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# Diffeomorphic theorems for open Riemannian manifolds with curvature decay

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**Abstract.** In this paper, we study the topology of complete non-compact Riemannian manifolds with curvature decay to a non-positive constant. We show that such a complete open manifold $M$ is diffeomorphic to a Euclidean $n$-space $\mathbb{R}^n$ if it contains enough rays starting from the base point. As applications, we also show that this kind of manifolds with Ricci curvature bounded from below by a non-positive constant are diffeomorphic to $\mathbb{R}^n$ if the volumes of geodesic balls in $M$ grow properly. Our results generalize the main theorems of Wang–Xia for manifolds with quadratic curvature decay to zero.

## 1. Introduction

Let $(M,g)$ be a complete non-compact $n(\geq 2)$-dimensional Riemannian manifold. For a fixed point $p \in M$ and any $r>0$, denote by $B(p,r)$ the open geodesic ball around $p$ with radius $r$ in $M$, and $S(p,r)$ the corresponding geodesic sphere.

Denote by $K_M$ the sectional curvature of $M$, and let

$$k_p(r) = \inf_{M\setminus B(p,r)} K_M,$$

where the infimum is taken over all the sections at all points on $M\setminus B(p,r)$. It is easy to see that we can choose $k_p(r)$ to be a non-positive monotone increasing function of $r$.

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In this paper, we consider complete open Riemannian manifolds with curvature decay to a non-positive constant, that is to say, there exists a positive monotone decreasing function $K(r)$ with $\lim_{r\to\infty} K(r) \geq 0$ satisfying

$$k_p(r) \geq -(K(r))^2, \quad \text{for all } r > 0.$$

Let $R_p$ denote the (point set) union of rays issuing from $p$. One can show that $R_p$ is a closed subset of $M$. Define a function $h_p$ on $M$ by

$$h_p(x) = d(x, R_p),$$

where $d$ is the distance function on $M$. We set for $r > 0$ (cf. [8], [17], [21])

$$\mathcal{H}(p,r) = \max_{x\in S(p,r)} d(x, R_p). \tag{1.1}$$

By definition, we always have

$$\mathcal{H}(p,r) \leq \max_{x\in S(p,r)} d(x,p) = r, \quad \text{for all } r > 0.$$

If $M$ is a complete simply connected Riemannian manifold with non-positive sectional curvature, then $\mathcal{H}(p,x) \equiv 0$. This follows from the fact that any point in $M$ lies in some ray starting from $p$.

For a constant $c \geq 0$, we denote by $M^n(-c)$ an $n$-dimensional complete simply connected Riemannian manifold of constant curvature $-c$. If the Ricci curvature of $M$ satisfies $\mathrm{Ric}_M \geq -(n-1)c$, the relative volume comparison theorem [4] tells us that the function $r \to \frac{\mathrm{vol}[B(p,r)]}{\alpha_n(r,-c)}$ is monotone decreasing, where $\mathrm{vol}[B(p,r)]$ is the volume of $B(p,r)$ and $\alpha_n(r,-c)$ the volume of a geodesic ball of radius $r$ in $M^n(-c)$. It is well known that

$$\alpha_n(r,-c) = \omega_{n-1}\int_0^r f_{-c}^{n-1}(t)dt,$$

where

$$f_{-c}(t) = \begin{cases} t, & c = 0,\\ \dfrac{\sinh(\sqrt{c}t)}{\sqrt{c}}, & c > 0, \end{cases} \tag{1.2}$$

and $\omega_m$ is the volume of $S^m(1)$.

