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Finsler angle-preserving connection in dimensions $N \geq 3$

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Abstract. The Finsler space is considered to be the deformated Riemannian space under the condition that the indicatrix of the Finsler space is a space of constant curvature. In this case, the Finslerian two-vector angle can explicitly be found, which gives rise to simple and explicit representation for the connection preserving the angle in the indicatrix-homogeneous case. The connection is metrical and the Finsler space is obtainable from the Riemannian space by means of the parallel deformation. Since also the transitivity of covariant derivative holds, in such Finsler spaces the metrical non-linear angle-preserving connection is the respective export of the metrical linear Riemannian connection. In case of the $\mathcal{F}S$-space, the example can be developed which entails the explicit connection coefficients and the metric function of the Finsleroid type.

1. Motivation and description

In any dimension $N \geq 3$ the Finsler metric function F geometrizes the tangent bundle TM over the base manifold M such that at each point $x \in M$ the tangent space T_xM is endowed with the curvature tensor constructed from the respective Finslerian metric tensor $g_{\{x\}}(y)$ by means of the conventional rule of the Riemannian geometry considering y to be the variable argument. There arises the Riemannian space $R_{\{x\}} = \{g_{\{x\}}(y), T_xM\}$ supported by the point $x \in M$ such that T_xM plays the role of the base manifold for the space. In the Riemannian limit of the Finsler space, the spaces $\mathcal{R}_{\{x\}}$ are Euclidean spaces and the tensor $g_{\{x\}}(y)$ is independent of y. The conformally flat structure of the spaces $\mathcal{R}_{\{x\}}$ can naturally be taken to treat as the next level of generality of the Finsler space.

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Can the metrical connection preserving the two-vector angle be introduced on that level?

The deformation of the Riemannian space to the Finsler space proves to be convenient invention to apply. Namely, when the Riemannian space can be deformated to the Finsler space characterized by the conformally flat structure of the spaces $\mathcal{R}_{\{x\}}$ the positive and clear answer to the above question can be arrived at.

Given an N-dimensional Riemannian space $\mathcal{R}^N = (M, S)$, where S denotes the Riemannian metric function, one may endeavor to obtain a Finsler space $\mathcal{F}^N = (M, F)$ by applying an appropriate *deformation* $\mathbf{C}$ of the space $\mathcal{R}^N$. The notation F stands for the Finsler metric function. The base manifold M is keeping the same for both the spaces, $\mathcal{R}^N$ and $\mathcal{F}^N$.

We assume that the transformation $\mathbf{C}$ is *restrictive*, in the sense that no point $x \in M$ is shifted under the transformation, so that in each tangent space T_xM the deformation maps tangent vectors $y \in T_xM$ into the tangent vectors of the same T_xM:

$$y = \mathbf{C}(x, \bar{y}), \quad y, \bar{y} \in T_xM. \tag{1.1}$$

In general, this transformation is non-linear with respect to $\bar{y}$. Non-singularity and sufficient smoothness are always implied.

We may evidence in the Riemannian space $\mathcal{R}^N$ the *metrical linear Riemannian connection* $\mathcal{R}L$, which in terms of local coordinates $\{x^i\}$ introduced in M is given by

$$\mathcal{R}L = \{L^m{}_j, L^m{}_{ij}\}: \qquad L^m{}_j = -a^m{}_{ij}y^i, \quad L^m{}_{ij} = a^m{}_{ij}, \tag{1.2}$$

with $a^m{}_{ij} = a^m{}_{ij}(x)$ standing for the Christoffel symbols constructed from the Riemannian metric tensor $a_{mn}(x)$ of the space $\mathcal{R}^N$. The indices $i, j, \ldots$ are specified on the range $(1, \ldots, N)$. The respective covariant derivative ∇ can be introduced in the natural way, namely by means of the definition (4.11) which involves the action of the operator

$$d_i^{\mathrm{Riem}} = \frac{\partial}{\partial x^i} + L^k{}_i \frac{\partial}{\partial y^k}, \tag{1.3}$$

considering tensors on the tangent bundle underlined the space $\mathcal{R}^N$. In the space, the scalar product $\langle y_1, y_2 \rangle_{\{x\}}^{\mathrm{Riem}} = a_{mn}(x) y_1^m y_2^n$ of two vectors y_1, y_2 supported by a fixed point $x \in M$ is linear with respect to each vector, which gives rise to the profound meaning of the connection (1.2) to preserve the product under the entailed parallel transports of the entered vectors along curves running on M.

For a given function F we can construct the covariant tangent vector $\hat{y} = \{y_i\}$ and the Finslerian metric tensor $\{g_{ij}\}$ in the ordinary way: $y_i := (1/2)\partial F^2/\partial y^i$ and $g_{ij} := \partial y_i/\partial y^j$. The contravariant tensor $\{g^{ij}\}$ defined by the reciprocity conditions $g_{ij}g^{jk} = \delta_i^k$, where δ stands for the Kronecker symbol.

In the Finsler space, the scalar product is essentially non-linear object with respect to the entered vectors, so that we may hope to meet similar preservation property in the Finslerian domains if only we apply the connection which is non-linear, in the sense that the involved connection coefficients depend on tangent vectors y in non-linear way. With this hope, we need the *metrical non-linear Finsler connection* $\mathcal{F}N$, such that

$$\mathcal{F}N = \{N^m{}_i, D^m{}_{ij}\}: \qquad N^m{}_i = N^m{}_i(x,y), \qquad D^m{}_{ij} = D^m{}_{ij}(x,y). \tag{1.4}$$

The adjective "metrical" means that the action of the entailed covariant derivative $\mathcal{D}$ on the Finsler metric function, and also on the Finsler metric tensor, yields identically zero. The coefficients $N^m{}_i$ and $D^m{}_{ij}$ are assumed to be positively homogeneous regarding the dependence on vectors y, respectively of degree 1 and degree 0.

In this respect, the most important object what should be lifted from the Riemannian to Finslerian space is the two-vector angle, to be denoted by $\alpha_{\{x\}}(y_1, y_2)$, where $y_1, y_2 \in T_xM$. Like to the Riemannian geometry proper, the underlined idea is to measure the angle by means of length of the respective geodesic arcs evidenced on the indicatrix. The Finsler space endows the vector pair y_1, y_2 with the scalar product

$$\langle y_1, y_2 \rangle_{\{x\}} = F(x, y_1)F(x, y_2)\cos\alpha_{\{x\}}(y_1, y_2) \tag{1.5}$$

on analogy of the Riemannian geometry.

The non-linear deformation

$$\mathcal{F}N = \mathbf{C} \cdot \mathcal{R}L \tag{1.6}$$

of the Riemannian connection may exist to yield the Finsler connection $\mathcal{F}N$ which preserves the Finslerian two-vector angle $\alpha_{\{x\}}(y_1, y_2)$ under the associated parallel transports of the vectors y_1, y_2.

In the theory of Finsler spaces, the key objects, the connection included, were introduced and studied on the basis of various convenient sets of axioms (see [1]–[5] and references therein). Regarding the significance of the angle notion, the important farther step was made in [6] were in processes of studying implications of the two-vector angle defined by area, the theorem was proved which states that a diffeomorphism between two Finsler spaces is an isometry iff it keeps the angle.

This Tamássy's theorem substantiates the idea to develop the Finsler connection from the Finsler two-vector angle, possibly on the analogy of the Riemannian geometry. The idea should influence the researchers to develop the Finsler geometry anew, namely by proceeding from the starting-point: *The Priority of the two-vector angle over the connection.*

To meet new methods of applications, the interesting chain of linear connections was introduced and studied in [3]. It was emphasized that in the Riemannian geometry we have naturally the metrical and linear connection. We depart from this connection to develop the Finsler connection by the help of an outstanding non-linear deformation.

Namely, we shall confine our attention to the case when the space $\mathcal{F}^N$ is obtainable from the space $\mathcal{R}^N$ by means of the deformation which is specified by the stipulations (2.1)–(2.3) of Section 2. We assume that under the used transformations the Finslerian indicatrix $\mathcal{I}F_{\{x\}} \in T_xM$ and the Riemannian sphere $\mathcal{S}_{\{x\}} \in T_xM$ are in correspondence (according to (2.2)). Also, we subject the applied transformations to the condition of positive homogeneity with respect to tangent vectors y, denoting the homogeneity degree by H.

Remarkably, such Finsler spaces of dimensions $N \geq 3$ can be characterized by the condition that the indicatrix is a space of constant curvature (see Proposition 2.1). *The indicatrix curvature value is the square of the homogeneity degree* H (which is indicated in Proposition 2.1). The relevant conformal multiplier p^2 is constructed from the Finsler metric function F, according to $p = (1/H)F^{1-H}$.

The condition has been realized, the Finslerian two-vector angle $\alpha_{\{x\}}(y_1, y_2)$ proves to be a factor of the angle operative traditionally in the Riemannian space, namely the simple equality

$$\alpha_{\{x\}}(y_1, y_2) = \frac{1}{H(x)}\alpha^{\mathrm{Riem}}_{\{x\}}(\bar{y}_1, \bar{y}_2) \tag{1.7}$$

(see (2.26)–(2.28)) is obtained.

The equality

$$S(x, \bar{y}) = (F(x, y))^H \tag{1.8}$$

is arisen (see the last part of the proof of Proposition 2.1), which validates the indicatrix correspondence principle (2.2); $S(x, \bar{y}) = \sqrt{a_{mn}(x)\bar{y}^m\bar{y}^n}$.

