Publ. Math. Debrecen
79/1-2 (2011), 211–229
DOI: 10.5486/PMD.2011.5006

Star order on operator and function algebras

By MARTIN BOHATA (Prague)

Abstract. The paper deals with the star order on proper $*$-algebras. Many results on the star order on matrix algebras and algebras of bounded operators acting on a Hilbert space are generalized to the C^*-algebraic context. We characterize the star order on partial isometries in proper $*$-algebras in terms of their initial and final projections. As a corollary, we present a new characterization of infinite C^*-algebras. Further, main results concern the infimum and supremum problem for the star order on a C^*-algebra $C(X)$ of all continuous complex-valued functions on a Hausdorff topological space X. We show that if X is locally connected or hyperstonean, then any upper bounded set in $C(X)$ has an infimum and a supremum in the star order.

1. Introduction

It was shown by Drazin [5] that the equations

$$a^*a = a^*b \quad \text{and} \quad aa^* = ba^*$$

define a partial order on a proper *-semigroup. This order is now known as the star order (or, in abbreviation, *-order). The star order has been intensely studied on algebras of matrices which has led to many interesting results of a matrix analysis and its applications [2], [3], [7], [13]. Recently, the study of the star order has been extended from matrices to a more general case of the set $\mathscr{B}(\mathscr{H})$ of all bounded operators on a Hilbert space $\mathscr{H}$ by Antezana and others [1]. It has not only brought new infinite dimensional results but it has also put older facts on the star order for matrices into a new perspective. We would like

---

Mathematics Subject Classification: 06A06, 46E25, 46J10, 46L05.
Key words and phrases: C^*-algebras, function algebras, Gudder order, star order.

to continue this research line to C^*-algebras and *-algebras. C^*-algebras are simultaneous extension of both matrix algebras and the algebra $\mathscr{B}(\mathscr{H})$. On the other hand, C^*-algebras together with algebras of continuous functions constitute important examples of *-algebras which have inspired general theory in many aspects. Having this in mind, the main goal of this paper is to study the star order on general *-algebras with a special attention devoted to C^*-algebras and algebras of continuous functions in the background.

The present work may be viewed as a part of a recent effort to study various types of order on matrices and operators, which has attracted many authors. In particular, the spectral order and the Gudder order are being intensively investigated at the present. They were introduced on the self-adjoint operators in $\mathscr{B}(\mathscr{H})$ by Olson [15] and Gudder [6], respectively. It was proved in [15], [16] that the set of all self-adjoint operators in $\mathscr{B}(\mathscr{H})$ endowed with the spectral order as well as Gudder order is a so-called boundedly complete poset. (By a boundedly complete poset, we mean a poset L such that every bounded subset of L has an infimum and a supremum.) Furthermore, maps preserving these orders were investigated by Dolinar, Hamhalter, Molnár, and Šemrl in papers [4], [9], [14]. We shall see later in Proposition 2.1 that the Gudder order is in fact the restriction of the star order which was pointed out by Pulmannová and Vinceková in [16].

Let us now recall basic terminology and fix the notation. By a **-algebra* $\mathcal{A}$ we shall mean an associative (not necessarily commutative) complex algebra with the *-operation satisfying the following conditions: (i) $a^{**} = a$, (ii) $(ab)^* = b^*a^*$, (iii) $(a+b)^* = a^* + b^*$, and (iv) $(\alpha a)^* = \overline{\alpha} a^*$ for all $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. A *-algebra $\mathcal{A}$ is called *proper* if $a^*a = 0$ implies $a = 0$ for any $a \in \mathcal{A}$. In the sequel, the symbol $\mathcal{A}$ will always denote a proper *-algebra. The important example of such an algebra $\mathcal{A}$ is a C^*-algebra and a *-algebra $C(X)$ of continuous complex-valued functions on a Hausdorff topological space X. (For basic elements of operator algebra theory we refer the reader to [8], [11].) The algebra $\mathcal{A}$ carries the multiplicative structure of the proper *-semigroup and so, following Drazin [5], we can define a partial order on $\mathcal{A}$ as follows.

Definition 1.1. We say that $a \in \mathcal{A}$ is less than or equal to $b \in \mathcal{A}$ in the *star order*, written $a \preceq b$, if

$$a^*a = a^*b \quad \text{and} \quad aa^* = ba^*.$$

We write $a \prec b$ whenever $a \preceq b$ and $a \neq b$.

The paper is organized as follows. The basic properties of the star order are summarized in Section 2. A number of them generalize the well known results for $\mathscr{B}(\mathscr{H})$ and matrix algebras. We observe that the star order is preserved by a *-homomorphism. This simple observation is especially useful in the case of an abelian C^*-algebra $\mathcal{C}$ because this algebra is *-isomorphic to an algebra of continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff topological space called the *spectrum* of $\mathcal{C}$. It motivates the investigation of the star order on function algebras in last two sections. We show that the star order on C^*-algebras is well behaved with respect to the function calculus. A similar result was proved for the star order on $\mathscr{B}(\mathscr{H})$ in [1] and also for the Gudder order in [6].

In Section 3, we discuss the star order on partial isometries in general proper *-algebras. Partial isometries play a significant role in the geometry of Hilbert spaces and in the theory of operator algebras. It is well known, for example, that the set of extreme points of the unit ball of a unital C^*-algebra consists of partial isometries. Further, partial isometries are important ingredients in the polar decomposition of operators. Using the polar decomposition it was shown in [1] that the star order on $\mathscr{B}(\mathscr{H})$ can be investigated on the set of partial isometries and on the set of positive elements separately. So the star order on partial isometries is an important component of the star order on general elements, which has motivated our interest. In this paper, we study the connection between the star order on partial isometries and properties of their initial and final projections. This way we describe the structure of the set of partial isometries which lie below or above a given partial isometry. In particular, we show that a proper *-algebra admits an infinite projection if and only if there are appropriate partial isometries u_1 and u_2 such that $u_1 \prec u_2$. This gives a characterization of infinite C^*-algebras and von Neumann algebras in terms of the star order.

Last two sections deal with the infimum and supremum problem (i.e., the question of the existence of an infimum and a supremum) for the star order. This problem has been solved in the affirmative for the Gudder order in [6], [16]. For example, it was shown that the infimum of two elements in the Gudder order always exists. However, the supremum of two elements exists if and only if they have a common upper bound. Similar results concerning infimum have also been proved for the star order on $\mathscr{B}(\mathscr{H})$ and matrix algebras in [1], [7]. The main goal of our work is to investigate the infimum and supremum problem for the star order on a *-algebra $C(X)$ of all continuous complex-valued functions on a Hausdorff topological space X. (We do not assume that X is locally compact and so we study a more general case than abelian C^*-algebras.) In Section 4, we show that the infimum problem has a positive solution in two seemingly opposite cases: (1) X is a locally connected space; (2) X is an extremely disconnected space.