For any $p \in M$, we set

$$v_{-c}(p) = \lim_{r\to\infty} \frac{\mathrm{vol}[B(p,r)]}{\alpha_n(r,-c)},$$

and define

$$v_{-c} = \inf_{p\in M} v_{-c}(p). \tag{1.3}$$

One always has

$$\frac{\mathrm{vol}[B(p,r)]}{\alpha_n(r,-c)} \geq v_{-c}(p) \geq v_{-c}, \quad \forall r>0,\ \forall p\in M.$$

Abresch [1] proved that if $\int_0^\infty rk_p(r)dr > -\infty$, then $M$ is of finite topological type. Xia [21] proved that if $M$ is an $n$-dimensional complete open Riemannian manifold with nonnegative sectional curvature in which there exists a $p\in M$ such that $\mathcal{H}(p,r) < r$, for all $r>0$, then $M$ is diffeomorphic to $\mathbb{R}^n$. Wang–Xia [19] proved that there exists a constant $\epsilon = \epsilon(n) > 0$ such that an $n$-dimensional open manifold with quadratic curvature decay to zero and $\mathcal{H}(p,r) < \epsilon r$ for all $r>0$ is diffeomorphic to $\mathbb{R}^n$. For recent progress on manifolds with quadratic curvature decay, we refer to the paper of Yeganefar [24] for more details.

In this paper, we first obtain the following pinching theorem, which generalizes the result of Wang–Xia in [19]

**Theorem 1.1.** *Given a positive monotone decreasing function $K(r)$, there are positive constants $\epsilon,\delta\in(0,1)$ with $\epsilon+\delta<1$, such that any $n$-dimensional complete open manifold $M$ satisfying*

$$k_p(r) \geq -(K(r))^2,$$

*and*

$$\mathcal{H}(p,r) < \epsilon r - \frac{1}{2K(\delta r)} \ln\left\{\cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1}\right\}, \tag{1.4}$$

*for some $p\in M$ and all $r>0$ is diffeomorphic to $\mathbb{R}^n$.*

Our second theorem is on a Riemannian manifold with large volume growth, i.e. $v_{-c}>0$. There have many articles studying complete noncompact Riemannian manifold with large volume growth (cf. [3], [7], [8], [11], [12], [14]–[23]). If $M$ has nonnegative Ricci curvature, it has been proven by Li [12] and Anderson [3] that $\pi_1(M)$ is finite. Perelman [15] has shown that there is a small constant $\epsilon(n)>0$ depending only on $n$ such that if $v_0 > 1-\epsilon(n)$, then $M$ is contractible, and Cheeger–Colding [7] showed that the manifold in Perelman's theorem is actually diffeomorphic to $\mathbb{R}^n$. Shen [17] has shown that $M$ has finite topological type, provided that $\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} = v_0 + o(\frac{1}{r^{n-1}})$ and, either the conjugate radius

$\mathrm{conj}_M \geq c > 0$ or the sectional curvature $K_M \geq K_0 > -\infty$. Sha–Shen [16] proved that these manifolds have finite topological type if in addition the manifolds have quadratic curvature decay to zero. One can find some other topological uniqueness theorems about $M$, e.g. in [8], [14], [22].

Given a point $p \in M$, we denote by $\mathrm{crit}_p$ the criticality radius of $M$ at $p$, i.e., $\mathrm{crit}_p$ is the smallest critical value for the distance function $d(p,.) : M \to \mathbb{R}$. Recall that $q$ is not a critical point of this distance function iff there exists a vector $\mathbf{v} \in S_q M$ such that for all minimizing geodesics $\sigma$ from $\sigma(0) = q$ to $p$, we have $\angle(\mathbf{v}, \sigma'(0)) > \frac{\pi}{2}$ (cf. [6, 9]). In view of Theorem 1.1, we have the following topological rigidity theorem for manifolds with Ricci curvature bounded from below by a non-positive constant and large volume growth.