We set forth the conventional requirement of preservation of the Finsler metric function $F(x, y)$, namely $d_iF = 0$ with

$$d_i = \frac{\partial}{\partial x^i} + N^k{}_i(x, y)\frac{\partial}{\partial y^k}. \tag{1.9}$$

With the natural definition $\mathcal{D}y^n := dy^n - N^n{}_j(x,y)dx^j$ of covariant displacement of the tangent vector, the parallel transport of the vector means the vanishing

$$\mathcal{D}y^n = 0. \tag{1.10}$$

We apply this observation to the two-vector angle $\alpha_{\{x\}}(y_1, y_2)$: the coefficients $N^k{}_i(x,y)$ fulfill the *angle preservation equation*

$$d_i \alpha_{\{x\}}(y_1, y_2) = 0, \quad y_1, y_2 \in T_x M \tag{1.11}$$

under the parallel displacements of the entered vectors y_1 and y_2, if the involved operator d_i is taken to read

$$d_i = \frac{\partial}{\partial x^i} + N^k{}_i(x, y_1)\frac{\partial}{\partial y_1^k} + N^k{}_i(x, y_2)\frac{\partial}{\partial y_2^k}. \tag{1.12}$$

The $N^k{}_i(x,y)$ thus appeared can naturally be interpreted as the *coefficients of the non-linear connection produced by angle.*

In this way we fulfill the canonical geometrical principle: the angle $\alpha_{\{x\}}(y_1, y_2)$ formed by two vectors y_1 and y_2 is left unchanged under the parallel displacements of the vectors y_1 and y_2, namely $\mathcal{D}\alpha \stackrel{\text{def}}{=} (dx^i)d_i\alpha = 0$, for $d_i\alpha = 0$.

When $d_i F = 0$ is fulfilled, the vanishings $d_i \alpha_{\{x\}}(y_1, y_2) = 0$ and $d_i \langle y_1, y_2\rangle_{\{x\}} = 0$ reciprocally determine each other (see (1.5)). We may say that the notion "the metrical Finsler angle-preserving connection" is tantamount to the notion "the connection which preserves the Finsler scalar product".

In general the indicatrix curvature value $\mathcal{C}_{\text{Ind.}}$ may depend on the points $x \in M$. We say that the space $\mathcal{F}^N$ is *indicatrix-homogeneous*, if the value is a constant. In view of the result $\mathcal{C}_{\text{Ind.}} \equiv H^2$ such spaces can be characterized by the condition that the homogeneity degree H of the underlined transformation is independent of x.

It proves that *in the indicatrix-homogeneous case of the studied space* $\mathcal{F}^N$ *the equations* (1.11)–(1.12) *can explicitly be solved for the coefficients* $N^k{}_i$ (see Proposition 2.2 and Note placed thereafter in Section 2).

From the obtained coefficients $N^k{}_m$ given by (2.30), the entailed coefficients

$$N^k{}_{mn} = \frac{\partial N^k{}_m}{\partial y^n}, \quad N^k{}_{mnj} = \frac{\partial N^k{}_{mn}}{\partial y^j} \tag{1.13}$$

can straightforwardly be evaluated (Section 3). Let us use the coefficients to construct the covariant derivative $\mathcal{D}_m g_{nj}$ of the Finsler metric tensor $g_{nj} = g_{nj}(x,y)$ of the considered space $\mathcal{F}^N$, namely

$$\mathcal{D}_m g_{nj} := d_m g_{nj} + N^k{}_{mj} g_{kn} + N^k{}_{mn} g_{kj}, \tag{1.14}$$

where d_m is given by (1.9). It proves that the covariant derivative introduced by (1.14) with the coefficients ${N^k}_m$ given by (2.30) possesses the property $\mathcal{D}_m g_{nj} = 0$ in the indicatrix-homogeneous case. The property can be verified by straightforward substitutions which result in the vanishing

$$y_k {N^k}_{mnj} = 0 \tag{1.15}$$

(see Proposition 3.1).

It is amazing but the fact that the last vanishing is an implication of the identity $y^k C_{knj} = 0$ shown by the Cartan tensor $C_{knj} = (1/2)\partial g_{kn}/\partial y^j$. Indeed, additional evaluation leads to the result

$${N^k}_{mnj} = -\mathcal{D}_m {C^k}_{nj} \tag{1.16}$$

in the indicatrix-homogeneous case (see Proposition 3.2), where

$$\mathcal{D}_m {C^k}_{nj} := d_m {C^k}_{nj} - {N^k}_{mt} {C^t}_{nj} + {N^t}_{mn} {C^k}_{tj} + {N^t}_{mj} {C^k}_{nt}. \tag{1.17}$$

The coefficients $N_{kmnj} = g_{kh} {N^h}_{mnj}$ can be written as $N_{kmnj} = -\mathcal{D}_m C_{knj}$ and, therefore, they are *symmetric* with respect to the subscripts k, n, j.

Thus, with the identification

$${D^k}_{in}(x,y) = -{N^k}_{in}(x,y), \tag{1.18}$$

in the Finsler space $\mathcal{F}^N$ of the indicatrix-homogeneous type (that is, when $H =$ const) the metrical angle-preserving connection (1.4) is given by the coefficients $\{N^k_i, {D^k}_{in}\}$ found explicitly. Recollecting the assumed homogeneity of the coefficients, from (1.18) we infer the equality

$${D^k}_{in} y^n = -{N^k}_i. \tag{1.19}$$

Realizing the **C**-transformation locally by $y^i = y^i(x,t)$ with $t^n \equiv \bar{y}^n$ (see (2.8)), it is possible to conclude that

$${N^n}_i = d_i^{\mathrm{Riem}} y^n \tag{1.20}$$

(see (2.41)). This representation of the coefficients ${N^n}_i$ manifests a clear geometrical and tensorial meaning and is alternative (and equivalent) to the representation (2.30). The derivation of the representation (1.20) uses the formula (1.19).

According to Proposition 2.3, the Finsler space $\mathcal{F}^N$ of the indicatrix-homogeneous type is obtained from the Riemannian space $\mathcal{R}^N$ by means of the *parallel deformation.*

Since also the transitivity of covariant derivative holds, namely $\mathcal{D}_n t^i = 0$ (see (2.33)), and $g_{kh} = C^m_k C^n_h a_{mn}$ (see (2.21)), we should conclude that in the Finsler space $\mathcal{F}^N$ of the indicatrix-homogeneous type the metrical angle-preserving connection is the **C**-export of the metrical linear Riemannian connection (1.2) applied conventionally in the background Riemannian space $\mathcal{R}^N$.

In Section 4 we perform the attentive comparison between the commutators of the involved Finsler covariant derivative $\mathcal{D}$ and the commutators of the underlined Riemannian covariant derivative ∇, assuming $H =$ const. By this method, we derive the associated curvature tensor $\rho_k{}^n{}_{ij}$. The skew-symmetry $\rho_{mnij} = -\rho_{nmij} = -\rho_{mnji}$ holds. The covariant derivative $\mathcal{D}_l$ of the tensor fulfills the cyclic identity, completely similar to the Riemannian case in which the cyclic identity is valid for the derivative $\nabla_l a_k{}^n{}_{ij}$ of the Riemannian curvature tensor $a_k{}^n{}_{ij}$. The tensor $M^n{}_{ij} = -y^k \rho_k{}^n{}_{ij}$ proves to be transitive to the Riemannian tensor $-t^h a_h{}^t{}_{ij}$, namely the equality $M^n{}_{ij} = -y^n_t t^h a_h{}^t{}_{ij}$ holds. The very tensor $\rho_k{}^n{}_{ij}$ is not transitive to the Riemannian precursor $a_h{}^m{}_{ij}$, instead the more general equality (4.19) is obtained. The difference between the curvature tensor $\rho_k{}^n{}_{ij}$ and the transitive term $y^n_m a_h{}^m{}_{ij} t^h_k$ is proportional to $(1 - H)$. The square $\rho^{knij}\rho_{knij}$ of the tensor is the sum (4.23) which is the $\mathcal{F}^N$-extension of the Riemannian term $a^{knij}a_{knij}$. The difference $\rho^{knij}\rho_{knij} - a^{knij}a_{knij}$ is proportional to $(H^{-2} - 1)$.

In Section 5 we develop an explicit and attractive particular case, namely we present the explicit example (5.9) of the transformations possessing the studied properties, specializing the Finsler space to be the $\mathcal{FS}$-space. The latter space is endowed with the Finsler metric function F which is constructed from a Riemannian metric function $S = \sqrt{a_{ij}(x)y^i y^j}$ and an 1-form $b = b_i(x)y^i$ according to the functional dependence

$$F(x, y) = \Phi\left(x; b, S, y\right), \tag{1.21}$$

where Φ is a sufficiently smooth scalar function. In step-by-step way, we derive the coefficients $N^m{}_i$ specified by (2.30), obtaining the explicit representation (5.31)–(5.32). It proves that the suitability of the transformation (5.9) imposes the severe restriction on the Finsler metric function, namely the function must be of the Finsleroid type (described in [7]). In the restricted case which implies independence of the function $\Phi\left(x; b, S, y\right)$ of x, assuming also that the Riemannian norm of the 1-form b is a constant, the obtained coefficients $N^m{}_i$ straightforwardly entail the vanishing set $\mathcal{D}_n F = \mathcal{D}_n y_j = \mathcal{D}_n g_{ij} = 0$ together with the angle preservation (1.11). The t^m which enter the transformation (5.9) are linear combinations of the unit vectors l^m and m^m (see (5.35) and (5.36)).

In Conclusions, Section 6, we emphasize several important ideas.

2. Initial observations

Below, *any dimension* $N \geq 3$ is allowable.