More specifically, we show that any set in $C(X)$ has an infimum in the star order provided that X is a locally connected space, which covers the important case of the algebra $C([0,1])$. Applying this result to an abelian C^*-algebra, we obtain that the infimum problem has a positive answer for all abelian C^*-algebras with locally connected spectra. On the other hand, it is shown that the infimum of any set in $C(X)$ exists if X is an extremely disconnected topological space. This gives a positive solution of the infimum problem for abelian von Neuman algebras. In concluding Section 5, we prove that the star order supremum of a subset A of $C(X)$ exists for X locally connected or hyperstonean if and only if there is an upper bound for the set A. In summary, the algebras specified above, when endowed with the star order, have a structure of boundedly complete poset.

2. Basic properties

The star order is closely related to the concept of the *-orthogonality which was first discovered by Hestenes [10] in the context of matrix algebras. We say that $a, b \in \mathcal{A}$ are **-orthogonal*, written $a \perp b$, if $a^*b = ba^* = 0$. It is easy to see that the *-orthogonality is a symmetric binary relation which, for self-adjoint elements, coincides with the usual notion of orthogonality. Let $\mathcal{C}$ be a C^*-algebra acting on a Hilbert space. We shall denote the range of operator $a \in \mathcal{C}$ by $\mathcal{R}(a)$ and the corresponding projection onto the closure $\overline{\mathcal{R}(a)}$ by p_a. By a *projection* we always mean an idempotent self-adjoint element. Clearly, the projection onto $\overline{\mathcal{R}(a)}$ is an element of the second commutant $\mathcal{C}''$ of the C^*-algebra $\mathcal{C}$. The null space of a will be denoted by $\mathcal{N}(a)$. The *-orthogonality on C^*-algebras acting on Hilbert spaces has natural characterizations in terms of ranges and range projections. In particular, the following conditions are equivalent: (i) $a \perp b$, (ii) $\overline{\mathcal{R}(b)} \subseteq \mathcal{N}(a^*)$ and $\overline{\mathcal{R}(a^*)} \subseteq \mathcal{N}(b)$, and (iii) $p_a p_b = p_{a^*} p_{b^*} = 0$.

Proposition 2.1. *If a, b are elements of $\mathcal{A}$, then the following conditions are equivalent:*

(i) $a \preceq b$.

(ii) *There is an element c of $\mathcal{A}$ such that $a \perp c$ and $b = c + a$.*

Proof. (i)$\Rightarrow$(ii). If $c = b - a$, then, using $a^*a = a^*b$ and $aa^* = ba^*$, it is clear that $ca^* = a^*c = 0$. Furthermore, $b = a + (b - a) = a + c$.

(ii)$\Rightarrow$(i). If there is c such that $ca^* = a^*c = 0$ and $b = a + c$, then, multiplying $b = a + c$ by a^* from the left, we obtain $a^*b = a^*a + a^*c = a^*a$. In the same way, we can prove that $aa^* = ba^*$. $\square$

Let us note that the equivalence of (i) and (ii) in the previous theorem was first proved by HESTENES [10] for matrix algebras. Further, the condition (ii) is a natural extension of definition of the Gudder type order (see [6]). Let us recall that the Gudder order, $\leq_G$, has been defined on the bounded self-adjoint operators acting on a Hilbert space $\mathscr{H}$ by $a \leq_G b$, if $(b-a)b = 0$. The condition (ii) also provides the useful insight into the star order. Loosely speaking, $a \preceq b$ means that a is an orthogonal part of b. Moreover, we can see that the second orthogonal part $c = b - a$ of b also satisfies $c \preceq b$ because the *-orthogonality is a symmetric relation.

The following proposition is a minor modification of the results known for the Gudder order [6] and the star order on $\mathscr{B}(\mathscr{H})$ [1].

Proposition 2.2. *Let $\mathcal{C}$ be a C^*-algebra acting on a Hilbert space $\mathscr{H}$ and let $a, b \in \mathcal{C}$. Then the following conditions are equivalent:*

(i) $a \preceq b$.

(ii) $a\xi = b\xi$ *for any* $\xi \in \overline{\mathcal{R}(a^*)}$ *and* $a^*\zeta = b^*\zeta$ *for any* $\zeta \in \overline{\mathcal{R}(a)}$.

(iii) $a = bp_{a^*}$ *and* $a^* = b^*p_a$.

PROOF. (i)$\Rightarrow$(ii). If $a \preceq b$ then $aa^*\psi = ba^*\psi$ for any $\psi \in \mathscr{H}$. Therefore $a\xi = b\xi$ for all $\xi \in \overline{\mathcal{R}(a^*)}$. Analogously, $a^*\zeta = b^*\zeta$ for all $\zeta \in \overline{\mathcal{R}(a)}$.

(ii)$\Rightarrow$(iii). By (ii), we immediately have $ap_{a^*}\xi = bp_{a^*}\xi$ and $a^*p_a\xi = b^*p_a\xi$ for all $\xi \in \mathscr{H}$. Further, $ap_{a^*} = (p_{a^*}a^*)^* = a$ and $a^*p_a = (p_a a)^* = a^*$ and so (iii) holds.

(iii)$\Rightarrow$(i). If (iii) holds, then

$$a^*b = b^*p_a b = b^*p_a p_a b = a^*a,$$

$$ba^* = bp_{a^*}b^* = bp_{a^*}p_{a^*}b^* = aa^*. \qquad \square$$

The next proposition describes the basic properties of the star order. Some of them were published in the case of $\mathcal{A}$ being a matrix algebra in [7]. The proof follows directly from the definition of the star order and will be omitted. Let us recall that by a *-*homomorphism* we shall mean an algebraic homomorphism preserving the involution.

Proposition 2.3. *Let a, b be elements of $\mathcal{A}$.*

(i) $a \preceq b$ *if and only if* $a^* \preceq b^*$.

(ii) $a \preceq b$ *if and only if* $\lambda a \preceq \lambda b$ *for any* $\lambda \in \mathbb{C} \setminus \{0\}$.

(iii) *If* $a \preceq b$ *and the element* $x \in \mathcal{A}$ *commutes with* a *and* b, *then* $ax \preceq bx$.

(iv) *If* $u, v \in \mathcal{A}$ *are unitary elements, then* $a \preceq b$ *if and only if* $uav \preceq ubv$.

