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On local semi CAP-subgroups of finite groups

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Abstract. A subgroup H of a finite group G is said to have the semi cover-avoiding property in G if there is a normal series of G such that H covers or avoids every normal factor of the series. In this paper we analyze how certain properties of semi cover-avoiding subgroups influence the structure of groups.

1. Introduction

A subgroup H of a group G is said to have the cover-avoiding property in G if H covers or avoids every chief factor of G. Since 1962, when Gaschütz introduced a certain conjugacy class of subgroups of a finite solvable group (see [2]) which have the cover-avoiding property, there has been much interest in investigating the topic on the cover-avoiding property. On the one hand, some authors continued to find some kind of subgroups of a finite solvable group having the cover-avoiding property [3], [16]. On the other hand, many authors hope to obtain structural insight into a finite group when some of its subgroups have the cover-avoiding property.

Recently, Fan, Guo and Shum introduced the semi cover-avoiding property and obtained some new results.

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Definition 1.1 ([1, Definition 2.1]). A subgroup H of a finite group G is said to have the semi cover-avoiding property in G if there is a normal series

$$G = G_0 > G_1 > \cdots > G_n = 1$$

of G such that for every $i = 0, 1, 2, \ldots, n-1$, H either covers G_i/G_{i+1} or avoids G_i/G_{i+1}. In this case, we also say that H is a semi cover-avoiding subgroup of G. For short, H is a semi CAP-subgroup of G.

Recall that a subgroup H of a finite group G is said to be c-normal in G if there is a normal subgroup N in G such that $G = HN$ and $H \cap N \leq \mathrm{core}_G(H)$. It is clear that the semi cover-avoiding property covers not only the cover-avoiding property but also c-normality. It has been proved that the concept of semi CAP-subgroup is suitable for describing the structure of groups [5], [7], [11], [17]. Our motivation in this paper comes from the following example.

Example 1.1. Let $N = \langle a \rangle \times \langle b \rangle$ be an elementary abelian 5-group of order 5^2 and $c \in \mathrm{Aut}(N)$ such that $a^c = b^2$, $b^c = a$. Then the semidirect product $G = N \rtimes \langle c \rangle$ is of order $5^2 \times 2^3$. It is clear that $K = \langle b \rangle$ is not a semi CAP-subgroup of G, but K is a semi CAP-subgroup of $H = N \rtimes \langle c^2 \rangle$.

Therefore, it is a natural question to ask how much information about the structure of the group G we can uncover knowing that some subgroups are semi CAP-subgroups of G. In this paper, we first study this question. Then we investigate the solvability of some normal subgroup by using certain maximal subgroups, which is a generalization of known results.

Throughout this paper, the terminology and notation not mentioned here agree with standard usage.

2. Basic definitions and preliminary results

From now on, let G be a finite group and let p be a prime. We denote the derived subgroup of a group G by G'. Let $M \lessdot G$ mean that M is a maximal subgroup of G. If $M \leq G$, then M_G denotes the core of M in G. Let N and K be normal subgroups of a group G with $K \leq N$. Then N/K is called a normal factor of G. It is clear that every chief factor of a group is a normal factor of the group. A subgroup H of G is said to cover N/K if $HN = HK$. On the other hand, if $H \cap N = H \cap K$, then H is said to avoid N/K.

Recall that a class of groups $\mathcal{F}$ is a formation if $\mathcal{F}$ contains all homomorphic images of groups in $\mathcal{F}$ and if, for all normal subgroups M, N of G with G/M and

G/N in $\mathcal{F}$, we also have $G/(M \cap N)$ in $\mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$ [10, VI, Satz 7.1 and 7.2]. Throughout this paper $\mathcal{U}$ will denote the class of supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation [10, VI, Satz 8.6].

For convenience, we write $\mathcal{CAP}_s(G)$ to denote the set of all semi CAP-subgroups of G.

Lemma 2.1. *A subgroup H of a group G is a semi CAP-subgroup of G if and only if there is a chief series of G such that H covers or avoids every chief factor of the series.*

Proof. It can be seen from [1, Lemma 1] and Definition 1.1. □

Lemma 2.2 ([5, Lemma 2.5]). *Let H be a subgroup of a group G. If $H \in \mathcal{CAP}_s(G)$, then $H \in \mathcal{CAP}_s(K)$ for every subgroup K of G with $H \leq K$.*

Lemma 2.3 ([5, Lemma 2.6]). *Let N be a normal subgroup of a group G and let $H \in \mathcal{CAP}_s(G)$. Then $HN/N \in \mathcal{CAP}_s(G/N)$ if one of the following holds:*

(1) $N \leq H$.

