad \tilde{t}_4 = \frac{15\pi}{8}\,. \end{aligned}$$

Thus the determining equations from (39):

$$(67)\qquad \begin{cases} -(a_1^{(2)} + 2)\sin t_i - 2a_2^{(1)}\sin 2t_i + b_1^{(1)}\cos t_i + 2b_2^{(1)}\cos 2t_i = \\ = [(a_0^{(2)} + a_2^{(2)} - a_2^{(1)} - 2) + (a_1^{(2)} + 2)\cos t_i + \\ \quad + a_2^{(1)}\cos 2t_i + b_1^{(1)}\sin t_i + b_2^{(1)}\sin 2t_i]^2 - \\ \quad - 2\cos t_i[(a_0^{(2)} + a_2^{(2)} - a_2^{(1)} - 2) + \\ \quad + (a_1^{(2)} + 2)\cos t_i + a_2^{(1)}\cos 2t_i + b_1^{(1)}\sin t_i + b_2^{(1)}\sin 2t_i] - \\ \quad - 2\sin t_i\,, \qquad i = 1,2,3,4\,, \\ -a_1^{(2)}\sin\tilde{t}_j + b_1^{(2)}\cos\tilde{t}_j - 2a_2^{(2)}\sin 2\tilde{t}_j + 2b_2^{(2)}\sin 2\tilde{t}_j = \\ = (a_0^{(2)} + a_1^{(2)}\cos\tilde{t}_j + b_1^{(2)}\sin\tilde{t}_j + a_2^{(2)}\cos 2\tilde{t}_j + b_2^{(2)}\sin 2\tilde{t}_j)^2 - \\ \quad - (2 + \sin\tilde{t}_j)(a_0^{(2)} + a_1^{(2)}\cos\tilde{t}_j + b_1^{(2)}\sin\tilde{t}_j + a_2^{(2)}\cos 2\tilde{t}_j + \\ \quad + b_2^{(2)}\sin 2\tilde{t}_j) + \cos\tilde{t}_j\,, \qquad j = 1,2,3,4\,. \end{cases}$$

An exact solution of (62)–(64) is the following

$$x = \begin{cases} x_1(t) = 2\cos t\,, & t \in [0,\pi)\,, \\ x_2(t) = 2 + \sin t\,, & t \in (\pi, 2\pi]\,. \end{cases}$$

	initial value	*approximate value*		*initial value*	*approximate value*
$a_0^{(1)}$	–	$-1.5 \cdot 10^{-11}$	$a_0^{(2)}$	1.9	2.0
$a_1^{(1)}$	–	2.0	$a_1^{(2)}$	0.0	$1.2 \cdot 10^{-12}$
$a_2^{(1)}$	-0.1	$8.7 \cdot 10^{-12}$	$a_2^{(2)}$	0.1	$-6.8 \cdot 10^{-12}$
$b_1^{(1)}$	-0.1	$3.0 \cdot 10^{-12}$	$b_1^{(2)}$	1.1	1.0
$b_2^{(1)}$	0.1	$1.0 \cdot 10^{-12}$	$b_2^{(2)}$	0.1	$-7.5 \cdot 10^{-12}$

The unknown coefficients (65), satisfying (67), are in the following table.

Remark 1. The solution of (67) is based upon an *ABS*-method (see [10], [11]).

Remark 2. Starting the *ABS*-method with other initial values, we got another solution of (67), that is

$$\begin{gathered} a_0^{(1)} = -0.24670\,,\; a_1^{(1)} = 1.85868\,,\; a_2^{(1)} = 0.03222\,, \\ b_1^{(1)} = -0.05326\,,\; b_2^{(1)} = -0.23316\,, \\ a_0^{(2)} = 1.50395\,,\; a_1^{(2)} = -0.14132\,,\; a_2^{(2)} = 0.28156\,, \\ b_1^{(2)} = 0.47541\,,\; b_2^{(2)} = 0.24488\,. \end{gathered}$$

As a second example consider a second order differential equation

$$\frac{d^2x}{dt} = \begin{cases} x^2 - \dot{x}^2 - x - 2\sin 2t\,, & t \in [0,\pi)\,, \\ -x\sin t + x^2 - 1 - 2\sin t\,, & t \in (\pi, 2\pi]\,, \end{cases} \tag{68}$$

under conditions

$$\begin{aligned} x(0) &= x(2\pi)\,, \\ \dot{x}(0) &= \dot{x}(2\pi)\,, \end{aligned} \tag{69}$$

and for $\tau = \pi$

$$\boldsymbol{E}\begin{bmatrix} x(\tau+0) \\ \dot{x}(\tau+0) \end{bmatrix} - \boldsymbol{E}\begin{bmatrix} x(\tau-0) \\ \dot{x}(\tau-0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \tag{70}$$