(v) *If a is a normal element and $a \preceq b$, then $a^*b = ba^*$. If, in addition, at least one of elements a and b is self-adjoint, then a commutes with b.*

(vi) *If $\Phi : \mathcal{A} \to \mathcal{B}$ is a *-homomorphism between proper *-algebras and $a \preceq b$, then $\Phi(a) \preceq \Phi(b)$.*

Further simple features of the star order will be discussed in a series of the following propositions. In particular, we show that the set of all upper bounds of a given element is convex. Further, we show that the star order is "preserved" by a tensor product. Finally, we shall discuss the behavior of the star order with respect to the strong-operator limit.

Proposition 2.4. *Let a, b_1, and b_2 be elements of $\mathcal{A}$. If $a \preceq b_1$ and $a \preceq b_2$, then $a \preceq \lambda b_1 + (1-\lambda)b_2$ for every $\lambda \in \mathbb{C}$.*

Proof. If $a \preceq b_1$ and $a \preceq b_2$, then

$$a^*(\lambda b_1 + (1-\lambda)b_2) = \lambda a^*b_1 + (1-\lambda)a^*b_2 = \lambda a^*a + (1-\lambda)a^*a = a^*a.$$

Similarly, $(\lambda b_1 + (1-\lambda)b_2)a^* = aa^*$. □

Proposition 2.5. *Let $\mathcal{C}$ be a C^*-algebra. If $a_i, b_i \in \mathcal{C}$ $(i = 1, 2)$ and $a_i \preceq b_i$, then $a_1 \otimes a_2 \preceq b_1 \otimes b_2$ in $\mathcal{C} \otimes \mathcal{C}$.*

Proof. From $a_i \preceq b_i$ and elementary properties of the tensor product, it follows that

$$(a_1 \otimes a_2)^*(b_1 \otimes b_2) = (a_1^*b_1) \otimes (a_2^*b_2) = (a_1^*a_1) \otimes (a_2^*a_2) = (a_1 \otimes a_2)^*(a_1 \otimes a_2).$$

Similarly, $(b_1 \otimes b_2)(a_1 \otimes a_2)^* = (a_1 \otimes a_2)(a_1 \otimes a_2)^*$. □

Proposition 2.6. *Let $\mathcal{C}$ be a C^*-algebra acting on a Hilbert space. Suppose that (b_α) is a net of elements from $\mathcal{C}$ whose limit in the strong-operator topology is $b \in \mathcal{C}$. If $a \preceq b_\alpha$ for all α, then $a \preceq b$.*

Proof. If $a \preceq b_\alpha$ for all α, then $a^*a = a^*b_\alpha$ and $aa^* = b_\alpha a^*$. Since the multiplication is separately continuous in the strong-operator topology, $a^*a = a^*b$ and $aa^* = ba^*$. □

We have seen in Proposition 2.3(v) that if $a \preceq b$ $(a, b \in \mathcal{A})$ and a is self-adjoint, then a commutes with b. If we restrict our attention to C^*-algebras, we can strengthen this result even for normal elements. Note that the following proposition and theorem were proved for the case of $\mathscr{B}(\mathscr{H})$ in [1]. The extension to C^*-algebras is straightforward. Nevertheless, we give alternative proofs based on C^*-algebraic viewpoint.

Proposition 2.7. *Let $\mathcal{C}$ be a C^*-algebra and let a and b be elements of $\mathcal{C}$. If a is normal and $a \preceq b$, then a commutes with b.*

Proof. By Gelfand-Naimark theorem, there is a faithful representation π of $\mathcal{C}$ on a Hilbert space $\mathscr{H}$. If $a \preceq b$, it follows from Proposition 2.3(vi) that $\pi(a) \preceq \pi(b)$. Since a is normal, $\pi(a)$ has to be normal and so the corresponding range projection $p_{\pi(a)}$ satisfies $p_{\pi(a)} = p_{\pi(a)^*}$. Using this and Proposition 2.2(iii), we obtain

$$\pi(a) = \pi(b)p_{\pi(a)} = p_{\pi(a)}\pi(b).$$

Therefore

$$\pi(a)\pi(b) = \pi(b)p_{\pi(a)}\pi(b) = \pi(b)\pi(a).$$

Consequently, a commutes with b. □

Theorem 2.8. *Let $\mathcal{C}$ be a unital C^*-algebra and let a and b be normal elements of $\mathcal{C}$ such that $a \preceq b$. Suppose that $f : \sigma(b) \cup \{0\} \to \mathbb{C}$, where $\sigma(b)$ is the spectrum of b, is a continuous function satisfying $f(0) = 0$. Then $f(a) \preceq f(b)$.*

Proof. Since a is normal and $a \preceq b$, it follows from Proposition 2.7 that a and b commute. Therefore a, b, and the unit element $\mathbf{1}$ of $\mathcal{C}$ generate a unital abelian C^*-algebra which is *-isomorphic to an algebra $C(X)$ of all continuous complex-valued functions on a compact Hausdorff topological space X. Let Φ be a corresponding *-isomorphism from the subalgebra of $\mathcal{C}$ onto $C(X)$. Since the spectrum of a continuous function is the range of the function, we obtain from simple properties of the star order on functions (see, for example, Proposition 4.1 below) that $\sigma(a) \subseteq \sigma(b) \cup \{0\}$, where $\sigma(a)$ and $\sigma(b)$ are the spectra of a and b, respectively. Further, Proposition 2.3(vi) implies that $\Phi(a) \preceq \Phi(b)$. In addition, it can be proved (see Proposition 4.1) that $\Phi(a) \preceq \Phi(b)$ if and only if $\Phi(a) = \chi_{\mathrm{Supp}(\Phi(a))}\Phi(b)$, where $\chi_{\mathrm{Supp}(\Phi(a))}$ is the characteristic function of $\mathrm{Supp}(\Phi(a)) = \{x \in X | \Phi(a)(x) \neq 0\}$. Using the assumption $f(0) = 0$, we obtain

$$f\big|_{\sigma(a)} \circ \Phi(a) = f\big|_{\sigma(a)} \circ (\chi_{\mathrm{Supp}(\Phi(a))}\Phi(b)) = (f\big|_{\sigma(b)} \circ \Phi(b))\chi_{\mathrm{Supp}(\Phi(a))}$$

which implies $f|_{\sigma(a)} \circ \Phi(a) \preceq f|_{\sigma(b)} \circ \Phi(b)$. Since $f|_{\sigma(a)} \circ \Phi(a) = \Phi(f|_{\sigma(a)}(a))$ and $f|_{\sigma(b)} \circ \Phi(b) = \Phi(f|_{\sigma(b)}(b))$ (see Proposition 4.4.7 in [11]), we have $\Phi(f|_{\sigma(a)}(a)) \preceq \Phi(f\big|_{\sigma(b)}(b))$. From this it follows, by Proposition 2.3(vi), that $f(a) \preceq f(b)$. □

As a direct consequence of the preceding theorem, it can be shown that the modulus of an element preserves the star order on a unital C^*-algebra $\mathcal{C}$. The proof is based on the same arguments which have been applied in the case $\mathscr{B}(\mathscr{H})$ (see [1]) and therefore will be omitted. Recall that the *modulus* $|a|$ of an element $a \in \mathcal{C}$ is defined by $|a| = \sqrt{a^*a}$.