(2) $\gcd(|H|, |N|) = 1$, *where* $\gcd(-,-)$ *denotes the greatest common divisor.*

Lemma 2.4. *Let N/K be a chief factor of a group G. Then:*

(1) *If a p-group H covers N/K, then N/K is an elementary abelian p-group.*

(2) *Let P be a Sylow p-subgroup of G and let P_1 be a maximal subgroup of P. If N/K is avoided by P_1 and N/K is a p-group, then N/K is of order p.*

Proof. (1) If $HN = HK$, then it follows from $|N/K| = |(HK \cap N) : K| = |(H \cap N)K : K| = |H \cap N : H \cap N \cap K| = |H \cap N : H \cap K|$ that N/K is an elementary abelian p-group.

(2) Since P_1 avoids N/K, $P_1K/K \cap N/K = 1$. By hypothesis, N/K is a p-group, hence P_1K/K is a maximal subgroup of PK/K. Therefore N/K is of order p. □

Lemma 2.5. *Let N be a normal subgroup of a group G and let P be a Sylow p-subgroup of G. If every maximal subgroup of P lies in $\mathcal{CAP}_s(N_G(P))$, then every maximal subgroup of PN/N lies in $\mathcal{CAP}_s(N_{G/N}(PN/N))$.*

Proof. Let M/N be a maximal subgroup of PN/N. Then $M = N(M \cap P)$ and $P \cap M$ is a maximal subgroup of P. Set $P_1 = P \cap M$. By hypothesis, there exists a chief series

$$N_G(P) = T_0 > T_1 > \cdots > T_n = 1$$

such that P_1 covers or avoids T_i/T_{i+1}, for every $i = 0, 1, 2, \dots, n-1$. It is easy to see that the following series

$$N_G(P)N/N = T_0N/N \geq T_1N/N \geq \cdots \geq T_{n-1}N/N \geq 1$$

is a normal series of $N_G(P)N/N$.

Since $N_G(P)$ is p-solvable, T_i/T_{i+1} is a p-group or p'-group. If T_i/T_{i+1} is a p'-group. Obviously, the p-group P_1N/N avoids $(T_iN/N)\big/(T_{i+1}N/N)$.

Now suppose that T_i/T_{i+1} is a p-group. If T_i/T_{i+1} was covered by P_1, obviously $(T_iN/N)\big/(T_{i+1}N/N)$ was covered by P_1N/N. If T_i/T_{i+1} was avoided by P_1, then T_i/T_{i+1} is of order p by Lemma 2.4 (2). Hence the order of $(T_iN/N)/(T_{i+1}N/N) = T_iN\big/T_{i+1}N$ is at most p. Therefore $(T_iN/N)\big/(T_{i+1}N/N)$ is covered or avoided by P_1N/N.

Hence $M/N = P_1N/N$ lies in $\mathcal{CAP}_s(N_{G/N}(PN/N)) = \mathcal{CAP}_s(N_G(P)N/N)$. This completes the proof of the lemma. □

If we replace the hypothesis that P is a Sylow p-subgroup of G in Lemma 2.5 by the hypothesis that P is a Sylow p-subgroup of some normal subgroup of G, we obtain the following result.

Lemma 2.6. *Let H be a normal subgroup of a group G and let P be a Sylow p-subgroup of H. If N is a normal p'-subgroup of G and every maximal subgroup of P lies in $\mathcal{CAP}_s(N_G(P))$, then every maximal subgroup of PN/N lies in $\mathcal{CAP}_s(N_{G/N}(PN/N))$.*

Proof. By using similar arguments as in the proof of Lemma 2.5, we only need to prove either $P_1T_iN = P_1T_{i+1}N$ or $P_1N \cap T_iN = P_1N \cap T_{i+1}N$ whenever both P_1 and $N_G(P) \cap N$ avoid T_i/T_{i+1}, where P_1 is a maximal subgroup of P and T_i/T_{i+1} is a chief factor of $N_G(P)$ as in the proof of Lemma 2.5.