Let M be an N-dimensional C^∞ differentiable manifold, T_xM denote the tangent space to M at a point $x \in M$, and $y \in T_xM \backslash 0$ mean tangent vectors. Suppose we are given on the tangent bundle TM a Riemannian metric S. Denote by $\mathcal{R}^N = (M, S)$ the obtained N-dimensional Riemannian space. Let additionally a Finsler metric function F be introduced on this TM, yielding a Finsler space $\mathcal{F}^N = (M, F)$. We shall study the Finsler space $\mathcal{F}^N$ specified according to the following definition.

Input Definition. The Finsler space $\mathcal{F}^N$ under consideration is the *deformated Riemannian space* $\mathcal{R}^N$:

$$\mathcal{F}^N = \mathbf{C} \cdot \mathcal{R}^N, \tag{2.1}$$

specified by the condition that in each tangent space T_xM the metric tensor $g_{\{x\}}(y)$ produced by the Finsler metric is conformal to the Euclidean metric tensor entailed by the Riemannian metric of the space $\mathcal{R}^N$. It is assumed that the applied $\mathbf{C}$-transformations do not influence any point $x \in M$ of the base manifold M and that they are invertible. It is also natural to require that the $\mathbf{C}$-transformations send unit vectors to unit vectors:

$$\mathcal{I}F_{\{x\}} = \mathbf{C} \cdot \mathcal{S}_{\{x\}}. \tag{2.2}$$

Additionally, we subject the $\mathbf{C}$-transformations to the condition of positive homogeneity with respect to tangent vectors y, denoting the degree of homogeneity by H. If $f(x,y)$ is the involved conformal multiplier, then from the equality $g_{\{x\}}(y) = f(x,y)u_{\{x\}}(y)$ we obtain the tensor $u_{\{x\}}(y)$ which possesses the property: if we construct from the tensor $u_{\{x\}}(y)$ the Riemannian curvature tensor $\widetilde{R}_{\{x\}}(y)$ regarding y^i as variables, we arrive at the vanishing $\widetilde{R}_{\{x\}}(y) = 0$. Finally, the dependence of the multiplier f on the variable y is assumed to be presented by the power of the Finsler metric function, such that

$$\begin{gathered} g_{\{x\}}(y) = p^2 u_{\{x\}}(y), \quad p = c_1(x)\,(F(x,y))^{a(x)}\,, \\ 1 > a(x) > 0, \quad c_1(x) > 0. \end{gathered} \tag{2.3}$$

Let the $\mathbf{C}$-transformation (2.1) be assigned locally by means of the differentiable functions

$$\bar{y}^m = \bar{y}^m(x,y), \tag{2.4}$$

subject to the required homogeneity

$$\bar{y}^m(x, ky) = k^H \bar{y}^m(x, y), \quad k > 0, \forall y. \tag{2.5}$$

This entails the identity

$$\bar{y}^m_k y^k = H\bar{y}^m, \tag{2.6}$$

where $\bar{y}^m_k = \partial\bar{y}^m/\partial y^k$. Fulfilling (2.1) means locally

$$g_{mn}(x, y) = c_{ij}(x, \bar{y})\bar{y}^i_m \bar{y}^j_n, \quad c_{ij}(x, \bar{y}) = (p(x, y))^2\, a_{ij}(x). \tag{2.7}$$

On every punctured tangent space $T_xM \setminus 0$, the Finsler metric function F is assumed to be positive, and also positively homogeneous of degree 1: $F(x, ky) = kF(x, y)$, $k > 0$, $\forall y$. The entailed Finslerian metric tensor is positively homogeneous of degree 0. Therefore, to comply the representation (2.7) with the stipulation (2.3), we must put

$$H = 1 - a.$$

The existence of the Finsler spaces under study is explained by the following proposition.

Proposition 2.1. *A Finsler space is of the claimed type $\mathcal{F}^N$ if and only if the indicatrix of the Finsler space is a space of constant curvature. Denoting the indicatrix curvature value by $\mathcal{C}_{\text{Ind.}}$, the equality $\mathcal{C}_{\text{Ind.}} \equiv H^2$ is obtained. The relevant conformal multiplier is given by p^2 with $p = (1/H)F^{1-H}$.*

Proof. Constructing from the tensor $u_{ij}(x,y){=}g_{ij}(x,y)/(c_1(x)(F(x,y))^{a(x)})$ the Riemannian curvature tensor $\widetilde{R}_n{}^h{}_{ij}$ by regarding y^i as variables, simple straightforward evaluations (which are presented in Appendix A in [10]) lead to the equality

$$F^2\widetilde{R}_n{}^m{}_{ij} = S_n{}^m{}_{ij} - a(2-a)(h_{nj}h^m_i - h_{ni}h^m_j),$$

where $h_{nj} = g_{nj} - (1/F^2)y_n y_j$ and $S_n{}^m{}_{ij} = -\left(C^h{}_{ni}C^m{}_{hj} - C^h{}_{nj}C^m{}_{hi}\right)F^2$. The tensor $S_n{}^m{}_{ij}$ describes the curvature of indicatrix. It is known that the indicatrix is a space of constant curvature if and only if the tensor fulfills the equality $S_n{}^m{}_{ij} = C(h_{nj}h^m_i - h_{ni}h^m_j)$ with the factor C which is independent of y, in which case $\mathcal{C}_{\text{Ind.}} = 1 - C$ (see Section 5.8 in [1]). Since the vanishing $\widetilde{R}_n{}^m{}_{ij} = 0$ is equivalent to the equality $S_n{}^m{}_{ij} = C(h_{nj}h^m_i - h_{ni}h^m_j)$ with $C = a(2-a)$, we get $\mathcal{C}_{\text{Ind.}} = (1-a)^2 = H^2$. Also, contracting the g_{mn} by $y^m y^n$ and noting the involved homogeneity, we get the equality $(F(x,y))^2 = (H(x))^2\,(p(x,y))^2\,(S(x,\bar{y}))^2$ (see (2.6) and (2.7)), so that

$$p(x, y) = \frac{1}{H(x)}\frac{F(x, y)}{S(x, \bar{y})}.$$

Therefore, to obey the indicatrix correspondence (2.2), we should put $c_1 = 1/H$ in (2.3), which enables us to have the equality $S(x, \bar{y}) = (F(x, y))^H$ indicated in (1.8). Proposition 2.1 is valid. $\square$

The proposition is of the local meaning in both the base manifold and the tangent space. The value $\mathcal{C}_{\mathrm{Ind.}}$ may vary from point to point of the manifold M, so that in general $H = H(x)$. We take $\mathcal{C}_{\mathrm{Ind.}} > 0$. Extension of the proposition to negative value of $\mathcal{C}_{\mathrm{Ind.}}$ would be a straightforward task.

Our consideration is based essentially on the notion of indicatrix and, therefore, the conclusions obtained can be addressed to the dimensions $N \geq 3$ excluding the two-dimensional case $N = 2$. The notion of indicatrix is unapplicable to two-dimensional Finsler spaces. Instead, in the latter case of spaces the theory displays the structural role of the so-called "main scalar" and a possibility to measure the angle by means of the arc-length of the indicatrix. This handy possibility can be exploited to propose the angle-preserving connection explicitly for arbitrary two-dimensional Finsler metric function, at least on the local level of consideration [9]. In Finsler spaces of dimensions $N \geq 3$ the situation is cardinally more complicated. Although the basic principles can still be outlined to define the angle-preserving connection, the required connection coefficients can in general be introduced in only implicit way. Under these circumstances, it is urgent to "do the start-up" aimed to single out the particular classes of Finsler spaces in which the respective connection coefficients can be proposed in an explicit way. Clearly, the knowledge of such coefficients is to be preceded by the knowledge of the involved two-vector angle. Since in the Finsler spaces characterized by the condition that the indicatrix is of constant curvature the angle appears explicitly (see (1.7)), such spaces are attractive cases to investigate with utmost attention in any dimension $N \geq 3$.

Denote by

$$y^i = y^i(x, t), \quad t^n \equiv \bar{y}^n, \tag{2.8}$$

the inverse transformation, so that

$$y^i(x, kt) = k^{1/H} y^i(x, t), \quad k > 0, \forall t,$$

and

$$y^i_n t^n = \frac{1}{H} y^i, \tag{2.9}$$

where $y^i_n = \partial y^i / \partial t^n$. The inverse to (2.7) reads:

$$g_{kh} y^k_m y^h_n = c_{mn}. \tag{2.10}$$

The following useful relations can readily be arrived at:

$$y_m y_n^m = \frac{F^2}{HS^2} t_n \equiv \frac{1}{H} F^{2(1-H)} t_n, \qquad t_n = a_{nh} t^h, \tag{2.11}$$

and $y_m y_{nl}^m t_j^l + g_{mj} y_n^m = 2\left(H^{-1} - 1\right) F^{-2H} y_j t_n + (1/H) F^{2(1-H)} a_{nh} t_j^h$, where $t_j^l = \bar{y}_j^l$ and $y_{nl}^m = \partial y_n^m / \partial y^l$. Alternatively,

$$t_h t_n^h = \frac{HS^2}{F^2} y_n \equiv H F^{2(H-1)} y_n \tag{2.12}$$

and

$$t_h t_{nu}^h y_i^u + a_{hi} t_n^h = 2(H-1) F^{-2} t_i y_n + H F^{2(H-1)} g_{nu} y_i^u, \tag{2.13}$$

where $t_{nu}^h = \partial t_n^h / \partial y^u$. We may also write

$$t_h t_{ni}^h = H(1-H) F^{2(H-1)} (g_{ni} - 2 l_n l_i). \tag{2.14}$$

From (2.10) it follows that $g_{nm} y_i^m = p^2 t_n^j a_{ij}$.