**Theorem 1.2.** *Given a positive monotone decreasing function $K(r)$, there are positive constants $\epsilon, \delta \in (0,1)$ with $\epsilon + \delta < 1$, such that any $n$-dimensional complete open manifold $M$ with $\mathrm{Ric}_M \geq -(n-1)c$ $(c \geq 0)$, $v_{-c} > 0$,*

$$k_p(r) \geq -(K(r))^2,$$

*and*

$$\frac{\mathrm{vol}[B(p,r)]}{\alpha_n(r,-c)} < \left\{ 1 + \frac{\int_0^A f_{-c}^{n-1}(t)dt}{\int_0^{2r} f_{-c}^{n-1}(t)dt} \right\} v_{-c}, \tag{1.5}$$

*for some $p \in M$ and all $r > 0$ is diffeomorphic to $\mathbb{R}^n$, where*

$$A = \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right\}.$$

Especially, if $M$ is of nonnegative Ricci curvature, we have

**Theorem 1.3.** *Given a positive monotone decreasing function $K(r)$, there are positive constants $\epsilon, \delta \in (0,1)$ with $\epsilon + \delta < 1$, such that any $n$-dimensional complete open manifold $M$ with $\mathrm{Ric}_M \geq 0$, $v_0 > 0$,*

$$k_p(r) \geq -(K(r))^2,$$

*and*

$$\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} < \left\{ 1 + 2^{2-3n} r^{1-n} \left[ \epsilon r - \frac{1}{2K(\delta r)} \ln \left( \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right) \right]^{n-1} \right\} v_0, \tag{1.6}$$

*for some $p \in M$ and all $r > 0$ is diffeomorphic to $\mathbb{R}^n$.*

*Remark 1.4.* (i) If the curvature of $M$ is quadratic decay to zero, i.e., there exists a constant $C > 0$ such that $K(r)r \leq C$ for all $r > 0$, then we can choose $\epsilon$ in Theorem 1.1 small enough so that (1.4) becomes

$$\mathcal{H}(p,r) < \tilde{\epsilon} r,$$

for some $\tilde{\epsilon} \in (0,1)$ depending on $\epsilon$, $\delta$ and $C$. Then by Theorem 1.1, $M$ is diffeomorphic to $\mathbb{R}^n$, and this recovers Wang–Xia's result in [19].

(ii) If the function $K(r) = Cr^{-\beta}$ for $C > 0, \beta \in [0,1]$ and the Ricci curvature of $M$ is nonnegative, (1.6) can be written in an explicit form, therefore we again obtain a Wang–Xia's type theorem in [19].

## 2. Preliminaries

Let $(M,g)$ be a complete non-compact $n$-dimensional Riemannian manifold. For a fixed point $p \in M$. We say that $K_p^{\min} \geq c$ if for any minimal geodesic $\gamma$ issuing from $p$ all sectional curvatures of planes which are tangent to $\gamma$ are greater than or equal to $c$. For $p, q \in M$, the excess function $e_{pq}(x)$ is defined by

$$e_{pq}(x) = d(p,x) + d(q,x) - d(p,q).$$

We denote by $M^2(c)$ the complete simply connected surface of constant curvature $c$. Throughout this paper, all geodesics are assumed to have unit speed. In [13], Machigashira proved the following Toponogov-type comparison theorem for complete manifolds with $K_p^{\min} \geq c$.

**Lemma 2.1.** *Let $M$ be a complete $n$-manifold and $p$ be a point of $M$ with $K_p^{\min} \geq c$. Let $\gamma_i : [0, l_i] \to M$, $i = 0,1,2$ be minimal geodesics with $\gamma_1(0) = \gamma_2(l_2) = p$, $\gamma_0(0) = \gamma_1(l_1)$ and $\gamma_0(l_0) = \gamma_2(0)$. Then there exist minimal geodesics $\tilde{\gamma}_i : [0, l_i] \to M^2(c)$, $i = 0,1,2$ with $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(l_2)$, $\tilde{\gamma}_0(0) = \tilde{\gamma}_1(l_1)$ and $\tilde{\gamma}_0(l_0) = \tilde{\gamma}_2(0)$ which are such that*

$$L(\gamma_i) = L(\tilde{\gamma}_i) \quad \textit{for } i = 0,1,2,$$

*and*

$$\angle(-\gamma_1'(l_1), \gamma_0'(0)) \geq \angle(-\tilde{\gamma}_1'(l_1), \tilde{\gamma}_0'(0)),$$
$$\angle(-\gamma_0'(l_0), \gamma_2'(0)) \geq \angle(-\tilde{\gamma}_0'(l_1), \tilde{\gamma}_2'(0)).$$