Now we consider the index $|P_1T_iN : P_1T_{i+1}N|$. On the one hand, we have $|P_1T_iN : P_1T_{i+1}N| = |T_i/T_{i+1}|\,|P_1 \cap T_{i+1}N : P_1 \cap T_iN|$. On the other hand, $|P_1T_iN : P_1T_{i+1}N| = |T_i/T_{i+1}|\,|N \cap P_1T_{i+1} : N \cap P_1T_i|$. Therefore $|P_1 \cap T_iN : P_1 \cap T_{i+1}N| = |N \cap P_1T_i : N \cap P_1T_{i+1}|$. However, $|P_1 \cap T_iN : P_1 \cap T_{i+1}N|$ is a p-number and $|N \cap P_1T_i : N \cap P_1T_{i+1}|$ is a p'-number. It follows that $|P_1 \cap T_iN : P_1 \cap T_{i+1}N| = 1$. Hence $P_1 \cap T_iN = P_1 \cap T_{i+1}N$ and therefore $P_1N \cap T_iN = P_1N \cap T_{i+1}N$, as desired. □

Lemma 2.7 ([9, Lemma 2.6])**.** *Let N be a solvable normal subgroup of a group G with $N \neq 1$. If every minimal normal subgroup of G which is contained in N is not contained in $\Phi(G)$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

3. Main results

In this section, we study the structure of a group G when some subgroups are semi CAP-subgroups of a local subgroup of G. Our first result is about p-nilpotency.

Theorem 3.1. *Let p be the smallest prime dividing the order of a group G and let P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if every maximal subgroup of P lies in $\mathcal{CAP}_s(N_G(P))$ and $P' \in \mathcal{CAP}_s(G)$.*

Remark 3.1. The hypothesis that $P' \in \mathcal{CAP}_s(G)$ in Theorem 3.1 is essential. In fact, Let $G = PSL(2,7)$ and P a Sylow 2-subgroup of G. Since $P = N_G(P)$, every maximal subgroup of P lies in $\mathcal{CAP}_s(N_G(P))$, but G is not 2-nilpotent.

Even if G is a solvable group and p is an odd prime, the hypothesis that $P' \in \mathcal{CAP}_s(G)$ in Theorem 3.1 is also essential. For example, let $H = C_3 \times C_3 \times C_3$ be an elementary abelian group of order 3^3. Then there is a subgroup $C_{13} \rtimes C_3$ in the automorphism group of H, where $C_{13} \rtimes C_3$ is a semidirect product. Let $G = (C_3 \times C_3 \times C_3) \rtimes (C_{13} \rtimes C_3)$ be the corresponding semidirect product and $P \in \mathrm{Syl}_3(G)$. Clearly, $P = N_G(P)$. It follows that every maximal subgroup of P lies in $\mathcal{CAP}_s(N_G(P))$, but G is not 3-nilpotent.

Furthermore we can not remove the hypothesis that p is the smallest prime dividing the order of a group G in Theorem 3.1. In fact, let P be a Sylow 3-subgroup of A_5, the alternating group of degree 5. Then every maximal subgroup of P lies in $\mathcal{CAP}_s(N_G(P))$ and $P' \in \mathcal{CAP}_s(G)$, but A_5 is not 3-nilpotent.

Proof of Theorem 3.1. Since every p-subgroup of a p-supersolvable group G is a semi CAP-subgroup of G, we only need to prove the sufficiency. Suppose that the theorem is not true and let G be a counterexample with smallest order. Then:

(1) $O_{p'}(G) = 1$.

Otherwise, $O_{p'}(G) \neq 1$. Since $(PO_{p'}(G)/O_{p'}(G))' = P'O_{p'}(G)/O_{p'}(G)$, $(PO_{p'}(G)/O_{p'}(G))'$ lies in $\mathcal{CAP}_s(G/O_{p'}(G))$ by Lemma 2.3. Furthermore, every maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$ lies in $\mathcal{CAP}_s(N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)))$ by Lemma 2.5. We can see that $G/O_{p'}(G)$ satisfies the hypothesis of our theorem. Thus, by the minimality of G, $G/O_{p'}(G)$ is p-nilpotent and therefore G is p-nilpotent, a contradiction.