Differentiating (2.7) with respect to y^k yields the following representation for the tensor $C_{mnk} = (1/2) \partial g_{mn} / \partial y^k$:

$$2C_{mnk} = (1-H) \frac{2}{F} l_k g_{mn} + p^2 (t_{mk}^i t_n^j + t_m^i t_{nk}^j) a_{ij}. \tag{2.15}$$

Contracting this tensor by y^n results in the equality

$$p^2 t_{mk}^i t^j a_{ij} = \left(\frac{1}{H} - 1\right) (h_{km} - l_k l_m), \tag{2.16}$$

where the vanishing $C_{mnk} y^n = 0$ and the homogeneity identity (2.6) have been taken into account. Symmetry of the tensor C_{mnk} demands

$$(1-H) \frac{2}{F} (l_k g_{mn} - l_m g_{kn}) + p^2 (t_m^i t_{nk}^j - t_k^i t_{nm}^j) a_{ij} = 0, \tag{2.17}$$

so that we may alternatively write

$$C_{mnk} = (1-H) \frac{1}{F} (l_k g_{mn} + l_n g_{mk} - l_m g_{nk}) + p^2 t_m^i t_{nk}^j a_{ij}. \tag{2.18}$$

Contracting the last tensor by g^{nk} yields

$$F C_m = -(N-2)(1-H) l_m + F g^{nk} p^2 t_{nk}^i t_m^j a_{ij}. \tag{2.19}$$

The space $\mathcal{F}^N$ is obtainable from the Riemannian space $\mathcal{R}^N$ by means of the deformation which, owing to (2.7), can be presented by the *conformal deformation tensor*

$$C^i_m := p\bar{y}^i_m, \tag{2.20}$$

so that

$$g_{mn} = C^i_m C^j_n a_{ij}. \tag{2.21}$$

The zero-degree homogeneity

$$C^i_m(x, ky) = C^i_m(x, y), \qquad k > 0, \forall y, \tag{2.22}$$

holds, together with

$$C^i_m(x, y)y^m = (F(x, y))^{1-H}\, \bar{y}^i. \tag{2.23}$$

The indicatrix correspondence (2.2) is a direct implication of the equality (1.8). We may apply the transformation (1.1) to the unit vectors:

$$l = \mathbf{C} \cdot L : l^i = y^i(x, L); \qquad L = \mathbf{C}^{-1} \cdot l : L^i = t^i(x, l), \tag{2.24}$$

where $l^i = y^i/F(x, y)$ and $L^i = t^i/S(x, t)$ are components of the respective Finslerian and Riemannian unit vectors, which possess the properties $F(x, l) = 1$ and $S(x, L) = 1$. We have $L^m = t^m(x, l)$. On the other hand, from (2.7) it just follows that

$$g_{mn}(x, l) = \frac{1}{H^2} a_{ij}(x) t^i_m(x, l) t^j_n(x, l), \tag{2.25}$$

so that under the transformation (2.24) we have

$$g_{mn}(x, l) dl^m dl^n = \frac{1}{H^2} a_{ij}(x) dL^i dL^j. \tag{2.26}$$

Note. The deformation performed by the formulas (2.20) and (2.21) is *unholonomic*, in the sense that

$$\frac{\partial C^i_m}{\partial y^n} - \frac{\partial C^i_n}{\partial y^m} \neq 0.$$

The vanishing appears if only the factor $p = F^{1-H}/H$ is independent of the vectors y, that is, when $H = 1$ (which is the Riemannian case proper). Regarding the y-dependence, the tensor C^i_m is homogeneous of degree zero, in accordance with (2.22). If we divide the tensor by p, we obtain from (2.20) the tensor $\bar{y}^i_m$ which is the derivative tensor, namely $\bar{y}^i_m = \partial\bar{y}^i/\partial y^m$. However, such a property cannot be addressed to the tensor C^i_m. It is the reason why we start with the stipulation that the underlined transformation (which is downloaded locally by the formulas (2.4)–(2.7)) be homogeneous of the degree H with respect to the variable y. By proceeding in this way, it proves possible to come to the conformal representation (2.26) of $g_{mn}(x,l)dl^m dl^n$ which is of the key significance to obtain the angle and the connection coefficients.

No support vector enters the right-hand part of (2.26). Therefore, any two nonzero tangent vectors $y_1, y_2 \in T_xM$ in a fixed tangent space T_xM form the $\mathcal{F}^N$*-space angle*

$$\alpha_{\{x\}}(y_1, y_2) = \frac{1}{H(x)} \arccos \lambda, \tag{2.27}$$

where the scalar

$$\lambda = \frac{a_{mn}(x)t_1^m t_2^n}{S_1 S_2}, \text{ with } t_1^m = t^m(x, y_1) \quad \text{and } t_2^m = t^m(x, y_2), \tag{2.28}$$

is of the entire Riemannian meaning in the space $\mathcal{R}^N$; the notation $S_1 = \sqrt{a_{mn}(x)t_1^m t_1^n}$, $S_2 = \sqrt{a_{mn}(x)t_2^m t_2^n}$ has been used.

From (2.28) it follows that

$$\frac{\partial \lambda}{\partial x^i} = \frac{a_{mn,i}t_1^m t_2^n}{S_1 S_2} + \frac{1}{S_1 S_2} a_{mn} \left(\frac{\partial t_1^m}{\partial x^i} t_2^n + t_1^m \frac{\partial t_2^n}{\partial x^i} \right)$$
$$-\frac{1}{2}\lambda \left[\frac{1}{S_1 S_1} \left(a_{mn,i}t_1^m t_1^n + 2a_{mn} \frac{\partial t_1^m}{\partial x^i} t_1^n \right) + \frac{1}{S_2 S_2} \left(a_{mn,i}t_2^m t_2^n + 2a_{mn} \frac{\partial t_2^m}{\partial x^i} t_2^n \right) \right],$$

where $a_{mn,i} = \partial a_{mn}/\partial x^i$, and

$$\frac{\partial \lambda}{\partial y_1^k} = \left[\frac{a_{mn}t_2^n}{S_1 S_2} - \frac{a_{mn}t_1^n}{S_1 S_1} \lambda \right] t_{1k}^m, \quad \frac{\partial \lambda}{\partial y_2^k} = \left[\frac{a_{mn}t_1^n}{S_2 S_1} - \frac{a_{mn}t_2^n}{S_2 S_2} \lambda \right] t_{2k}^m.$$

Let the coefficients $N^k{}_i$ be subjected to the equation

$$d_i \lambda = 0, \tag{2.29}$$

where d_i is the operator (1.12). Using the above representation of the derivatives of λ, it proves possible to establish the validity of the following proposition.

Proposition 2.2. *When* $d_iF = 0$ *and* $H = \text{const}$*, the equation* (2.29) *can be solved for the coefficients* $N^m{}_n$*, yielding*

$$N^m{}_n = -y_i^m \left(\frac{\partial t^i}{\partial x^n} + a^i{}_{kn} t^k \right). \tag{2.30}$$

See Appendix B in [10].

In (2.30), the $a^i{}_{kn} = a^i{}_{kn}(x)$ are the Christoffel symbols

$$a^i{}_{kn} = \frac{1}{2} a^{ih} \Big(\frac{\partial a_{hk}}{\partial x^n} + \frac{\partial a_{hn}}{\partial x^k} - \frac{\partial a_{kn}}{\partial x^h} \Big) \tag{2.31}$$

of the Riemannian space $\mathcal{R}^N$.

Note. When $H = \text{const}$, from (2.27) it just follows that the angle $\alpha_{\{x\}}(y_1, y_2)$ fulfills the vanishing which is completely similar to (2.29), namely the vanishing (1.11) claimed in Section 1.

With the covariant derivative

$$\mathcal{D}_n t^i := d_n t^i + a^i{}_{kn} t^k \tag{2.32}$$

the representation (2.30) can be interpreted as the manifestation of the *transitivity*

$$\mathcal{D}_n t^i = 0 \tag{2.33}$$

of the connection under the transformation (2.1).

By differentiating (2.32) with respect to y^m we may conclude that the covariant derivative

$$\mathcal{D}_n t^i_m := d_n t^i_m - D^h{}_{nm} t^i_h + a^i{}_{nl} t^l_m, \qquad D^h{}_{nm} = -N^h{}_{nm}, \tag{2.34}$$

vanishes identically:

$$\mathcal{D}_n t^i_m = 0. \tag{2.35}$$

Since $y^n_k t^k_j = \delta^n_j$, the previous identity can be transformed to

$$d^{\text{Riem}}_i y^n_k + D^n{}_{is} y^s_k - a^h{}_{ik} y^n_h = 0, \tag{2.36}$$

which can be interpreted as the covatiant derivative vanishing:

$$\mathcal{D}_i y^n_k = 0. \tag{2.37}$$

This formula entails

$$\mathcal{D}_i y^n = 0 \tag{2.38}$$

(because of (2.33)), where

$$\mathcal{D}_i y^n := d^{\text{Riem}}_i y^n + D^n{}_{is} y^s. \tag{2.39}$$

Here, y^n mean the functions $y^n(x,t)$ introduced by (2.8).

We have used the Riemannian operator

$$d_i^{\text{Riem}} = \frac{\partial}{\partial x^i} + L^k{}_i \frac{\partial}{\partial t^k}, \qquad L^k{}_i = -a^k{}_{ih} t^h \tag{2.40}$$

(cf. (1.3)).

Since $D^n{}_{is} y^s = -N^n{}_i$, from (2.38)–(2.39) we may conclude that the representation

$$N^n{}_i = d_i^{\text{Riem}} y^n \equiv \frac{\partial y^n(x,t)}{\partial x^i} + y^n_h L^h{}_i \tag{2.41}$$

is valid which is alternative to (2.30).