Corollary 2.9. *Let $\mathcal{C}$ be a unital C^*-algebra and a, b be elements of $\mathcal{C}$. If $a \preceq b$, then $|a| \preceq |b|$.*

3. Star order and partial isometries

In this section, we study the star order on partial isometries of the algebra $\mathcal{A}$. Recall that $u \in \mathcal{A}$ is called *partial isometry* if $u = uu^*u$. The following statement, first pointed out by Drazin [5], says that any element less than or equal to a partial isometry has to be a partial isometry.

Proposition 3.1. *If u, v are elements of $\mathcal{A}$, $u \preceq v$, and v is a partial isometry, then u is also partial isometry.*

Proof. Since v is a partial isometry, $v = vv^*v$. By $u \preceq v$, we have

$$u^*u = u^*v = u^*vv^*v = u^*uu^*u.$$

Thanks to this

$$(u - uu^*u)^*(u - uu^*u) = u^*u - u^*uu^*u - u^*uu^*u + u^*uu^*uu^*u = 0.$$

Hence $u = uu^*u$. □

It is easy to see that $u \in \mathcal{A}$ is a partial isometry if and only if u^*u is a projection. Indeed, if u is a partial isometry, then $u^*u = u^*uu^*u$. The converse implication was shown in the course of the proof of the previous result. Another equivalent condition is that uu^* is a projection. The projection u^*u is called *initial projection* of u and uu^* is called *final projection* of u. We can interpret the next proposition as an analogue of Proposition 2.2 for partial isometries in $\mathcal{A}$ where the range projections are replaced by initial and final projections.

Proposition 3.2. *Let $u, v \in \mathcal{A}$ be partial isometries. Then $u \preceq v$ if and only if $u = fv = ve$, where $e = u^*u$ and $f = uu^*$.*

Proof. If $u \preceq v$, then

$$u = uu^*u = uu^*v = fv,$$

$$u = uu^*u = vu^*u = ve.$$

Conversely, if $u = fv = ve$, then

$$u^*u = u^*fv = u^*uu^*v = u^*v,$$

$$uu^* = veu^* = vu^*uu^* = vu^*.$$ □

Projections e and f in $\mathcal{A}$ are said to be *equivalent*, written $e \sim f$, if there is a partial isometry u in $\mathcal{A}$ such that $u^*u = e$ and $uu^* = f$. It turns out that the order of partial isometries implies the equivalence of certain projections and also the order of the corresponding initial as well as final projections. Let us note that it follows immediately from the definition of the star order that the projections e and f satisfy $e \preceq f$ if and only if $e \leq f$, where $e \leq f$ is the standard order on projections defined by $ef = e$.

Theorem 3.3. *Suppose that $u_i \in \mathcal{A}$ $(i = 1, 2)$ are partial isometries such that $u_i^*u_i = e_i$ and $u_iu_i^* = f_i$. If $u_1 \preceq u_2$, then*

$$e_1 \leq e_2,\ f_1 \leq f_2,\ \text{and } e_2 - e_1 \sim f_2 - f_1.$$

Proof. From $u_1 \preceq u_2$, it follows that $u_1^*u_1 = u_1^*u_2$ and $u_1u_1^* = u_2u_1^*$. By this and Proposition 3.2, we have

$$e_1e_2 = e_1u_2^*u_2 = u_1^*u_2 = u_1^*u_1 = e_1,$$

$$f_1f_2 = f_1u_2u_2^* = u_1u_2^* = u_1u_1^* = f_1.$$

Therefore $e_1 \leq e_2$ and $f_1 \leq f_2$. Using Proposition 2.1 and Proposition 3.1, we see that $u = u_2 - u_1$ is a partial isometry satisfying $u \preceq u_2$. Now we show that $u^*u = e_2 - e_1$ and $uu^* = f_2 - f_1$. We have

$$u^*u = (u_2 - u_1)^*(u_2 - u_1) = u_2^*u_2 - u_2^*u_1 - u_1^*u_2 + u_1^*u_1 = u_2^*u_2 - u_1^*u_1 = e_2 - e_1,$$

$$uu^* = (u_2 - u_1)(u_2 - u_1)^* = u_2u_2^* - u_2u_1^* - u_1u_2^* + u_1u_1^* = u_2u_2^* - u_1u_1^* = f_2 - f_1.$$

Thus $e_2 - e_1 \sim f_2 - f_1$. □

In the following theorem, partial isometries, which are above a given partial isometry, are characterized in terms of initial and final projections. A similar result for partial isometries which are below a fixed partial isometry is proved in Theorem 3.5.

Theorem 3.4. *Suppose that $e_i, f_i \in \mathcal{A}$ $(i = 1, 2)$ are projections. Let u_1 be a partial isometry such that $u_1^*u_1 = e_1$ and $u_1u_1^* = f_1$. Then the following conditions are equivalent:*

(ii) *There is a partial isometry u_2 such that $u_2^*u_2 = e_2$, $u_2u_2^* = f_2$, and $u_1 \preceq u_2$.*

(ii) *$e_1 \leq e_2$, $f_1 \leq f_2$, and $e_2 - e_1 \sim f_2 - f_1$.*

Proof. (i)⇒(ii) follows immediately from the previous theorem.