(2) If H is a proper subgroup of G with $P \leq H$, then H is p-nilpotent. In particular, $N_G(P)$ is p-nilpotent.

It is clear that $N_H(P) \leq N_G(P)$, and hence every maximal subgroup of P lies in $\mathcal{CAP}_s(N_H(P))$ and $P' \in \mathcal{CAP}_s(H)$ by Lemma 2.2. It follows that H satisfies the hypothesis of our theorem. Now, by the minimality of G, H is p-nilpotent. If $N_G(P) = G$, then G is p-nilpotent by [5, Theorem 3.2] or [11, Theorem 3.1]. So $N_G(P) < G$ and $N_G(P)$ is p-nilpotent.

(3) $O_p(G) \neq 1$ and there exists a minimal normal subgroup N of G such that G/N is p-nilpotent and G is p-solvable.

In fact, if $P' = 1$, then $N_G(P) = C_G(P)$ since $N_G(P)$ is p-nilpotent, so G is p-nilpotent by Burnside's Theorem [8, Theorem 7.2.1], a contradiction. Thus $P' \neq 1$. By the hypothesis, there exists a chief series

$$G = G_0 > G_1 > \cdots > N > G_n = 1 \tag{$*$}$$

of G such that P' covers or avoids G_i/G_{i+1}, where $1 \leq i \leq n-1$. If P' covers $N/1$, then $N \leq P'$, whence $N \leq O_p(G)$. So we may assume that P' avoids $N/1$. Then $P' \cap N = P' \cap 1 = 1$. Thus $(P \cap N)' \leq P' \cap N = 1$, hence $P \cap N$ is abelian. If $N_G(P \cap N) = G$, then $P \cap N$ is normal in G. The minimal normality of N means that $N \leq P$ and therefore $N \leq O_p(G)$, as desired. Hence we may assume that $N_G(P \cap N)$ is a proper subgroup of G. Since $P \leq N_G(P \cap N)$, we have that $N_G(P \cap N)$ is p-nilpotent by (2) and so $N_N(P \cap N)$ is p-nilpotent. It follows that $N_N(N \cap P) = C_N(N \cap P)$. Now Burnside's Theorem [8, Theorem 7.2.1] implies that N is p-nilpotent and therefore the minimal normality of N means that $N \leq O_p(G)$.

According to $(*)$, we can see that the following series

$$G/N = G_0/N > G_1/N > \cdots > 1$$

is a chief series of G/N. By our hypothesis, P' covers or avoids G_i/G_{i+1}, where $0 \leq i \leq n-1$. If P' covers G_i/G_{i+1}, then $P'N/N$ covers $G_i/N\big/G_{i+1}/N$. If P' avoids G_i/G_{i+1}, then $P'N/N$ avoids $G_i/N\big/G_{i+1}/N$. This means that $P'N/N$ lies in $\mathcal{CAP}_s(G/N)$. Moreover, every maximal subgroup of PN/N lies in $\mathcal{CAP}_s(N_{G/N}(PN/N))$ by Lemma 2.5. Hence G/N satisfies the hypothesis of the theorem, the minimality of G implies that G/N is p-nilpotent. Therefore G is p-solvable, as desired.

(4) There exists a Sylow q-subgroup Q of G with $q \neq p$ such that $G = PQ$.

Since G is p-solvable, there exits a Sylow q-subgroup Q of G such that $PQ = QP$ for any prime $q \neq p$ by [4, Theorem 6.3.5]. If $PQ < G$, then PQ is p-nilpotent by (2). It follows that $Q \leq C_G(O_p(G)) \leq O_p(G)$ by [13, Theorem 9.3.1], a contradiction.

(5) $N_G(P) = P$ is a maximal subgroup of G.

Let M be a maximal subgroup of G with $N_G(P) \leq M$. By (2) M is p-nilpotent and therefore $O_{p'}(M) \leq C_G(O_p(G)) \leq O_p(G)$. So $N_G(P) = P$ is a maximal subgroup of G.

(6) QN/N is a minimal normal subgroup of G/N and Q is an elementary abelian group.

Let K/N be a minimal normal subgroup of G/N contained in QN/N. It follows from (5) that $PK = G$ and therefore claim (6) is true.

(7) $P' \cap N = 1$ and $N \leq Z(P)$.