Owing to the equalities $p = (1/H)F^{1-H}$, $\mathcal{D}_n y^m_k = 0$, and (2.20), we are entitled to formulate the following proposition.

Proposition 2.3. *When* $d_i F = 0$ *and* $H = \text{const}$, *the deformation tensor* (2.20) *is parallel*

$$\mathcal{D}_n C^m_k = 0, \tag{2.42}$$

where

$$\mathcal{D}_n C^m_k = d_n C^m_k - D^h{}_{nk} C^m_h + a^m{}_{nl} C^l_k. \tag{2.43}$$

The coefficients $N^k{}_i(x,y)$ can also be obtained by means of the transitivity map

$$\{N^k{}_i\} = \mathbf{C} \cdot \{L^k{}_i\}. \tag{2.44}$$

Indeed, with an arbitrary differentiable scalar $w(x,y)$, we can apply the transformation $\{y^i = y^i(x,t),\, t^n \equiv \bar{y}^n\}$ indicated in (2.8) and consider the $\mathbf{C}$-transform

$$W(x,t) = w(x,y), \quad \text{which entails} \quad \frac{\partial W}{\partial t^n} = y^k_n \frac{\partial w}{\partial y^k}, \tag{2.45}$$

thereafter postulating that the $\mathbf{C}$-transformation is *covariantly transitive*, namely

$$\left(\frac{\partial}{\partial x^i} + N^k{}_i(x,y)\frac{\partial}{\partial y^k}\right) w(x,y) = \left(\frac{\partial}{\partial x^i} + L^k{}_i(x,t)\frac{\partial}{\partial t^k}\right) W(x,t). \tag{2.46}$$

Since the field w is arbitrary, the last equality is fulfilled if and only if

$$N^k{}_i = d_i^{\text{Riem}} y^k \equiv \frac{\partial y^k(x,t)}{\partial x^i} + y^k_h L^h{}_i. \tag{2.47}$$

This is the representation which is required to realize the map (2.44). We have again arrived at the coefficients (2.41).

With the knowledge of the coefficients $N^k{}_i(x,y)$, we can use the formulas (2.34) and (2.35) to express the Finslerian connection coefficients $D^h{}_{nm}$ through the Riemannian Christoffel symbols $a^i{}_{nl}$. Thus we have induced the connection in the Finsler space $\mathcal{F}^N$ from the metrical linear Riemannian connection (1.2) meaningful in the background Riemannian space $\mathcal{R}^N$.

3. Properties of connection coefficients

The derivative coefficients (1.13) can straightforwardly be evaluated from (2.30). We obtain explicitly

$$N^k{}_{mn} = -y^k_{sl}t^l_n T^s_m - y^k_s T^s_{n,m} \quad \text{with} \quad T^s_m = \frac{\partial t^s}{\partial x^m} + a^s{}_{mh}t^h,$$

$$T^s_{n,m} = \frac{\partial t^s_n}{\partial x^m} + a^s{}_{mh}t^h_n, \tag{3.1}$$

which entails the contractions

$$y_k N^k{}_{mn} = -\left(2\left(\frac{1}{H}-1\right)F^{-2H}y_n t_s + \frac{1}{H}F^{2(1-H)}a_{sh}t^h_n - g_{ln}y^l_s\right)T^s_m - \frac{F^2}{HS^2}t_s T^s_{n,m} \tag{3.2}$$

and

$$y_k N^k{}_{mn} + g_{ln}N^l{}_m = -\left(2\left(\frac{1}{H}-1\right)F^{-2H}y_n t_s + \frac{1}{H}F^{2(1-H)}a_{sh}t^h_n\right)T^s_m - \frac{F^2}{HS^2}t_s T^s_{n,m},$$

together with

$$\begin{aligned}
&y_k N^k{}_{mni} + g_{ki}N^k{}_{mn} + g_{ln}N^l{}_{mi} + 2C_{lni}N^l{}_m \\
&= -\left(\frac{1}{H}-1\right)2F^{-2H}\left[(g_{ni}-2Hl_n l_i)t_s + (y_n a_{sl}t^l_i + y_i a_{sl}t^l_n)\right]T^s_m \\
&- \frac{1}{H}F^{2(1-H)}a_{sh}t^h_{ni}T^s_m \\
&- \left(2\left(\frac{1}{H}-1\right)F^{-2H}y_n t_s + \frac{1}{H}F^{2(1-H)}a_{sh}t^h_n\right)T^s_{i,m} \\
&- \left(2\left(\frac{1}{H}-1\right)F^{-2H}y_i t_s + \frac{1}{H}F^{2(1-H)}a_{sh}t^h_i\right)T^s_{n,m} \\
&- \frac{1}{H}F^{2(1-H)}t_s\left(\frac{\partial t^s_{ni}}{\partial x^m} + a^s{}_{mh}t^h_{ni}\right).
\end{aligned}$$

The attentive calculation of the entered terms (carried out in Appendix C in [10]) leads to the following remarkable result.

Proposition 3.1. *If the coefficients $N^k{}_m$ are taking according to* (2.30) *and the vanishing $d_m F = 0$ is implied, then the vanishing $y_k N^k{}_{mnj} = 0$ holds identically.*

In performing involved calculation it is necessary to note that in view of (2.11) and (2.30), we can write

$$d_m F = \frac{\partial F}{\partial x^m} + N^k{}_n \frac{\partial F}{\partial y^k} = \frac{\partial F}{\partial x^m} + N^k{}_n l_k = \frac{\partial F}{\partial x^m} - \frac{1}{FH} F^{2(1-H)} t_s T^s_m$$

so that, because of $d_m F = 0$, the equality

$$\frac{\partial F}{\partial x^m} = \frac{1}{FH} F^{2(1-H)} t_s T^s_m \tag{3.3}$$

is valid.

It is also possible to evaluate the covariant derivative $D_m C^k{}_{nj}$ (see (1.17)), using the equality $d_m g_{hn} = -N^t{}_{mh} g_{tn} - N^t{}_{mn} g_{th}$ entailed by the metricity (see (1.14)). This way leads to the following result.

Proposition 3.2. *The representation $N^k{}_{mnj} = -D_m C^k{}_{nj}$ is valid, whenever $d_m F = 0$ and $H = \text{const}$.*

Proof of this proposition can be arrived at during a long chain of straightforward substitutions (see Appendix D in [10]).

4. Entailed curvature tensor

Throughout the present section we assume that $H = \text{const}$.

Given a tensor $w^n{}_k = w^n{}_k(x, y)$ of the tensorial type (1,1), commuting the covariant derivative

$$\mathcal{D}_i w^n{}_k := d_i w^n{}_k + D^n{}_{ih} w^h{}_k - D^h{}_{ik} w^n{}_h \tag{4.1}$$

yields straightforwardly the equality

$$(\mathcal{D}_i \mathcal{D}_j - \mathcal{D}_j \mathcal{D}_i)\, w^n{}_k = M^h{}_{ij} S_h w^n{}_k - \rho_k{}^h{}_{ij} w^n{}_h + \rho_h{}^n{}_{ij} w^h{}_k \tag{4.2}$$

with the *curvature tensor*

$$\rho_k{}^n{}_{ij} = d_i D^n{}_{jk} - d_j D^n{}_{ik} + D^m{}_{jk} D^n{}_{im} - D^m{}_{ik} D^n{}_{jm} - M^h{}_{ij} C^n{}_{hk} \tag{4.3}$$

and the tensor

$$M^n{}_{ij} := d_i N^n{}_j - d_j N^n{}_i \equiv \frac{\partial N^n{}_j}{\partial x^i} - \frac{\partial N^n{}_i}{\partial x^j} - N^h{}_i D^n{}_{jh} + N^h{}_j D^n{}_{ih}. \tag{4.4}$$

The definition

$$\mathcal{S}_h w^n{}_k := \frac{\partial w^n{}_k}{\partial y^h} + C^n{}_{hs} w^s{}_k - C^m{}_{hk} w^n{}_m \tag{4.5}$$

was introduced which has the meaning of the covariant derivative in the tangent space supported by the point $x \in M$. In particular,

$$\mathcal{S}_h g_{nk} := \frac{\partial g_{nk}}{\partial y^h} - C^m{}_{hn} g_{mk} - C^m{}_{hk} g_{nm} = 0. \tag{4.6}$$

The choice $D^k{}_{in} = -N^k{}_{in}$ is implied (cf. (1.18)) and we may use the equality $N^j{}_i = -D^j{}_{ik} y^k$.

By applying the commutation rule (4.2) to the particular choices $\{F, y^n, y_k, g_{nk}\}$ and noting the vanishing $\{\mathcal{D}_i F = \mathcal{D}_i y^n = \mathcal{D}_i y_k = \mathcal{D}_i g_{nk} = 0\}$, we obtain the relations

$$y_n M^n{}_{ij} = 0, \quad y^k \rho_k{}^n{}_{ij} = -M^n{}_{ij}, \quad y_n \rho_k{}^n{}_{ij} = M_{kij}, \quad \rho_{mnij} = -\rho_{nmij}, \tag{4.7}$$

where $M_{kij} = g_{nk} M^n{}_{ij}$ and $\rho_{mnij} = g_{nl} \rho_m{}^l{}_{ij}$.