(ii)⇒(i). If $e_2 - e_1 \sim f_2 - f_1$, then there is a partial isometry v such that $v^*v = e_2 - e_1$ and $vv^* = f_2 - f_1$. Since

$$(u_1^*v)^*(u_1^*v) = v^*u_1u_1^*v = v^*f_1v = v^*f_1vv^*v = v^*f_1(f_2-f_1)v = v^*(f_1-f_1)v = 0,$$

$$(vu_1^*)^*(vu_1^*) = u_1v^*vu_1^* = u_1(e_2 - e_1)u_1^* = u_1e_1(e_2 - e_1)u_1^* = u_1(e_1 - e_1)u_1^* = 0,$$

we have $u_1^*v = vu_1^* = 0$. Now we set $u_2 = u_1 + v$. It is easily seen (e.g., by Proposition 2.1) that $u_1 \preceq u_2$. Moreover,

$$u_2^*u_2 = (u_1 + v)^*(u_1 + v) = u_1^*u_1 + u_1^*v + v^*u_1 + v^*v = e_1 + e_2 - e_1 = e_2,$$

$$u_2u_2^* = (u_1 + v)(u_1 + v)^* = u_1u_1^* + u_1v^* + vu_1^* + vv^* = f_1 + f_2 - f_1 = f_2.$$

Thanks to this u_2 is a partial isometry with required properties. □

Theorem 3.5. *Suppose that $e_i, f_i \in \mathcal{A}$ $(i = 1, 2)$ are projections. Let u_2 be a partial isometry such that $u_2^*u_2 = e_2$ and $u_2u_2^* = f_2$. Then the following conditions are equivalent:*

(i) *There is a partial isometry u_1 such that $u_1^*u_1 = e_1$, $u_1u_1^* = f_1$, and $u_1 \preceq u_2$.*

(ii) *$e_1 \le e_2$, $f_1 \le f_2$, and $u_2e_1 = f_1u_2$.*

Proof. (i)⇒(ii). It follows from Theorem 3.3 that $e_1 \le e_2$ and $f_1 \le f_2$. Since $u_1 \preceq u_2$, we obtain, by Proposition 3.2, that $f_1u_2 = u_2e_1$.

(ii)⇒(i). Let us set $u_1 = u_2e_1$. Then

$$u_1^*u_1 = e_1u_2^*u_2e_1 = e_1e_2e_1 = e_1,$$

$$u_1u_1^* = u_2e_1e_1u_2^* = u_2e_1u_2^* = f_1u_2u_2^* = f_1f_2 = f_1.$$

Thus u_1 is a partial isometry and, by Proposition 3.2, $u_1 \preceq u_2$. □

Let us remark that, in the light of Proposition 3.1, the previous theorem describes the set of all elements underneath a given partial isometry.

A projection $e \in \mathcal{A}$ is called *infinite projection* if $e \sim e_0 < e$ for some projection e_0 in $\mathcal{A}$. The following theorem describes the connection between infinite projections and the star order on partial isometries.

Theorem 3.6. *Assume that $\mathcal{A}$ has a unit element $\mathbf{1}$ and $f_i \in \mathcal{A}$ $(i = 1, 2)$ are projections. Then the following conditions are equivalent:*

(i) *$f_2 \sim f_1 < f_2$.*

(ii) *There are partial isometries* $u_i \in \mathcal{A}$ $(i = 1, 2)$ *such that* $u_1 \prec u_2$, $u_1 u_1^* = u_2^* u_2 = f_1$, *and* $u_2 u_2^* = f_2$.

PROOF. (i)⇒(ii). By $f_2 \sim f_1$, there is a partial isometry u_2 such that $u_2^* u_2 = f_1$ and $u_2 u_2^* = f_2$. Put $e = u_2^* f_1 u_2$. Now we show that e is a projection. Indeed, $e^* = e$. Furthermore,

$$e^2 = u_2^* f_1 u_2 u_2^* f_1 u_2 = u_2^* f_1 f_2 f_1 u_2 = u_2^* f_1 u_2.$$

Thus e is a projection. Further we show that $e < f_1$. Since

$$u_2^*(\mathbf{1} - f_1)u_2 = ((\mathbf{1} - f_1)u_2)^*((\mathbf{1} - f_1)u_2),$$

we have $u_2^*(\mathbf{1} - f_1)u_2 \geq 0$. Now suppose that $u_2^*(\mathbf{1} - f_1)u_2 = 0$. Then

$$0 = u_2 u_2^*(\mathbf{1} - f_1)u_2 u_2^* = f_2(\mathbf{1} - f_1)f_2 = f_2 - f_1,$$

which is a contradiction with $f_1 < f_2$. Therefore $u_2^*(\mathbf{1}-f_1)u_2 = u_2^* u_2 - u_2^* f_1 u_2 > 0$ and so $e < f_1$. Now let us put $u_1 = u_2 e$. It remains to show that u_1 is a partial isometry with the required properties. We can compute $u_1 = u_2 u_2^* f_1 u_2 = f_2 f_1 u_2 = f_1 u_2$. Further,

$$u_1^* u_1 = e u_2^* u_2 e = e f_1 e = e,$$

$$u_1 u_1^* = f_1 u_2 u_2^* f_1 = f_1 f_2 f_1 = f_1.$$

Hence u_1 is a partial isometry. Using Proposition 3.2, we obtain that $u_1 \preceq u_2$. Since $f_1 < f_2$, we have $u_1 \neq u_2$.

Now we show that $(ii) \Rightarrow (i)$. From $u_1 \prec u_2$ we have, applying Theorem 3.3, $e \leq f_1$, $f_1 \leq f_2$, and $f_1 - e \sim f_2 - f_1$, where e is the initial projection of u_1. If we suppose that $f_1 = f_2$, then, using Proposition 3.2, we can compute

$$u_1 = f_1 u_2 = f_2 u_2 = u_2,$$

which is a contradiction. Thus $f_1 < f_2$. □

Our results developed in the general context of proper *-algebras may be applied to a comparison theory of C^*-algebras and von Neumann algebras. This enables us, for example, to characterize infiniteness of C^*-algebras in Murray-von Neumann comparison theory. Let us recall that an element u of a unital C^*-algebra is called *coisometry* if $uu^* = \mathbf{1}$. A unital C^*-algebra is called *infinite* if its unit element $\mathbf{1}$ is infinite.

Corollary 3.7. *A unital* C^**-algebra* $\mathcal{C}$ *is infinite if and only if there are a partial isometry* $u_1 \in \mathcal{C}$ *and a coisometry* $u_2 \in \mathcal{C}$ *such that* $u_1 \prec u_2$ *and* $u_1 u_1^* = u_2^* u_2$.

4. Infimum problem for function algebras

Let $C(X)$ be a *-algebra of all continuous complex-valued functions on a Hausdorff topological space X and $f \in C(X)$. We shall denote $\mathrm{Supp}(f) = \{x \in X | f(x) \neq 0\}$ and $\mathrm{Null}(f) = \{x \in X | f(x) = 0\}$. The characteristic function of a set M is denoted by χ_M. Let us remark that, in the case of $C(X)$, the definition of $f \preceq g$ ($f, g \in C(X)$) is reduced to only one equation $\overline{f} f = \overline{f} g$, where $\overline{f}$ is the complex conjugate of f. This definition can be expressed in useful equivalent ways which are summarized in the following proposition analogous to Proposition 2.1 and Proposition 2.2. A similar result for random variables can be found in [6].