Since P' covers or avoids $N/1$, $N \leq P'$ or $N \cap P' = 1$. If $N \leq P'$, then $P \cap NQ = N \leq P' \leq \Phi(P)$. Thus NQ is p-nilpotent by a famous theorem of Tate [15] and therefore G is p-nilpotent, in contradiction to the choice of G. Thus $N \cap P' = 1$. It follows from $[P,N] \leq [P,P] = P'$ that $[P,N] \leq P' \cap N = 1$, as desired.

(8) Q is cyclic of order q and has no fixed points on N.

In fact, if there exists a non-trivial element x in N and a non-trivial element y in Q such that $[x,y] = 1$, then $\langle P, y\rangle \leq C_G(x)$. Since P is a maximal subgroup of G, $x \in Z(G)$ and hence $N = \langle x\rangle$. In this case, $NQ = N \times Q$, hence Q is a normal p-complement in G, a contradiction. So Q has no fixed points on N. It follows from Theorem 8.3.2 in [8] that Q is a cyclic group of order q and the claim (8) is true.

(9) Final contradiction.

Let $a \in N$ and let $b \in Q$ be such that $Q = \langle b\rangle$. Since NQ is normal in G, $G = NN_G(Q)$ by the Frattini argument. As N is an elementary abelian subgroup of G, we see that $N \cap N_G(Q)$ is normal in G. It follows that $N \cap N_G(Q) = 1$ or N. If the latter is true, then Q is a normal p-complement in G, a contradiction. Hence $N \cap N_G(Q) = 1$. Let $P_1 \in \mathrm{Syl}_p(N_G(Q))$ such that $P_1 \leq P$ and let $g \in P_1$. By (7), we have $N \leq Z(P)$ and hence $P_1 \leq C_G(N)$. As $C_G(N)$ is normal in G and P_1 normalizes Q, it follows that $[P_1, Q] \leq C_G(N) \cap Q = 1$, by (8). Therefore $P_1 \leq C_G(Q)$, and Q is abelian by (8). Together with (4) we deduce that $N_G(Q) = P_1Q = C_G(Q)$ and hence Burnside's Theorem [8, Theorem 7.2.1] implies that $G = N_G(P)$. This final contradiction completes our proof. □

As an immediate consequence of Theorem 3.1, we have:

Theorem 3.2. *Let p be the smallest prime of the order of a group G and let H a normal subgroup of G such that G/H is p-nilpotent. Let $P \in \mathrm{Syl}_p(H)$ and suppose that every maximal subgroup of P lies in $\mathcal{CAP}_s(N_G(P))$ and that $P' \in \mathcal{CAP}_s(G)$. Then G is p-nilpotent.*

Proof. We argue by induction on the order of G. By Lemma 2.2, every maximal subgroup of P lies in $\mathcal{CAP}_s(N_H(P))$, and $P' \in \mathcal{CAP}_s(H)$. By Theorem 3.1, H is p-nilpotent. Let K be a normal p-complement in H. Then K is normal in G and $(G/K)\big/(H/K) \cong G/H$ is p-nilpotent. In view of Lemma 2.3 and 2.6, we conclude that G/K satisfies the hypothesis of the theorem for its normal subgroup H/K.

If $K \neq 1$, then G/K is p-nilpotent by induction and so G is p-nilpotent. Hence we may assume that $K = 1$ and therefore $H = P$ is a p-group. For any prime q dividing the order of G with $p \neq q$ and $Q \in \mathrm{Syl}_q(G)$, it is clear that PQ satisfies the hypothesis of Theorem 3.1 and therefore $PQ = P \times Q$. Let S/H be a normal p-complement in G/H. The above arguments imply that $S = H \times S_1$ where S_1 is a Hall p'-subgroup of S. Hence S_1 is a normal p-complement in G. This completes the proof. □

Corollary 3.1. *For every Sylow subgroup P of a group G, suppose that every maximal subgroup of P lies in $\mathcal{CAP}_s(N_G(P))$ and that $P' \in \mathcal{CAP}_s(G)$. Then G is a Sylow tower group of supersolvable type.*

We can now prove:

Theorem 3.3. *Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups $\mathcal{U}$ and let H be a normal subgroup of a group G such that $G/H \in \mathcal{F}$. Suppose that, for all primes p dividing the order of H and for all $P \in \mathrm{Syl}_p(H)$, every maximal subgroup of P lies in $\mathcal{CAP}_s(N_G(P))$ and suppose that $P' \in \mathcal{CAP}_s(G)$. Then $G \in \mathcal{F}$.*

Proof. Suppose that the theorem is false and let G be a counterexample with smallest order.