The curvature tensor obeys also the cyclic identity

$$\mathcal{D}_l \rho_k{}^n{}_{ij} + \mathcal{D}_j \rho_k{}^n{}_{li} + \mathcal{D}_i \rho_k{}^n{}_{jl} = 0, \tag{4.8}$$

where

$$\mathcal{D}_l \rho_k{}^n{}_{ij} = d_l \rho_k{}^n{}_{ij} + D^n{}_{lt} \rho_k{}^t{}_{ij} - D^t{}_{lk} \rho_t{}^n{}_{ij} - a^s{}_{li} \rho_k{}^n{}_{sj} - a^s{}_{lj} \rho_k{}^n{}_{is}. \tag{4.9}$$

Let us realize the action of the **C**-transformation (2.1) on tensors by the help of the *transitivity rule*, that is,

$$\{w^n{}_m(x,y)\} = \mathbf{C} \cdot \{W^n{}_m(x,t)\} : w^n{}_m = y^n_h t^j_m W^h{}_j, \tag{4.10}$$

where $W^n{}_m$ is a tensor of type (1,1). The metrical linear connection $\mathcal{R}L$ introduced by (1.2) may be used to define the covariant derivative ∇ in $\mathcal{R}^N$ according to the conventional rule:

$$\nabla_i W^n{}_m = \frac{\partial W^n{}_m}{\partial x^i} + L^k{}_i \frac{\partial W^n{}_m}{\partial t^k} + L^n{}_{hi} W^h{}_m - L^h{}_{mi} W^n{}_h, \tag{4.11}$$

where $L^k{}_i = -t^h L^k{}_{hi}$ and $L^n{}_{hi} = a^n{}_{hi}$. It follows that $\nabla_i S = 0$, $\nabla_i y^j = 0$, and $\nabla_i a_{mn} = 0$.

Due to the nullifications $\mathcal{D}_i y^n_h = 0$ and $\mathcal{D}_i t^j = 0$ see (2.37) and (2.33)), we have the transitivity property

$$\mathcal{D}_i w^n{}_m = y^n_h t^j_m \nabla_i W^h{}_j \tag{4.12}$$

for the covariant derivatives.

In the commutator

$$[\nabla_i \nabla_j - \nabla_j \nabla_i] W^n{}_k = -y^m a_m{}^h{}_{ij} \frac{\partial W^n{}_k}{\partial y^h} - a_k{}^h{}_{ij} W^n{}_h + a_h{}^n{}_{ij} W^h{}_k \tag{4.13}$$

the associated Riemannian curvature tensor is constructed in the conventional way

$$a_n{}^i{}_{km} = \frac{\partial a^i{}_{nm}}{\partial x^k} - \frac{\partial a^i{}_{nk}}{\partial x^m} + a^u{}_{nm} a^i{}_{uk} - a^u{}_{nk} a^i{}_{um}. \tag{4.14}$$

With the ordinary Riemannian covariant derivative

$$\nabla_k a_h{}^t{}_{ij} = \frac{\partial a_h{}^t{}_{ij}}{\partial x^k} + a^t{}_{ku} a_h{}^u{}_{ij} - a^u{}_{kh} a_u{}^t{}_{ij} - a^u{}_{ki} a_h{}^t{}_{uj} - a^u{}_{kj} a_h{}^t{}_{iu}, \tag{4.15}$$

the cyclic identity

$$\nabla_k a_m{}^n{}_{ij} + \nabla_j a_m{}^n{}_{ki} + \nabla_i a_m{}^n{}_{jk} = 0 \tag{4.16}$$

holds.

Under these conditions, by comparing the Finslerian commutator (4.2) with the Riemannian precursor (4.13), we obtain

$$M^n{}_{ij} = -y^n_t t^h a_h{}^t{}_{ij} \tag{4.17}$$

and $\rho_k{}^n{}_{ij} = \left(y^n_h t^h_{km} - C^n{}_{mk}\right) M^m{}_{ij} + y^n_m a_h{}^m{}_{ij} t^h_k$. Inserting here the tensor $C^n{}_{mk}$ taken from (2.15), noting the vanishing $l_m M^m{}_{ij} = 0$ (see (4.7)), using the equality $g_{nm} y^m_i = p^2 t^j_n a_{ij}$ (ensued from (2.10)), and applying the skew-symmetry relation (2.17), we obtain after short simplifications the representation

$$\rho_{knij} = -(1-H)\frac{1}{F}(l_k M_{nij} - l_n M_{kij}) + p^2 a_{hlij} t^h_k t^l_n, \tag{4.18}$$

where $a_{hlij} = a_{lr} a_h{}^r{}_{ij}$, which entails

$$\rho_k{}^n{}_{ij} = -(1-H)\frac{1}{F}(l_k \delta^n_m - l^n g_{mk}) M^m{}_{ij} + y^n_m a_h{}^m{}_{ij} t^h_k. \tag{4.19}$$

The totally contravariant representation $\rho^{knij} = g^{pk} a^{mi} a^{nj} \rho_p{}^n{}_{mn}$ reads

$$\rho^{knij} = -(1-H)\frac{1}{F}(l^k M^{nij} - l^n M^{kij}) + \frac{1}{p^2} y^k_h y^n_r a^{hrij}, \tag{4.20}$$

where $a^{hrij} = a^{hl} a^{mi} a^{nj} a_l{}^r{}_{mn}$ and $M^{mij} = a^{hi} a^{nj} M^m{}_{hn}$.

Similarly, we can conclude from (4.17) that the tensor $M_{nij} = g_{nm}M^m{}_{ij}$ reads

$$M_{nij} = -p^2 t^h t^m_n a_{hmij}. \tag{4.21}$$

Squaring yields

$$M^{nij}M_{nij} = p^2 t^l a_l{}^{nij} t^h a_{hnij}. \tag{4.22}$$

From the representation (4.19) it follows directly that the cyclic identity (4.8) is a direct consequence of the Riemannian cyclic identity (4.16), for $\mathcal{D}_l F = \mathcal{D}_l l_k = \mathcal{D}_l t^h_k = \mathcal{D}_l p = \mathcal{D}_l t^m = 0$.

Using the representations (4.18) and (4.20), we can square the ρ-tensor, obtaining

$$\rho^{knij}\rho_{knij} = a^{knij}a_{knij} + \frac{2}{S^2}\left(\frac{1}{H^2} - 1\right) t^l a_l{}^{nij} t^h a_{hnij}. \tag{4.23}$$

Because of the transitivity (4.12), from (4.17) it follows that

$$\mathcal{D}_l M^n{}_{ij} = -y^n_t t^h \nabla_l a_h{}^t{}_{ij} \tag{4.24}$$

and from (4.19) we can conclude that

$$\mathcal{D}_l \rho_k{}^n{}_{ij} = (1-H)\frac{1}{F}(l_k\delta^n_m - l^n g_{mk}) y^m_t t^h \nabla_l a_h{}^t{}_{ij} + y^n_m t^h_k \nabla_l a_h{}^m{}_{ij}. \tag{4.25}$$

5. $\mathcal{FS}$-space example of the space $\mathcal{F}^N$

Let us also assume that the manifold M admits a non-vanishing 1-form $b = b(x,y)$ of the unit norm $\|b\|_{\text{Riemannian}} = 1$. With respect to natural local coordinates x^i we have the local representations $b = b_i(x)y^i$ and $a^{ij}(x)b_i(x)b_j(x) = 1$. The reciprocity $a^{in}a_{nj} = \delta^i{}_j$ is assumed, where $\delta^i{}_j$ stands for the Kronecker symbol. The covariant index of the vector b_i will be raised by means of the Riemannian rule $b^i = a^{ij}b_j$, which inverse reads $b_i = a_{ij}b^j$.

We may conveniently use the scalar

$$q := \sqrt{a_{ij}y^i y^j - b^2}. \tag{5.1}$$

With the variable

$$w = \frac{q}{b}, \qquad b > 0, \tag{5.2}$$

we obtain

$$\frac{\partial w}{\partial y^i} = \frac{q e_i}{b^2}, \quad e_i = -b_i + \frac{b}{q^2}v_i, \quad y^i e_i = 0, \tag{5.3}$$

where $v_i = a_{ij}y^j - bb_i$, so that $y^i v_i = q^2$ and $b^i v_i = 0$.

The Finsler metric function F of the $\mathcal{FS}$-space is specified by (1.21). When $b > 0$, we can conveniently use the *generating metric function* $V = V(x, w)$ to have the representation

$$F = bV(x, w). \tag{5.4}$$

The unit vector $l_m = \partial F/\partial y^m$ is given by $l_m = b_m V + (w^2/\tau)V e_m$, where

$$\tau = \frac{wV}{V'}, \qquad V' = \frac{\partial V}{\partial w}. \tag{5.5}$$

It follows that $b^m l_m = V\left(1 - (w^2/\tau)\right)$.

We say that the $\mathcal{FS}$-space is *special*, if $\partial V/\partial x^n = 0$, that is when

$$V = V(w). \tag{5.6}$$

Taking two differentiable scalars $C = C(x)$, $C_1 = C_1(x)$, $C > 0$, $C > |C_1|$, we construct the scalars

$$H = \sqrt{C^2 - (C_1)^2}, \quad \breve{k} = \sqrt{\frac{C - C_1}{C + C_1}}. \tag{5.7}$$

Let a positive function $\mu = \mu(x, y)$ be introduced according to $\sqrt{\mu} = (H/2\breve{k})\big[1 + \breve{k}^2 + (1 - \breve{k}^2)\cos\varrho\big]$, where $\varrho = \varrho(x, y)$ is an input scalar. We can write

$$\sqrt{\mu} = C + C_1 \cos\varrho. \tag{5.8}$$

Consider the transformation $t^m = t^m(x, y)$ with

$$t^m = \left[i^m \sin\varrho + \frac{1}{2\breve{k}}[1 - \breve{k}^2 + (1 + \breve{k}^2)\cos\varrho]b^m\right]\frac{H}{\sqrt{\mu}}F^H, \tag{5.9}$$

where $i^m = (y^m - bb^m)/q$. We have $b_m i^m = 0$, $a_{mn}i^m i^n = 1$, $a_{mn}y^m i^n = q$, and

$$b^* = (C_1 + C\cos\varrho)\frac{1}{\sqrt{\mu}}S, \tag{5.10}$$

where $S = \sqrt{a_{mn}t^m t^n}$ and $b^* = t^m b_m$.