**Lemma 2.2** ([22])**.** *Let $M$ be a complete $n$-manifold with $\mathrm{Ric}_M \geq 0$ and $v_0 > 0$. Then for any $r > 0$ and any $x \in S(p, r)$, we have*

$$d(x, R_p) \leq 2v_0^{-\frac{1}{n}} \left\{ \frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} - v_0 \right\}^{\frac{1}{n}} r.$$

**Lemma 2.3** ([2])**.** *Let $(M, g)$ be a complete $n$-manifold with $\mathrm{Ric}_M \geq 0$. Let $\gamma : [0, a] \to M$ be a minimal geodesic from $p$ to $q$. Then for any $x \in M$*

$$e_{pq}(x) \leq 8 \left( \frac{s^n}{r} \right)^{\frac{1}{n-1}},$$

*where $s = d(x, \gamma)$ and $r = \min(d(p, x), d(q, x))$.*

Let $\Sigma$ be a closed subset of the unit tangent sphere $S_pM$ of $M$ at $p$. Denote by $B_\Sigma(p, r)$ the set of points $x \in B(p, r)$ such that there is a minimizing geodesic $\gamma$ from $p$ to $x$ with $\gamma'(0) \in \Sigma$. For $0 < r \leq \infty$, let $\Sigma_p(r)$ denote the set of unit vectors $\mathbf{v} \in \Sigma$ such that the geodesic $\gamma(t) = \exp_p(t\mathbf{v})$ is minimizing on $[0, r)$. Notice that

$$\Sigma_p(r_2) \subset \Sigma_p(r_1), \text{ for } 0 < r_1 < r_2; \quad \Sigma_p(\infty) = \bigcap_{r>0} \Sigma_p(r).$$

The standard argument [4, 5] gives the following generalized Bishop's comparison theorem.

**Lemma 2.4** ([22])**.** *Let (M, g) be a complete $n$-manifold with $\mathrm{Ric}_M \geq 0$ and $v_0 > 0$. Then*

$$\frac{\mathrm{vol}[B_{\Sigma_p(\infty)}(p, r)]}{\alpha_n(r, 0)} \geq v_0.$$

**Lemma 2.5** ([23])**.** *Let (M, g) be a complete $n$-manifold with $\mathrm{Ric}_M \geq -(n-1)$ and $v_{-1} > 0$. Then*

$$\frac{\mathrm{vol}[B_{\Sigma_p(\infty)}(p, r)]}{\alpha_n(r, -1)} \geq v_{-1}.$$

It is not difficult to check that Lemma 2.5 also holds for $\mathrm{Ric}_M \geq -(n-1)c$ ($c > 0$). We then have the following corollary.

**Corollary 2.6.** *Let (M, g) be a complete $n$-manifold with $\mathrm{Ric}_M \geq -(n-1)c$, $c \geq 0$ and $v_{-c} > 0$. Then for any $p \in M$ and any $r > 0$, we have*

$$\int_0^{\mathcal{H}(p,r)} f_{-c}^{n-1}(t)dt \leq v_{-c}^{-1} \int_0^{2r} f_{-c}^{n-1}(t)dt \left\{ \frac{\mathrm{vol}[B(p,r)]}{\alpha_n(r,-c)} - v_{-c} \right\},$$