Rewriting (68) into a system of first order differential equations, we get

$$\boldsymbol{x}(t) = \begin{cases} \boldsymbol{x}_1(t) = \begin{bmatrix} x_{1,1}(t) \\ x_{1,2}(t) \end{bmatrix}, & t \in [0,\pi)\,, \\ \boldsymbol{x}_2(t) = \begin{bmatrix} x_{2,1}(t) \\ x_{2,2}(t) \end{bmatrix}, & t \in (\pi, 2\pi]\,, \end{cases}$$

$$\dot{\boldsymbol{x}}(t) = \begin{cases} \dot{\boldsymbol{x}}_1(t) = \begin{bmatrix} x_{1,2} \\ x_{1,1}^2 - x_{1,2}^2 - x_{1,1} - 2\sin 2t \end{bmatrix}, & t \in [0,\pi)\,, \\ \dot{\boldsymbol{x}}_2(t) = \begin{bmatrix} x_{2,2} \\ -x_{2,1}\sin t + x_{2,1}^2 - 1 - 2\sin t \end{bmatrix}, & t \in (\pi, 2\pi]\,. \end{cases}$$

The new form of (69), (70) is

$$\begin{bmatrix} x_{1,1}(0) \\ x_{1,2}(0) \end{bmatrix} = \begin{bmatrix} x_{2,1}(2\pi) \\ x_{2,2}(2\pi) \end{bmatrix},$$

$$\begin{bmatrix} x_{2,2}(\pi+0) - x_{1,1}(\pi-0) \\ x_{2,2}(\pi+0) - x_{1,2}(\pi-0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The approximate trigonometric polynomials are

$$\boldsymbol{x}_{1m}(t) = \boldsymbol{a}_0^{(1)} + \sum_{k=1}^{m} (\boldsymbol{a}_k^{(1)} \cos kt + \boldsymbol{b}_k^{(1)} \sin kt)\,,\ t \in [0,\pi)\,,$$
$$\boldsymbol{x}_{2m}(t) = \boldsymbol{a}_0^{(2)} + \sum_{k=1}^{m} (\boldsymbol{a}_k^{(2)} \cos kt + \boldsymbol{b}_k^{(2)} \sin kt)\,,\ t \in (\pi,2\pi]\,.$$

The unknown coefficients are evaluated in the cases $m = 2$ and $m = 3$. The results are shown in the following two tables.

For comparison, an exact solution of the problem is

$$\boldsymbol{x}(t) = \begin{cases} \boldsymbol{x}_1(t) = \begin{bmatrix} \sin t + \cos t \\ \cos t - \sin t \end{bmatrix}, & t \in [0,\pi)\,, \\ \boldsymbol{x}_2(t) = \begin{bmatrix} 1 + \sin t \\ \cos t \end{bmatrix}, & t \in (\pi, 2\pi]\,. \end{cases}$$

$m = 2$

	initial value	*approximate value*		*initial value*	*approximate value*
$\boldsymbol{a}_0^{(1)}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1.8\cdot10^{-12}\\-2.5\cdot10^{-12}\end{bmatrix}$	$\boldsymbol{a}_0^{(2)}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1.0\\-1.5\cdot10^{-12}\end{bmatrix}$
$\boldsymbol{a}_1^{(1)}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1.0\\1.0\end{bmatrix}$	$\boldsymbol{a}_1^{(2)}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}-1.4\cdot10^{-13}\\1.0\end{bmatrix}$
$\boldsymbol{a}_2^{(1)}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}-4.2\cdot10^{-13}\\-3.0\cdot10^{-12}\end{bmatrix}$	$\boldsymbol{a}_2^{(2)}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}-2.1\cdot10^{-13}\\-1.5\cdot10^{-12}\end{bmatrix}$
$\boldsymbol{b}_1^{(1)}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1.0\\-1.0\end{bmatrix}$	$\boldsymbol{b}_1^{(2)}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1.0\\-3.\cdot10^{-12}\end{bmatrix}$
$\boldsymbol{b}_2^{(1)}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}-5.\cdot10^{-13}\\-7.9\cdot10^{-14}\end{bmatrix}$	$\boldsymbol{b}_2^{(2)}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}-2.4\cdot10^{-13}\\1.4\cdot10^{-12}\end{bmatrix}$