Case 1: H is a p-group for some prime p.

If $\Phi(H) \neq 1$, then, since $\Phi(H) \trianglelefteq G$ and $G/\Phi(H)\big/H/\Phi(H) \cong G/H$, we see that $G/\Phi(H)$ satisfies the hypothesis of our theorem by Lemma 2.3. Thus $G/\Phi(H) \in \mathcal{F}$ by the minimal choice of G, it follows that $G/\Phi(G)$ belongs to $\mathcal{F}$ and so does G, a contradiction. Hence $\Phi(H) = 1$.

Let N be a minimal normal subgroup of G contained in H. G/N satisfies the hypothesis of the theorem and the minimality of G implies that $G/N \in \mathcal{F}$. Noticing that $N \not\leq \Phi(G)$ and with Lemma 2.7 in mind, we may assume that

$$H = N_1 \times N_2 \times \cdots \times N_s$$

where $N_1, \ldots, N_s$ are all the minimal normal subgroups of G. By the above argument, $G/N_i \in \mathcal{F}$ for all $i \in \{1, \ldots, s\}$. If $s > 1$, then

$$G \cong G\big/(N_1 \cap N_2) \in \mathcal{F}.$$

Hence $H = N_1$ is a minimal normal subgroup of G.

Let P_1 be a maximal subgroup of H. Since H is itself a p-group, by the hypothesis, there exists a chief series of $G = N_G(H)$

$$G = G_0 > G_1 > \cdots > G_{n-1} > G_n = 1$$

such that P_1 covers or avoids G_j/G_{j+1} for $j = 0, 1, \ldots, n-1$. Since $H \le G_0$ and $H \nleq G_n P_1$, there exists an integer i such that $H \le G_i P_1$, but $H \nleq G_{i+1}P_1$, where $1 \le i \le n-1$. Noticing that P_1 covers or avoids G_i/G_{i+1} and $G_i P_1 \ne G_{i+1}P_1$, we have $P_1 \cap G_i = P_1 \cap G_{i+1}$. Since $H = H \cap G_i P_1 = (H \cap G_i)P_1$ and $H \cap G_i$ is a normal subgroup of G, $H \cap G_i = H$ by the minimal normality of H. It follows that $P_1 \cap G_{i+1} = P_1 \cap G_i = P_1$. On the other hand, $H \nleq G_{i+1}P_1$ implies that $H \cap G_{i+1} < H$. Hence $H \cap G_{i+1} = P_1$. The minimality of H implies that $P_1 = 1$. It follows that H is a cyclic group of order p and hence $G \in \mathcal{F}$, a contradiction.

Case 2: H is not of prime power order.

In view of Corollary 3.1 and Lemma 2.2, H is a Sylow tower group of supersolvable type. Let r be the largest prime dividing the order of H and let R be a Sylow r-subgroup of H. Then R is normal in G and $G/R\big/H/R \cong G/H \in \mathcal{F}$. By Lemma 2.3 and 2.6, we can see that G/R satisfies the hypothesis of the theorem for its normal subgroup H/R. By the minimality of G, $G/R \in \mathcal{F}$. It follows from the first case that $G \in \mathcal{F}$. The proof of the theorem is now complete. □

Finally we study the solvability of some normal subgroup of G by looking at certain maximal subgroups, leading to generalizations of known results.