*where $f_{-c}(t)$ is defined in (1.2), and $v_{-c}$ is defined in (1.3).*

PROOF. Fix an point $x \in S(p,r)$, and set $s = d(x, R_p)$, then $s \leq r$ and

$$B(p,r) \bigcup B_{\Sigma_p(\infty)}(p,2r) \subset B(p,2r).$$

The left hand side is a disjoint union. We have

$$\mathrm{vol}[B(p,r)] \geq v_{-c}\{\alpha_n(s,-c)\}.$$

From Lemma 2.4 and 2.5, one obtains

$$\begin{aligned} \mathrm{vol}[B(p,2r)] &\geq \mathrm{vol}[B(x,s)] + \mathrm{vol}[B_{\Sigma_p(\infty)}(p,2r)] \\ &\geq v_{-c}\alpha_n(s,-c) + v_{-c}\alpha_n(2r,-c). \end{aligned}$$

By Bishop's comparison theorem, we have

$$\frac{\mathrm{vol}[B(p,2r)]}{\alpha_n(2r,-c)} \leq \frac{\mathrm{vol}[B(p,r)]}{\alpha_n(r,-c)}.$$

We then obtain

$$\int_0^{\mathcal{H}(p,r)} f_{-c}^{n-1}(t)dt \leq v_{-c}^{-1} \int_0^{2r} f_{-c}^{n-1}(t)dt \left\{ \frac{\mathrm{vol}[B(p,r)]}{\alpha_n(r,-c)} - v_{-c} \right\}. \qquad \square$$

## 3. Proofs of Theorems

PROOF OF THEOREM 1.1. We shall prove that $M$ contains no critical points of the distance function $d(p,\cdot)$ other than $p$, and therefore it is diffeomorphic to $\mathbb{R}^n$ (cf. [9], Disk Theorem). We refer to [6], [9], [10] for the notion of critical points of the distance functions and their applications.

For any $\epsilon, \delta \in (0,1)$ with $\epsilon + \delta < 1$ and any $r > 0$, we see

$$\cosh(2K(\delta r)\epsilon r) - \cosh^2(K(\delta r)\epsilon r) = \cosh^2(K(\delta r)\epsilon r) - 1 > 0. \tag{3.1}$$

By definition, $\cosh(2K(\delta r)\epsilon r) = \frac{1}{2}(e^{2K(\delta r)\epsilon r} + e^{-2K(\delta r)\epsilon r})$, we have from (3.1)

$$(e^{2K(\delta r)\epsilon r})^2 - 2\cosh^2(K(\delta r)\epsilon r)e^{2K(\delta r)\epsilon r} + 1 > 0,$$

which implies

$$\epsilon r - \frac{1}{2K(\delta r)} \ln\left\{\cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1}\right\} > 0. \tag{3.2}$$

Take an arbitrary point $x(\neq p) \in M$ and set $r = d(p,x)$. By (1.1), $\mathcal{H}(p,r)$ must be nonnegative. By (3.2), we see that our condition (1.4) is reasonable, and this enables us to find a ray $\gamma : [0,+\infty) \to M$ such that

$$s = d(x,\gamma) < \epsilon r - \frac{1}{2K(\delta r)} \ln\left\{\cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1}\right\}. \quad (3.3)$$

Fix a minimizing geodesic $\sigma$ from $x$ to $q = \gamma(2r)$. For any minimal geodesic $\sigma_1$ from $x$ to $p$, let $\tilde{p} = \sigma_1(\epsilon r)$ and $\tilde{q} = \sigma(\epsilon r)$. The choice of $\epsilon$ and $\delta$ indicates that $\sigma|_{[0,\epsilon r]}$ and $\sigma_1|_{[0,\epsilon r]}$ are disjoint with $B(p,\delta r)$. Moreover, the sectional curvature of $M$ satisfies $K_M \geq -(K(\delta r))^2$ on $M \setminus B(p,\delta r)$. Applying the Toponogov comparison theorem Lemma 2.1 to the hinge $(\sigma|_{[0,\epsilon r]}, \sigma_1|_{[0,\epsilon r]})$ in $M \setminus B(p,\delta r)$, we have

$$\cos\theta \sinh^2(K(\delta r)\epsilon r) \leq \cosh^2(K(\delta r)\epsilon r) - \cosh(K(\delta r)d(\tilde{p},\tilde{q})), \quad (3.4)$$

where $\theta = \angle(\sigma'(0), \sigma_1'(0))$ is the angle of $\sigma$ and $\sigma_1$ at $x$.