The functions (5.9) obviously fulfill the H-degree homogeneity condition (2.5). The validity of the equality $S = F^H$ (see (1.8)) can readily be verified. The property $t^m(x, b(x)) \sim b^m(x)$ holds.

The following useful equalities can readily be obtained:

$$\cos\varrho = -\frac{(1-\breve{k}^2)S-(1+\breve{k}^2)b^*}{(1+\breve{k}^2)S-(1-\breve{k}^2)b^*}, \quad \sqrt{\mu} = \frac{2H\breve{k}S}{(1+\breve{k}^2)S-(1-\breve{k}^2)b^*},$$
$$\cos\varrho = -\frac{\sqrt{\mu}}{2H\breve{k}S}[(1-\breve{k}^2)S-(1+\breve{k}^2)b^*],$$
$$\sin^2\varrho = 4\breve{k}^2\frac{S^2-(b^*)^2}{\left[(1+\breve{k}^2)S-(1-\breve{k}^2)b^*\right]^2},$$

together with

$$\frac{\sin^2\varrho}{\mu} = \frac{1}{H^2}\frac{S^2-(b^*)^2}{S^2}. \tag{5.11}$$

With these observations, from (5.9) we find that the derivative coefficients $t_k^m = \partial t^m/\partial y^k$ can be given by

$$\frac{1}{H}\sqrt{\mu}t_k^m = \left[\cos\varrho i^m - \frac{1}{2\breve{k}}(1+\breve{k}^2)\sin\varrho b^m\right]\varrho'\frac{w}{b}e_kF^H$$
$$+\sin\varrho\frac{\partial i^m}{\partial y^k}F^H + \left[\sqrt{\mu}\frac{1}{F}l_k + \frac{1}{2\breve{k}}(1-\breve{k}^2)\sin\varrho\,\varrho'\frac{w}{b}e_k\right]t^m. \tag{5.12}$$

We can straightforwardly evaluate the contraction $a_{mn}t_k^m t_h^n$, which leads to the expression which is a linear combination of g_{kh}, e_ke_h, l_kl_h, and $e_kl_h + e_hl_k$. To obtain the conformal result, we must achieve cancelation of the terms l_kl_h, which proves possible if and only if the function μ is taken to be

$$\mu = \frac{1}{w^2}\tau\sin^2\varrho, \tag{5.13}$$

and define the ρ by means of the equation

$$\frac{\partial\varrho}{\partial w} = \frac{1}{w}\sqrt{\frac{\tau - w(\tau'-w)}{\tau}}\sin\varrho, \tag{5.14}$$

where $\tau' = \partial\tau/\partial w$. In so doing, we obtain the representation of the form (2.7) after required evaluation (which was presented in detail in [10]).

Therefore, the following assertion is valid.

Proposition 5.1. *With choosing the function μ to be given by* (5.13) *and subjecting the function ϱ to the equation* (5.14), *the transformation $t^m = t^m(x,y)$ introduced by* (5.9) *fulfills the input stipulations* (2.1)–(2.3).

Under these conditions, evaluation of the coefficients $N^m{}_i$ leads to the following proposition.

Proposition 5.2. *If in the special case of the* $\mathcal{F}S$*-space the transformation* (5.9) *obeys* (2.1)–(2.3), *then the coefficients* (2.30) *can explicitly be given by means of the representation*

$$N^m{}_i = \frac{1}{H}\frac{1}{q}(y^h\nabla_i b_h)m^m - \frac{1}{H}\sqrt{B - H^2q^2}\,\beta_i^m \\ - (y^h\nabla_i b_h)b^m + b\nabla_i b^m - a^m{}_{ih}y^h, \tag{5.15}$$

with

$$m^m = \frac{w}{\sqrt{\tau - w(\tau' - w)}}\left[y^m - \frac{B}{q^2}\,(y^m - bb^m)\right] \tag{5.16}$$

and

$$\beta_i^m = \nabla_i b^m - \frac{1}{q^2}(y^h\nabla_i b_h)\,(y^m - bb^m)\,, \quad B = b^2\tau, \tag{5.17}$$

whenever $H = \text{const}$.

The following proposition is valid.

Proposition 5.3. *The transformation* (5.9) *fulfills* (2.1)–(2.3) *iff*

$$\tau = \breve{C}^2 + 2\breve{C}\sqrt{1 - H^2}\,w + w^2. \tag{5.18}$$

It follows that $\tau - w(\tau' - w) = \breve{C}^2$. In these formulas, $\breve{C}$ is an integration scalar $\breve{C} = \breve{C}(x)$. It can readily be seen that when $|\breve{C}| \neq 1$, the entailed Finsler metric function F can vanish at various values of tangent vectors y. To agree with the condition that F vanishes only at zero-vectors $y = 0$, we admit strictly the particular values $\breve{C} = 1$ and $\breve{C} = -1$. In this case we can write the above τ as follows:

$$\tau = 1 + gw + w^2, \quad -2 < g < 2. \tag{5.19}$$

Generally, the g may depend on x. We obtain

$$B - H^2q^2 = \left(b + \frac{1}{2}gq\right)^2. \tag{5.20}$$

The function τ given by (5.19) represents the $\mathcal{F}F_g^{PD}$-Finsleroid space described in [7]. To comply with the representations used in [7], we should replace the notation H by the notation h:

$$h = \sqrt{1 - \frac{g^2}{4}}. \tag{5.21}$$

The g plays the role of the characteristic parameter. The $\mathcal{F}F_g^{PD}$-Finsleroid metric function K can be given as it follows:

$$K = \sqrt{B}\, J, \quad \text{with } J = \mathrm{e}^{-\frac{1}{2}g\chi}, \tag{5.22}$$

where

$$\chi = \frac{1}{h}\Big(-\arctan\frac{G}{2} + \arctan\frac{L}{hb}\Big), \text{ if } b \ge 0;$$
$$\chi = \frac{1}{h}\Big(\pi - \arctan\frac{G}{2} + \arctan\frac{L}{hb}\Big), \text{ if } b \le 0, \tag{5.23}$$

with the function $L = q + (g/2)b$ fulfilling the identity $L^2 + h^2b^2 = B$, with B issuing from (5.20):

$$B = b^2 + gbq + q^2; \tag{5.24}$$

$G = g/h$. The definition range $0 \le \chi \le \pi/h$ is of value to describe all the tangent space. The normalization in (5.23) is such that $\chi\big|_{y=b} = 0$. The quantity χ can conveniently be written as

$$\chi = \frac{1}{h}f \tag{5.25}$$

with the function

$$f = \arccos\frac{A(x,y)}{\sqrt{B(x,y)}}, \quad A = b + \frac{1}{2}gq, \tag{5.26}$$

ranging as follows:

$$0 \le f \le \pi. \tag{5.27}$$

The function K is the solution for the equation (5.19).

The Finsleroid-axis vector b^i relates to the value $f = 0$, and the opposed vector $-b^i$ relates to the value $f = \pi$: $f = 0 \ \sim\ y = b$; $f = \pi \ \sim\ y = -b$. The normalization is such that $K(x, b(x)) = 1$ (notice that $q = 0$ at $y^i = b^i$). The positive (not absolute) homogeneity holds: $K(x, \gamma y) = \gamma K(x, y)$ for any $\gamma > 0$ and all admissible (x, y).

The entailed components $y_i = (1/2)\partial K^2/\partial y^i$, $g_{ij} = (1/2)\partial^2 K^2/\partial y^i \partial y^j$, and $A_i = K\partial \ln(\sqrt{\det(g_{ij})})/\partial y^i$ can readily found, yielding

$$y_i = (a_{ij}y^j + gqb_i)J^2, \quad \det(g_{ij}) = \Big(\frac{K^2}{B}\Big)^N \det(a_{ij}) > 0, \quad A^iA_i = \frac{N^2g^2}{4}. \tag{5.28}$$

The vector (5.16) reduces to

$$m^m = w\left[y^m - \frac{B}{q^2}\,(y^m - bb^m)\right] \equiv \frac{C^m}{\sqrt{g^{kh}C_kC_h}}.$$

Under these conditions, we obtain the $\mathcal{F}F_g^{PD}$*-Finsleroid space*

$$\mathcal{F}F_g^{PD} := \{M;\ a_{ij}(x);\ b_i(x);\ g(x);\ K(x,y)\}. \tag{5.29}$$

Within any tangent space T_xM, the metric function $K(x,y)$ produces the $\mathcal{F}F_g^{PD}$*-Finsleroid* $\mathcal{F}F^{PD}_{g;\,\{x\}} := \{y \in \mathcal{F}F^{PD}_{g;\,\{x\}} : y \in T_xM, K(x,y) \leq 1\}$, which is an extension of the Euclidean unit ball. The $\mathcal{F}F_g^{PD}$*-Indicatrix* $\mathcal{I}F^{PD}_{g;\,\{x\}} \subset T_xM$ is the boundary of the $\mathcal{F}F_g^{PD}$-Finsleroid, that is,

$$\mathcal{I}F^{PD}_{g\,\{x\}} := \{y \in \mathcal{I}F^{PD}_{g\,\{x\}} : y \in T_xM, K(x,y) = 1\}. \tag{5.30}$$

The scalar $g(x)$ is called the *Finsleroid charge.* The 1-form $b = b_i(x)y^i$ is called the *Finsleroid-axis* 1*-form.*

In this case, with the tensor

$$\mathcal{H}^{mj} := g^{mj} - l^m l^j - m^m m^j, \tag{5.31}$$

the coefficients (5.15) take on the form

$$\begin{aligned} N^m{}_i = -l^m \frac{\partial K}{\partial x^i} + \Bigg[& \left(b - \frac{1}{h}\left(b + \frac{g}{2}q\right)\right) \mathcal{H}^{mj}\frac{K^2}{B} \\ & + \left(\frac{1}{hq} - \frac{b^2+q^2}{qB}\right) K m^m y^j \Bigg] \nabla_i b_j - h^m_t a^t{}_{ij} y^j, \end{aligned} \tag{5.32}$$

where $h^m_t = \delta^m_t - l^m l_t$ (see [10]).