Let H be a normal subgroup of a group G. We define the following families of subgroups:

$$\mathcal{M}(G) = \{M \mid M \lessdot G\}$$

$$\mathcal{M}_{pc}(G) = \{M \mid M \in \mathcal{M}(G),\ |G:M|_p = 1 \text{ and } |G:M| \text{ is composite}\}$$

$$\mathcal{M}^{pcn}(G) = \{M \mid M \in \mathcal{M}(G),\ N_G(P) \le M \text{ for a Sylow } p\text{-subgroup } P \text{ of } G,\ M \text{ is non-nilpotent and } |G:M| \text{ is composite}\}$$

$$\mathcal{M}_H(G) = \{M \mid M \in \mathcal{M}(G) \text{ and } H \nleq M\}$$

Theorem 3.4. *Let H be a normal subgroup of a group G and let p be the largest prime dividing the order of G. If every member of $\mathcal{M}_{pc}(G) \cap \mathcal{M}_H(G)$ lies in $\mathcal{CAP}_s(G)$, then H is solvable.*

Proof. Suppose that $\mathcal{M}_{pc}(G) \cap \mathcal{M}_H(G) = \emptyset$, then we claim that H is solvable. In fact, if $\mathcal{M}_{pc}(G) = \emptyset$, by [12, Theorem 8], G is solvable and so is H.

If $\mathcal{M}_{pc}(G) \neq \emptyset$, then H is contained in every member of $\mathcal{M}_{pc}(G)$. Applying [12, Theorem 8] again, H is solvable. Now we may assume that $\mathcal{M}_{pc}(G) \cap \mathcal{M}_H(G) \neq \emptyset$.

Let N be a minimal normal subgroup of G, and let M/N be a maximal subgroup of G/N with $M/N \in \mathcal{M}_{pc}(G/N) \cap \mathcal{M}_H(G/N)$, where $\mathcal{M}_H(G/N) = \{M/N \mid M/N \in \mathcal{M}(G/N) \text{ and } HN/N \nleq M/N\}$. Then $M \in \mathcal{M}_{pc}(G) \cap \mathcal{M}_H(G)$. Furthermore, M/N lies in $\mathcal{CAP}_s(G/N)$ by Lemma 2.3. It is clear that G/N satisfies the hypothesis of the theorem for the normal subgroup HN/N and so HN/N is solvable by induction. If $N \nleq H$, then $H \cong HN/N$ is solvable, as desired. Suppose that $N \leq H$. If G has two minimal normal subgroups N_1 and N_2, then both H/N_1 and H/N_2 are solvable and so is $H/(N_1 \cap N_2)$. This implies that the group H is solvable. Hence we may assume that G has a unique minimal normal subgroup N.

Suppose that N is non-solvable. Let q be the largest prime dividing the order of N and Q a Sylow q-subgroup of N. If $N_G(Q) = G$, then Q is normal in G. The minimality of N means that $N = Q$, contradicting the fact N is non-solvable. Then $G = N_G(Q)N$ by the Frattini argument. So there exists a maximal subgroup M of G which contains $N_G(Q)$, but $N \nleq M$. By hypothesis, $p \geq q$. If $p > q$, it is clear that $|G : M|_p = |N : M \cap N|_p$=1. If $p = q$, then $N_G(Q)$ contains a Sylow p-subgroup of G. Thus we conclude that $|G : M|_p = 1$ in these two cases. Furthermore, we can see that $|G : M|_q = 1$ because $N_G(Q) \leq M$ contains a Sylow q-subgroup of G. If $|G : M| = r$ for some prime r, then, since $M_G = 1$, we have that G is isomorphic to a subgroup of the symmetric group S_r of degree r. This implies that $|G| \mid r!$ and so $|N| \mid r!$, in contradiction to that q is the largest prime in $\pi(N)$. Hence we conclude that $M \in \mathcal{M}_{pc}(G) \cap \mathcal{M}_H(G)$. By the hypothesis, $M \in \mathcal{CAP}_s(G)$ and so $MN = M$ or $M \cap N = 1$. However, neither of this is possible because $N_G(Q)$ is contained in M and $N \nleq M$, a contradiction. This shows that N is solvable and therefore H is solvable. □

From Theorems 3.4, we have the following corollary:

Corollary 3.2. *Let p be the largest prime dividing the order of a group G. If every member of $\mathcal{M}_{pc}(G)$ lies in $\mathcal{CAP}_s(G)$, then G is solvable.*

Proof. Let $G = H$ in Theorem 3.4. Then we have the corollary. □

Remark 3.2. In Theorem 3.4 the group G may not be necessarily solvable. For example:

Let K, H be the alternating group of degree 5 and 4, respectively and let $G = K \times H$. Suppose that $M = K \times C_3$, where C_3 is a cyclic group of order 3 of H. Then M is a maximal subgroup of G. It is clear that $H \nleq M$ and $|G : M| = 4$.