Proposition 4.1. *If $f, g \in C(X)$, then the following conditions are equivalent:*

(i) $f \preceq g$.

(ii) $f(x) = g(x)$ *for all* $x \in \mathrm{Supp}(f)$.

(iii) $f = g\chi_{\mathrm{Supp}(f)}$.

(iv) *There is a function* $h \in C(X)$ *such that* $\overline{f} h = 0$ *and* $g = f + h$.

Proof. (i)⇔(iv) is a special case of Proposition 2.1.

(i)⇒(ii). If $f \preceq g$, then $\overline{f(x)} f(x) = \overline{f(x)} g(x)$ for any $x \in X$. Consequently, $f(x) = g(x)$ for $x \in \mathrm{Supp}(f)$.

(ii)⇒(iii). If $x \in \mathrm{Supp}(f)$, then

$$f(x) = g(x) = g(x)\chi_{\mathrm{Supp}(f)}(x).$$

If $x \in \mathrm{Null}(f)$, then

$$f(x) = 0 = g(x)\chi_{\mathrm{Supp}(f)}(x).$$

Since $X = \mathrm{Supp}(f) \cup \mathrm{Null}(f)$, we obtain $f = g\chi_{\mathrm{Supp}(f)}$.

(iii)⇒(iv) We have

$$\overline{f} f = \overline{f} g \chi_{\mathrm{Supp}(f)} = \overline{f \chi_{\mathrm{Supp}(f)}} g = \overline{f} g. \qquad \square$$

Motivated by the condition (iii) in the preceding proposition, let us concentrate on the question of when the function $\chi_M f$, where $f \in C(X)$ and M is an open subset of X, is continuous.

Proposition 4.2. *Let X be a Hausdorff topological space and let $M \subseteq X$ be open. Assume that $f : X \to \mathbb{C}$ is continuous. Then $\chi_M f$ is a continuous function if and only if f vanishes on ∂M.*

Proof. Suppose that $\chi_M f$ is a continuous function. Let z be an element of ∂M. Then there is a net (x_α) such that $x_\alpha \in M$ and $x_\alpha \to z$. Moreover, $z \notin M$ because M is an open set. Since $\chi_M f$ is a continuous function, we have

$$f(x_\alpha) = (\chi_M f)(x_\alpha) \to (\chi_M f)(z) = 0.$$

As f is a continuous function, $f(x_\alpha) \to f(z)$. Hence $f(z) = 0$.

For the converse implication suppose now that f vanishes on ∂M. Let us put $g = \chi_{\overline{M}} f$. It is easy to see that $g|_{\overline{M}}$ and $g|_{X \setminus M}$ are continuous functions. It implies (see, for example, [18, Theorem 7.6]) that the function $g = \chi_{\overline{M}} f = \chi_M f$ is continuous. □

Consider the functions $f_1, f_2 \in C(X)$. Now we would like to describe the set of all elements $g \in C(X)$ satisfying $g \preceq f_1$ and $g \preceq f_2$ since the infimum $f_1 \wedge f_2$ is a maximal element of this set. Denote

$$\Omega = \{x \in X | f_1(x) = f_2(x), f_1(x) \neq 0\},$$
$$\mathcal{M} = \{M | M \subseteq \Omega, \chi_M f_1 \in C(X)\}.$$

It is clear, by Proposition 4.1, that g has the form $g = \chi_M f_1$, where $M \in \mathcal{M}$. Conversely, any element M of $\mathcal{M}$ uniquely determines the function g satisfying $g \preceq f_1$ and $g \preceq f_2$. Therefore we can investigate the set $\mathcal{M}$ instead of the set of all common lower bounds for f_1 and f_2. Note that $\mathcal{M}$ is a nonempty set because $\emptyset \in \mathcal{M}$.

Lemma 4.3. *If M is an element of $\mathcal{M}$, then M is an open set.*

Proof. If $M \in \mathcal{M}$, then $\chi_M f_1$ is a continuous function and therefore $\mathrm{Null}(\chi_M f_1)$ is closed. Moreover, $\mathrm{Supp}(\chi_M f_1) = M$. From this it follows that $M = X \setminus \mathrm{Null}(\chi_M f_1)$ is an open set. □

Proposition 4.4. *If M, N are elements of $\mathcal{M}$, then $M \cup N$ is an element of $\mathcal{M}$.*

Proof. As M and N are open, it is clear that $\partial(M \cup N) \subseteq \partial M \cup \partial N$. By Proposition 4.2, we have that $f(x) = 0$ for any $x \in \partial(M \cup N)$ and therefore $\chi_{M \cup N} f_1$ is continuous. Clearly, $M \cup N \subseteq \Omega$. Hence $M \cup N \in \mathcal{M}$. □

Let us recall some topological concepts. We say that a family of neighborhoods of x in a topological space X is a *local base at x* if every neighborhood of x contains a member of the family. A topological space X is called *locally connected* if each point of X has a local base consisting of connected open sets. It can be shown that X is locally connected if and only if connected components of each open set are open. The topological space X is called *extremely disconnected* if closure of every open set is open. A more detailed treatment of these notions can be found, for example, in [12], [18].

We have seen in Proposition 4.4 that the family $\mathcal{M}$ is closed under forming finite unions. It implies that the infimum $f_1 \wedge f_2$ exists in the case when the family $\mathcal{M}$ has finitely many elements. However, the situation is more complicated if the family $\mathcal{M}$ is infinite. In the sequel, we shall prove that the infimum $f_1 \wedge f_2$ exists in locally connected spaces. Before doing this, we need some auxiliary topological results.