In the dimension $N = 2$ we would have $\mathcal{H}^{mj} = 0$.

Regarding regularity of the global y-dependence, it should be noted that the $\mathcal{F}F_g^{PD}$-Finsleroid metric function K given by the formulas (5.21)–(5.25) involves the scalar $q = \sqrt{r_{mn}y^m y^n}$ with $r_{mn} = a_{mn} - b_m b_n$. Since the 1-form b is of the unit norm $\|b\| = 1$, the scalar q is zero when $y = b$ or $y = -b$, that is, in the directions of the north pole or the south pole of the Finsleroid. The derivatives of K may involve the fraction $1/q$ which gives rise to the *pole singularities* when $q = 0$. This just happens in the right-hand part of the representation (5.32) for the coefficients $N^m{}_i$.

Therefore, we may apply the coefficients on but the b-slit tangent bundle $\mathcal{T}_bM := TM \setminus 0 \setminus b \setminus -b$ (obtained by deleting out in $TM \setminus 0$ all the directions which point along, or oppose, the directions given rise to by the 1-form b), on which the coefficients $N^m{}_i$, as well as the function K, are smooth of the class C^∞ regarding the y-dependence. On the punctured tangent bundle $TM\setminus 0$, the metric function K is smooth globally of the class C^2 and not of the class C^3 regarding the y-dependence. With the function (5.19) the equation (5.14) can readily be solved, leading to the conclusion the the transformation angle ρ entered (5.9) is given by the function f which was indicated in (5.25)–(5.26), so that

$$\rho = f \equiv -\frac{h}{2N\sqrt{1-h^2}} \ln\left(\frac{\det(g_{ij})}{\det(a_{mn})}\right). \tag{5.33}$$

We obtain $\sin\varrho = hq/\sqrt{B}$ and $\cos\varrho = (b+(1/2)gq)/\sqrt{B}$. The function (5.13) becomes the constant, namely $\mu = h^2$, so that from (5.8) we may conclude that $C_1 = 0$. The transformation (5.9) reduces to

$$t^m = \left[h(y^m - bb^m) + \left(b + \frac{1}{2}gq\right)b^m\right]\frac{K^h}{\sqrt{B}}. \tag{5.34}$$

Thus we have

Proposition 5.4. *In the $\mathcal{F}\mathcal{F}^{PD}_g$-Finsleroid space the transformation* (5.34) *fulfills the stipulations* (2.1)–(2.3). *When* $h = \text{const}$, *the coefficients* (2.30) *of the angle-preserving connection can explicitly be given by means of the representation* (5.31)–(5.32).

The t^m of (5.34) is equivalent to the ζ^m of (6.26) of [7]. The coefficients (5.32) are equivalent to (6.62) of [7]. Therefore, with the substitution $\zeta^m = t^m$ all the relations among curvature tensors which were established in [7] are applicable to the approach developed in the present section,

With respect to the unit vectors $\{l^m, m^m\}$, (5.34) is the linear expansion

$$t^m = (T_1 l^m + T_2 m^m)\frac{K^2}{B}\frac{K^{h-1}}{\sqrt{B}}, \tag{5.35}$$

where

$$T_1 = -(1-h)q^2 + B + \frac{1}{2}gq(b+gq), \quad T_2 = \left((1-h)b + \frac{1}{2}gq\right)q. \tag{5.36}$$

6. Conclusions

In the two-dimensional approach, $N = 2$, the general representation for the coefficients $N^m{}_i = N^m{}_i(x,y)$ entailing the property of preservation of two-vector angle can be indicated locally for arbitrary sufficiently smooth Finsler metric function [8], [9]. Such a general possibility can doubtfully be meet in the dimensions $N \geq 3$, for in these dimensions the two-vector is of complicated nature except for rare particular cases. However, the lucky cases are just proposed by the Finsler spaces which can be characterized by the constancy of the indicatrix curvature. The respective two-vector angle is explicit, namely is given by the simple formulas (1.7) and (2.27)–(2.28). In each tangent space, the indicatrix curvature value $\mathcal{C}_{\text{Ind.}} = H^2$ is obtained and the relevant conformal multiplier is given by p^2 with $p = (1/H)F^{1-H}$. The H plays also the role of the homogeneity degree of the involved transformation to the Riemannian space.

In the indicatrix-homogeneous case, the required connection coefficients are presented by the pair $\{N^j{}_i,\, D^j{}_{ik}\}$, where $D^j{}_{ik} = -\partial N^j{}_i/\partial y^k$. The equality $N^j{}_i = -D^j{}_{ik}y^k$ holds.

In the Riemannian geometry the two-vector angle is $\alpha^{\text{Riem}}_{\{x\}}(y_1, y_2) = a_{mn}(x)y_1^m y_2^n/S_1S_2$, where $S_1 = \sqrt{a_{mn}(x)y_1^m y_1^n}$ and $S_2 = \sqrt{a_{mn}(x)y_2^m y_2^n}$. Starting with the fundamental property of the metrical linear Riemannian connection that the Riemannian angle is preserving under the parallel displacements of the involved vectors, which in terms of our notation can be written as

$$d_i^{\text{Riem}} \alpha^{\text{Riem}}_{\{x\}}(y_1, y_2) = 0, \quad y_1, y_2 \in T_xM,$$

with

$$d_i^{\text{Riem}} = \frac{\partial}{\partial x^i} + L^k{}_i(x, y_1)\frac{\partial}{\partial y_1^k} + L^k{}_i(x, y_2)\frac{\partial}{\partial y_2^k},$$

where $L^k{}_i(x, y_1) = -a^k{}_{ij}(x)y_1^j$, $L^k{}_i(x, y_2) = -a^k{}_{ij}(x)y_2^j$, and $a^k{}_{ij}$ are the Riemannian Christoffel symbols fulfilling the Riemannian Levi–Civita connection, the important question can be set forth: Can we have the similar vanishing in the Finsler space? It proves that the respective extension of the Riemannian equation $d_i^{\text{Riem}}\alpha^{\text{Riem}} = 0$ to the equation $d_i\alpha = 0$ applicable to the Finsler space under consideration can straightforwardly be solved giving the required coefficients $N^j{}_i$ indicated in (2.30). They admit the remarkable alternative representation $N^n{}_i = d_i^{\text{Riem}} y^n$ (see (1.20)). In this way we obtain the connection $\{N^j{}_i,\, D^j{}_{ik}\}$ which is metrical and simultaneously angle-preserving. The key vanishing $y_k N^k{}_{mnj} = 0$ holds fine.

Remarkably, the Finsler connection presented by this pair $\{N^j{}_i,\, D^j{}_{ik}\}$ is the image of the metrical linear Riemannian connection under the desired transformations. When going from the considered Finsler space to the underlined Riemannian space, the covariant derivative behaves transitively and the nonlinear deformation which materializes the transformation is parallel. In particular, the Riemannian vanishing $d_m^{\text{Riem}} S = 0$ just entails the Finslerian counterpart $d_m F = 0$.

Also, the involved coefficients $N^m{}_i$ fulfill the representation
$N^k{}_{mnj} = -\mathcal{D}_m C^k{}_{nj}$ (see Proposition 3.2). Just the same representation is valid

in the two-dimensional Finsler spaces (see (2.14) in [8], [9]). Is the equation

$$\frac{\partial^2 N^k{}_m}{\partial y^n \partial y^j} = -\mathcal{D}_m C^k{}_{nj}$$

meaningful in other (in any?) Finsler spaces to find the coefficients $N^k{}_m$ required to preserve the two-vector angle? The question is addressed to readers.

The curvature tensor $\rho_k{}^n{}_{ij}$ has been explicated from commutators of arisen covariant derivatives which is attractive to develop in future the theory of curvature for the Finsler space $\mathcal{F}^N$.

For the $\mathcal{FS}$-space specialized by (1.21) we have got at our disposal the simple example of the parallel deformation transformation, namely proposing by (5.9), which entails the coefficients $N^m{}_i$ possessing the property of angle preservation. The coefficients are given by the explicit representation (5.15)–(5.17), which admits the alternative form (5.31)–(5.32). The space proves to be of the Finsleroid type, with the Finsleroid characteristic parameter g manifesting the meaning: $h = \sqrt{1-(g^2/4)}$ is the homogeneity degree (denoted above by H) of the underlined transformations.

The Finsleroid metric function K when considered on the b-slit tangent bundle $\mathcal{T}_b M := TM \setminus 0 \setminus b \setminus -b$ is smooth of the class C^∞ regarding the global y-dependence. The same regularity property is valid for the coefficients $N^m{}_i$ given by (5.32).

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