Thus $M \in \mathcal{M}_{pc}(G) \cap \mathcal{M}_H(G)$ and we can also see that $\mathcal{M}_{pc}(G) \cap \mathcal{M}_H(G) = \{M^g \mid g \in G\}$. Furthermore, it is easy to see that the following series

$$G > K \times K_4 > K > 1$$

is a chief series of G and that M^g avoids $(K_4 \times K)/K$ and covers the rest, where K_4 is the Klein four group contained in H. That is, $M^g \in \mathcal{CAP}_s(G)$. This shows that G satisfies the hypothesis of Theorem 3.4 for the normal subgroup H. However, G is not solvable.

Theorem 3.5. *Let H be a normal subgroup of a group G and let p be the largest prime dividing the order of G. If every member of $\mathcal{M}^{pcn}(G) \cap \mathcal{M}_H(G)$ lies in $\mathcal{CAP}_s(G)$, then H is p-solvable.*

Proof. If $\mathcal{M}^{pcn}(G) \cap \mathcal{M}_H(G) = \emptyset$, then we can see that H is p-solvable by [6, Lemma 2.4]. Now, we may assume that $\mathcal{M}^{pcn}(G) \cap \mathcal{M}_H(G) \neq \emptyset$. Let $P \in \mathrm{Syl}_p(G)$. If P is normal in G, then G is certainly p-solvable and so is H. So we may assume that $N_G(P) < G$.

Let N be a minimal normal subgroup of G. It is clear that G/N satisfies the hypothesis of the theorem for the normal subgroup HN/N and so HN/N is p-solvable by induction. By a routine argument, we can assume that N is contained in H and N is the unique minimal normal subgroup of G.

Suppose that N is not p-solvable. Then p is a divisor of the order of N. We know that $N \cap P \in \mathrm{Syl}_p(N)$ and $P \cap N$ is not normal in G. With a Frattini argument, we have that $G = N_G(P \cap N)N$. So there exists a maximal subgroup M of G which contains $N_G(P \cap N)$ and $M \not\geq N$. It is clear that $N_G(P) \leq M$. If $|G : M| = q$ is a prime, then, using the permutation representation of G on $S = \{Mg \mid g \in G\}$, we see that G is isomorphic to a subgroup of the symmetric group S_q of degree q. This implies that $|G| \mid q!$ and so $|N| \mid q!$. This contradicts p being the largest prime which divides the order of N. Hence $|G : M|$ must be a composite number. If M is nilpotent, then M has non-trivial Sylow 2-subgroups by [13, Theorem 10.4.2]. Let $M_{2'}$ be a Hall $2'$-subgroup of M. By [14, Theorem 1], $M_{2'}$ is normal in G and therefore $P \trianglelefteq G$ since P is a characteristic subgroup of $M_{2'}$. It follows that $P \cap N \trianglelefteq G$, a contradiction. Thus, $M \in \mathcal{M}^{pcn}(G) \cap \mathcal{M}_H(G)$. By the hypothesis, $M \in \mathcal{CAP}_s(G)$ and so $MN = M$ or $M \cap N = 1$. However, these two situations are impossible. This shows that N is p-solvable and therefore H is p-solvable. The proof of the theorem is now complete. □

Corollary 3.3 ([6, Theorem 3.8]). *Let G be a group and let p be the largest prime dividing the order of G. If every member of $\mathcal{M}^{pcn}(G)$ has the cover-avoiding property in G, then G is p-solvable.*

In Theorem 3.5, the group G need not be p-solvable as the following example shows.

Example 3.1. Let $H = C_2 \times C_2 \times C_2 \times C_2$ be an elementary abelian group of order 2^4. Then there is a subgroup $M = A_5$ in the automorphism group of H, where A_5 is the alternating group of degree 5. Let $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ be the corresponding semidirect product. We can deduce that $\mathcal{M}^{pcn}(G) \cap \mathcal{M}_H(G) = \{M^g \mid g \in G\}$. Furthermore, there exists a chief series

$$G > H > 1$$

such that M^g covers G/H and avoids $H/1$. Thus, $M^g \in \mathcal{CAP}_s(G)$. That is, G satisfies the hypothesis of Theorem 3.5 with the normal subgroup H. However, G is not 5-solvable.

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