Let $m \in \gamma$ be such that $d(x,m) = d(x,\gamma)$, then $m \in \gamma(0,2r)$. It follows from the triangle inequality that

$$\begin{aligned} d(\tilde{p},\tilde{q}) &\geq d(p,q) - d(p,\tilde{p}) - d(q,\tilde{q}) \\ &= d(p,m) + d(q,m) - [d(p,x) - d(\tilde{p},x)] - [d(x,q) - d(x,\tilde{q})] \\ &= 2\epsilon r + [d(p,m) - d(p,x)] + [d(q,m) - d(q,x)] \geq 2\epsilon r - 2d(x,m). \end{aligned}$$

Introducing (3.3) into the above inequality we see that

$$d(\tilde{p},\tilde{q}) > \frac{1}{K(\delta r)} \ln\left\{\cosh^2(K\epsilon r) + \sqrt{\cosh^4(K\epsilon r) - 1}\right\}. \quad (3.5)$$

This implies that

$$\cosh^2(K(\delta r)\epsilon r) - \cosh(K(\delta r)d(\tilde{p},\tilde{q})) < 0. \quad (3.6)$$

From (3.4) and (3.6), we obtain

$$\cos\theta \sinh^2(K(\delta r)\epsilon r) \leq \cosh^2(K(\delta r)\epsilon r) - \cosh(K(\delta r)d(\tilde{p},\tilde{q})) < 0.$$

Thus

$$\theta > \frac{\pi}{2}.$$

Therefore any minimizing geodesic $\sigma_1$, from $x$ to $p$ has $\angle(\sigma_1'(0), \sigma'(0)) > \frac{\pi}{2}$, which implies that $x$ is not a critical point of $d(p,\cdot)$. Theorem 1.1 follows. $\square$

PROOF OF THEOREM 1.2. We take the constants $\epsilon$ and $\delta$ in Theorem 1.2 to be the same as in Theorem 1.1. Therefore in order to prove Theorem 1.2, it suffices to show that

$$\mathcal{H}(p,r) < \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right\}.$$

Since $\mathrm{Ric}_M \geq -(n-1)c$, where $c \geq 0$, we have by Corollary 2.6

$$\int_0^{\mathcal{H}(p,r)} f_{-c}^{n-1}(t)dt \leq v_{-c}^{-1} \int_0^{2r} f_{-c}^{n-1}(t)dt \left\{ \frac{\mathrm{vol}[B(p,r)]}{\alpha_n(r,-c)} - v_{-c} \right\}.$$

Substituting the assumption (1.5) into the above inequality, we have

$$\int_0^{\mathcal{H}(p,r)} f_{-c}^{n-1}(t)dt < \int_0^{A} f_{-c}^{n-1}(t)dt.$$

Thus

$$\mathcal{H}(p,r) < \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right\}.$$

This completes the proof of Theorem 1.2. □

Before proving Theorem 1.3, we need the following lemma.

**Lemma 3.1.** *Given a positive monotone decreasing function $K(r)$, there are positive constants $\epsilon, \delta \in (0,1)$ with $\epsilon + \delta < 1$, such that any $n$-dimensional complete open manifold $M$ with* $\mathrm{Ric}_M \geq 0$,

$$k_p(r) \geq -(K(r))^2,$$

*and*

$$\mathcal{H}(p,r)] < r^{\frac{1}{n}} \left[ \frac{1}{4}\epsilon r - \frac{1}{8K(\delta r)} \ln \left( \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right) \right]^{n-1},$$

*for some $p \in M$ and all $r > 0$ is diffeomorphic to $\mathbb{R}^n$.*

PROOF. We take the constants $\epsilon$ and $\delta$ as same in Theorem 1.1. Fix any point $q \in M$ and set $r = d(p,q)$. We only need to show that $q$ is not a critical point of $d(p,\cdot)$.