Lemma 4.5. *Let S be an open set in a topological space X and let U be an open connected subset of X such that $S \cap U \neq \emptyset$ and $U \setminus S \neq \emptyset$. Then there is $z \in \partial S$ such that $z \in U$.*

Proof. For a contradiction, suppose that there is no $z \in \partial S$ such that $z \in U$. It implies that

$$\emptyset \neq S \cap U = \overline{S} \cap U.$$

Thus $\overline{S} \cap U$ and $U \cap (X \setminus \overline{S}) = U \setminus \overline{S}$ are nonempty open disjoint sets with union U. It is a contradiction with connectedness of U. □

Lemma 4.6. *Let X be a locally connected Hausdorff topological space and let $\mathcal{N}$ be a family of open sets in X. Assume that (x_α) is a net of elements of $\bigcup_{N\in\mathcal{N}} N$ satisfying $x_\alpha \to x$, where $x \in X \setminus \bigcup_{N\in\mathcal{N}} N$. Then there is a net (y_β) such that $y_\beta \in \bigcup_{N\in\mathcal{N}} \partial N$ and $y_\beta \to x$.*

Proof. Let $\mathscr{B}_x$ be a local base at x consisting of connected open sets. If $x_\alpha \to x$, then for any element $U \in \mathscr{B}_x$ there is α_0 such that x_α is an element of U for any α satisfying $\alpha_0 \leq \alpha$. For the given element $U \in \mathscr{B}_x$ there exists $N \in \mathcal{N}$ such that $U \cap N \neq \emptyset$ and $U \setminus N \neq \emptyset$ since $x_\alpha \in \bigcup_{M\in\mathcal{N}} M$ and $x \in X \setminus \bigcup_{M\in\mathcal{N}} M$. By Lemma 4.5, we have that there exists a point $y_U \in \partial N$ such that $y_U \in U$. In this way, we can construct the net (y_U), indexed by the elements of $\mathscr{B}_x$, such that $y_U \in \bigcup_{N\in\mathcal{N}} \partial N$ and $y_U \to x$. □

Using the previous result, we prove the following theorem which plays a key role in further discussion. A simple consequence of this theorem is Corollary 4.8 which implies that $\bigcup_{M\in\mathcal{M}} M$ is an element of $\mathcal{M}$ in the case of a locally connected topological space. It ensures the existence of the infimum $f_1 \wedge f_2$ in locally connected spaces (see Theorem 4.9).

Theorem 4.7. *Let X be a locally connected Hausdorff topological space and let $f : X \to \mathbb{C}$ be a continuous function. If $\mathcal{N}$ is a family of open sets such that $\chi_N f$ is continuous for every $N \in \mathcal{N}$, then $\chi_M f$, where $M = \bigcup_{N\in\mathcal{N}} N$, is continuous.*

Proof. Let $M = \bigcup_{N\in\mathcal{N}} N$. Using Lemma 4.6, we obtain that for $z \in \partial M$ there exists a net (y_α) such that $y_\alpha \in \bigcup_{N\in\mathcal{N}} \partial N$ and $y_\alpha \to z$. Since $\chi_N f$ is continuous for any $N \in \mathcal{N}$, we have, by Proposition 4.2, that f vanishes on $\bigcup_{N\in\mathcal{N}} \partial N$. Thus $0 = f(y_\alpha) \to f(z)$ and so f vanishes on ∂M. It follows from Proposition 4.2 that $\chi_M f$ is a continuous function. □

Corollary 4.8. *If X is a locally connected Hausdorff topological space and $\mathcal{N} \subseteq \mathcal{M}$, then $\bigcup_{N\in\mathcal{N}} N \in \mathcal{M}$.*

Proof. Denote $M = \bigcup_{N\in\mathcal{N}} N$. From Theorem 4.7, it follows that $\chi_M f_1$ is a continuous function. Moreover, it is clear that $M \subseteq \Omega$. Hence $M \in \mathcal{M}$. □

Theorem 4.9. *Suppose that $f_1, f_2 \in C(X)$, where X is a locally connected Hausdorff topological space. Then $f_1 \wedge f_2$ exists in $C(X)$. Moreover, $f_1 \wedge f_2 = \chi_N f_1$, where $N = \bigcup_{M\in\mathcal{M}} M$.*

Proof. Let us put $N = \bigcup_{M\in\mathcal{M}} M$. It follows from Corollary 4.8 that $N \in \mathcal{M}$ and so $g = \chi_N f_1$ satisfies $g \preceq f_1, f_2$. Now we prove that if $h \in C(X)$ and $h \preceq f_1, f_2$, then $h \preceq g$. Clearly, $h = \chi_A f_1$ where $A \in \mathcal{M}$. However, since $N = \bigcup_{M\in\mathcal{M}} M$, $A \subseteq N$ which implies, by Proposition 4.1, that $h \preceq g$. □

The next result says that the infimum exists in an abelian C^*-algebra which has locally connected spectrum. It follows immediately from Theorem 4.9 and Proposition 2.3(vi).

Corollary 4.10. *Suppose that $\mathcal{C}$ is an abelian C^*-algebra whose spectrum is locally connected. Let a and b be elements of $\mathcal{C}$. Then $a \wedge b$ exists.*

We have seen that the infimum problem in $C(X)$ has a positive answer when X is locally connected. Next, we show that there is a positive answer also in the case of an extremely disconnected space.

Theorem 4.11. *Let X be an extremely disconnected Hausdorff topological space and $f_1, f_2 \in C(X)$. Then $f_1 \wedge f_2$ exists. Moreover, $f_1 \wedge f_2 = \chi_N f_1$, where $N = \bigcup_{M\in\mathcal{M}} M$.*

Proof. Denote $N = \bigcup_{M\in\mathcal{M}} M$. Since X is an extremely disconnected Hausdorff topological space, $\overline{N}$ is a clopen set. Thanks to this, $\chi_{\overline{N}}$ is a continuous function and therefore $\chi_{\overline{N}} f_1$ is continuous. Put $S = \{x \in \overline{N} | f_1(x) \neq 0\}$. It is clear that $\chi_{\overline{N}} f_1 = \chi_S f_1$ and $S \subseteq \Omega$. Hence $S \in \mathcal{M}$ and so $S \subseteq N$. It can be easily verified that $N \subseteq S$. Therefore $N = S$ which implies $\chi_{\overline{N}} f_1 = \chi_N f_1$. The same arguments as in the proof of Theorem 4.9 show that $g = \chi_N f_1$ is the infimum $f_1 \wedge f_2$. □

The next consequence of Theorem 4.11 follows immediately from the fact that an abelian AW^*-algebra is *-isomorphic to $C(X)$ where X is an extremely disconnected compact Hausdorff topological space.

Corollary 4.12. *Let $\mathcal{C}$ be an abelian AW^*-algebra and $a, b \in \mathcal{C}$. Then $a \wedge b$ exists.*

Note that, in the special case of abelian von Neuman algebras, the preceding result can be proved without using Theorem 4.11. It is well known that an abelian von Neumann algebra is *-isomorphic to $L^\infty(\Gamma, \mu)$ for which the proof is straighforward.