$m = 3$

	initial value	*approximate value*		*initial value*	*approximate value*
$\boldsymbol{a}_0^{(1)}$	$\begin{bmatrix}0.01\\0.01\end{bmatrix}$	$\begin{bmatrix}-2.1\cdot10^{-12}\\4.2\cdot10^{-13}\end{bmatrix}$	$\boldsymbol{a}_0^{(2)}$	$\begin{bmatrix}0.99\\0.01\end{bmatrix}$	$\begin{bmatrix}1.0\\-3.1\cdot10^{-12}\end{bmatrix}$
$\boldsymbol{a}_1^{(1)}$	$\begin{bmatrix}0.99\\1.01\end{bmatrix}$	$\begin{bmatrix}1.0\\1.0\end{bmatrix}$	$\boldsymbol{a}_1^{(2)}$	$\begin{bmatrix}-0.01\\0.98\end{bmatrix}$	$\begin{bmatrix}-3.0\cdot10^{-12}\\1.0\end{bmatrix}$
$\boldsymbol{a}_2^{(1)}$	$\begin{bmatrix}0.01\\0.01\end{bmatrix}$	$\begin{bmatrix}3.7\cdot10^{-12}\\-3.2\cdot10^{-12}\end{bmatrix}$	$\boldsymbol{a}_2^{(2)}$	$\begin{bmatrix}-0.01\\-0.01\end{bmatrix}$	$\begin{bmatrix}-1.6\cdot10^{-12}\\-1.8\cdot10^{-13}\end{bmatrix}$
$\boldsymbol{a}_3^{(1)}$	$\begin{bmatrix}-0.01\\-0.01\end{bmatrix}$	$\begin{bmatrix}-9.3\cdot10^{-13}\\-4.8\cdot10^{-12}\end{bmatrix}$	$\boldsymbol{a}_3^{(2)}$	$\begin{bmatrix}0.01\\0.01\end{bmatrix}$	$\begin{bmatrix}1.5\cdot10^{-13}\\-4.6\cdot10^{-13}\end{bmatrix}$
$\boldsymbol{b}_1^{(1)}$	$\begin{bmatrix}1.01\\-1.01\end{bmatrix}$	$\begin{bmatrix}1.0\\-1.0\end{bmatrix}$	$\boldsymbol{b}_1^{(2)}$	$\begin{bmatrix}-0.01\\0.01\end{bmatrix}$	$\begin{bmatrix}1.0\\-2.8\cdot10^{-12}\end{bmatrix}$
$\boldsymbol{b}_2^{(1)}$	$\begin{bmatrix}0.01\\-0.01\end{bmatrix}$	$\begin{bmatrix}-1.6\cdot10^{-12}\\-1.1\cdot10^{-11}\end{bmatrix}$	$\boldsymbol{b}_2^{(2)}$	$\begin{bmatrix}0.0\\0.01\end{bmatrix}$	$\begin{bmatrix}-1.6\cdot10^{-12}\\2.0\cdot10^{-12}\end{bmatrix}$
$\boldsymbol{b}_3^{(1)}$	$\begin{bmatrix}0.01\\-0.01\end{bmatrix}$	$\begin{bmatrix}-1.2\cdot10^{-12}\\2.2\cdot10^{-12}\end{bmatrix}$	$\boldsymbol{b}_3^{(2)}$	$\begin{bmatrix}-0.01\\-0.01\end{bmatrix}$	$\begin{bmatrix}-1.0\cdot10^{-13}\\-4.1\cdot10^{-13}\end{bmatrix}$

References

[1] A. M. Samoilenko, N. I. Ronto and O. O. Kurbanbaev, Collocation method for boundary value problems of ordinary differential equations with impulse effect, *Preprint* (1989), (in Russian).

[2] A. M. Samoilenko, N. I. Ronto and O. O. Kurbanbaev, Finding of solutions of boundary value problems with impulse effect, *Preprint* (1990), (in Russian).

[3] A. M. Samoilenko and N. A. Perestyuk, Differential equations with impulse effect, *Visca Skola*, *Kiev*, 1987, (in Russian).

[4] A. M. Samoilenko, N. A. Perestyuk and S. I. Trofimchuk, The problem of "beating" for systems with impulse effect, *Preprint* (1990), (in Russian).

[5] G. K. Kulev and D. D. Bainov, Stability of sets for systems with impulses, *Bull. of the Inst. of Math. Acad. Linica* **17** (1989), 313–326.

[6] D. D. Bainov, S. I Kostadinov and A. D. Myshkis, Asymptotic equivalence of differential equations with impulse effect in Banach spaces, *Publ. Matemátiquer* **34** (1990), 249–257.

[7] M. A. Najmark, Linear differential operators, *Nauka*, *Moscow*, 1969, (in Russian).

[8] G. Vainikko, On convergency of collocation method for nonlinear differential equations, *Zh. vycisl. Mat. Fiz.* **6** (1966), 35–42, (in Russian).

[9] A. M. Samoilenko and N. I. Ronto, Numerical-analytical methods for investigating periodic solutions, *Mir publishers*, *Moscow*, 1979.

[10] J. Abaffy, A. Galántai and E. Spedicato, The local convergence of ABS methods for nonlinear algebraic equations, *Numerische Mathematic* **51** (1987), 429–439.

[11] A. Jeney, Numerical investigation of ABS-methods for solving systems of nonlinear algebraic equations, *Alkalmazott Matematikai Lapok* **15** (1990-1991), 353–364, (in Hungarian).

N. I. RONTO
INSTITUTE OF MATHEMATICS
UNIVERSITY OF MISKOLC
3515 HUNGARY

Á. TUZSON
INSTITUTE OF MATHEMATICS
UNIVERSITY OF MISKOLC
3515 HUNGARY

(Received February 3, 1993; revised September 9, 1993)