Take a point $m \in R_p$ so that $d(q,m) = d(q,R_p)$. Set $s = d(q,m)$. It then follows from the assumption of Lemma 3.1 that

$$s < r^{\frac{1}{n}} \left[ \frac{1}{4}\epsilon r - \frac{1}{8K(\delta r)} \ln \left( \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right) \right]^{n-1}. \tag{3.7}$$

Take a ray $\gamma : [0, +\infty) \to M$ starting from $p$ and passing through $m$. It follows from the triangle inequality that $\min(d(p, q), d(\gamma(t), q)) = r$ for all $t \geq 2r$. Thus $m \in \gamma(0, 2r))$ and so $d(q, \gamma|_{[0,2r]}) = s$. By Lemma 2.3 and (3.7) we have

$$e_{p,\gamma(2r)}(q) < 2\epsilon r - \frac{1}{K(\delta r)} \ln \left( \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right). \quad (3.8)$$

Set $z = \gamma(2r)$ and take a minimal geodesic $\tilde{\sigma}$ from $q$ to $z$. For any minimal geodesic $\tilde{\sigma}_1$ from $q$ to $p$, let $p' = \tilde{\sigma}_1(\epsilon r)$ and $z' = \tilde{\sigma}(\epsilon r)$, and set $\tilde{\theta} = \angle(\tilde{\sigma}'(0), \tilde{\sigma}_1'(0))$. Since $K_M \geq -(K(\delta r))^2$ on $M \setminus B(p, \delta r)$ we can apply the Toponogov comparison theorem to the hinge $\left(\tilde{\sigma}|_{[0,\epsilon r]}, \tilde{\sigma}_1|_{[0,\epsilon r]}\right)$ in $M \setminus B(p, \delta r)$ to get

$$\cos\tilde{\theta} \sinh^2(K(\delta r)\epsilon r) \leq \cosh^2(K(\delta r)\epsilon r) - \cosh(K(\delta r) d(p', z')). \quad (3.9)$$

It follows from the triangle inequality that

$$\begin{aligned} d(p', z') &\geq -d(p, p') - d(z, z') + d(p, z) \\ &= -d(p, q) + d(p', q) - d(q, z) + d(z', q) + d(p, z) = 2\epsilon r - e_{p,z}(q). \end{aligned}$$

Inserting (3.8) into the above inequality and noticing (3.9), one obtains

$$\cos\tilde{\theta} \sinh^2(K(\delta r)\epsilon r) \leq \cosh^2(K(\delta r)\epsilon r) - \cosh(K(\delta r)(2\epsilon r - e_{p,z}(q))) < 0.$$

Thus $\tilde{\theta} > \frac{\pi}{2}$. Consequently, $q$ is not a critical point of $d(p, \cdot)$. Lemma 3.1 follows. □

Proof of theorem 1.3. We take the constants $\epsilon$ and $\delta$ to be the same as in Lemma 3.1. In order to prove Theorem 1.3, it suffices to show that

$$\begin{aligned} \mathcal{H}(p, r) < r^{\frac{1}{n}} \Bigg[ \frac{1}{4}\epsilon r - \frac{1}{8K(\delta r)} \ln \Bigg( & \cosh^2(K(\delta r)\epsilon r) \\ & + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \Bigg) \Bigg]^{n-1}. \end{aligned} \quad (3.10)$$

By Lemma 2.2

$$d(x, R_p) \leq 2 v_0^{-\frac{1}{n}} \left\{ \frac{\mathrm{vol}[B(p, r)]}{\omega_n r^n} - v_0 \right\}^{\frac{1}{n}} r.$$

By the definition (1.1), it then follows from the assumption (1.6) of Theorem 1.3 that (3.10) holds, and therefore we complete the proof of the theorem. □

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