The results of this section, concerning the infimum problem for two functions, can be easily generalized to the infimum problem for an arbitrary subset of $C(X)$. Indeed, let $(f_\alpha)_{\alpha \in \mathbb{A}}$ be a family of elements of $C(X)$. For fixed $\beta \in \mathbb{A}$, we can denote

$$\Omega = \{x \in X | f_\beta(x) = f_\alpha(x) \text{ for all } \alpha \in \mathbb{A} \setminus \{\beta\}, f_\beta(x) \neq 0\},$$

$$\mathcal{M} = \{M | M \subseteq \Omega, \chi_M f_\beta \in C(X)\}$$

and repeat the discussion given in this section. From this, we conclude that the infimum of an arbitrary subset of $C(X)$ exists if X is a locally connected or an extremely disconnected Hausdorff topological space.

5. Supremum problem for function algebras

Let us investigate the supremum problem for a *-algebra $C(X)$ of all continuous complex-valued functions on a Hausdorff topological space. In the following theorem, we show that the supremum of $f_1, f_2 \in C(X)$ exists if and only if there is a common upper bound for f_1 and f_2.

Theorem 5.1. *Let f_1 and f_2 be elements of $C(X)$. The supremum $f_1 \vee f_2$ exists if and only if there is $h \in C(X)$ such that $f_1, f_2 \preceq h$.*

Proof. If $f_1 \vee f_2$ exists, then $f_1, f_2 \preceq f_1 \vee f_2$.

Let us prove the reverse implication. Denote $M_1 = \operatorname{Supp}(f_1)$ and $M_2 = \operatorname{Supp}(f_2)$. Obviously, M_1 and M_2 are open. If there is $h \in C(X)$ such that

$f_1, f_2 \preceq h$, then $f_1 = \chi_{M_1} h$ and $f_2 = \chi_{M_2} h$. By Proposition 4.2, h vanishes on $\partial M_1 \cup \partial M_2$ and so the inclusion $\partial(M_1 \cup M_2) \subseteq \partial M_1 \cup \partial M_2$ implies that $g = \chi_{M_1 \cup M_2} h$ is continuous. Now we prove that $g = f_1 \vee f_2$. It is clear that $f_1, f_2 \preceq g$. If there is $\widetilde{g} \in C(X)$ such that $f_1, f_2 \preceq \widetilde{g}$, then $h(x) = f_1(x) = \widetilde{g}(x)$ for $x \in M_1$ and $h(x) = f_2(x) = \widetilde{g}(x)$ for $x \in M_2$. Therefore $\widetilde{g}(x) = h(x)$ for $x \in M_1 \cup M_2$. Hence $g = \chi_{M_1 \cup M_2} h = \chi_{M_1 \cup M_2} \widetilde{g}$. Thus $g \preceq \widetilde{g}$. $\square$

We have seen in the previous theorem that, unlike the infimum problem, there is no restriction on the topological space X in the case of the supremum problem of two (and so finitely many) functions. In the sequel, we shall see that the supremum problem for an arbitrary subset of $C(X)$ has a positive answer if X is locally connected or hyperstonean (for definition of the hyperstonean space see, for example, [8]).

Theorem 5.2. *Let X be a locally connected Hausdorff space. Suppose that $(f_\alpha)_{\alpha \in \mathbb{A}}$ is a family of elements of $C(X)$. Then the following conditions are equivalent:*

(i) *There exists $\bigvee_{\alpha \in \mathbb{A}} f_\alpha$.*

(ii) *There is $h \in C(X)$ such that $f_\alpha \preceq h$ for any $\alpha \in \mathbb{A}$.*

Proof. (i)$\Rightarrow$(ii). Let us put $\mathcal{N} = \{N | N = \mathrm{Supp}(f_\alpha), \alpha \in \mathbb{A}\}$ and $M = \bigcup_{N \in \mathcal{N}} N$. Since $f_\alpha \preceq h$ for each $\alpha \in \mathbb{A}$, we have $\chi_N h$ is continuous for every $N \in \mathcal{N}$. By Theorem 4.7, the function $g = \chi_M h$ is continuous. Similarly to the proof of Theorem 5.1, we can verify that the function g is the supremum of the family $(f_\alpha)_{\alpha \in \mathbb{A}}$.

(ii)$\Rightarrow$(i) is clear. $\square$

Corollary 5.3. *Suppose that $\mathcal{C}$ is an abelian C^*-algebra with a locally connected spectrum and $(a_\alpha)_{\alpha \in \mathbb{A}}$ is a family of elements of $\mathcal{C}$. Then the following conditions are equivalent:*

(i) *There exists $\bigvee_{\alpha \in \mathbb{A}} a_\alpha$.*

(ii) *There is $b \in \mathcal{C}$ such that $a_\alpha \preceq b$ for any $\alpha \in \mathbb{A}$.*

Let us note that we can easily prove the analogue of the preceding result in the case of abelian von Neumann algebras (i.e., in the case of $C(X)$, where X is hyperstonean) using the fact that an abelian von Neumann algebra is *-isomorphic to $L^\infty(\Gamma, \mu)$.

We say that a poset L is *boundedly complete* if any bounded subset of L has an infimum and a supremum. Since a lower bound (with respect to the star order) of any subset of $C(X)$ is a function identically equal to zero, every subset, which has upper bound, is bounded. Thus, combining results of last two sections, we obtain that the poset $C(X)$, where X is a locally connected Hausdorff topological space, is boundedly complete. On the other hand, it is well known (see [11], [17]) that $C(X)$ endowed with the usual order is boundedly complete if and only if X is an extremely disconnected topological space. We can conclude that $C(X)$ endowed with the star order forms a boundedly complete poset under the quite different condition than in the case of the usual order.

Acknowledgment. I would like to thank to J. Hamhalter for continuous support and helpful discussions during the preparation of this paper. This work was supported by the Grant Agency of Czech Technical University in Prague, grant No. SGS10/264/OHK3/3T/13.

References

[1] J. Antezana, C. Cano, I. Musconi and D. Stojanoff, A note on the star order in Hilbert spaces, *Lin. Multilin. Alg.* **58** (2010), 1037–1051.

[2] J. K. Baksalary, J. Hauke, X. Liu and S. Liu, Relationships between partial orders of matrices and their powers, *Lin. Alg. Appl.* **379** (2004), 277–287.

[3] J. K. Baksalary, F. Pukelsheim and and G. P. H. Styan, Some properties of matrix partial orderings, *Lin. Alg. Appl.* **119** (1989), 57–85.

[4] G. Dolinar and L. Molnár, Maps on quantum observables preserving the Gudder order, *Rep. Math. Phys.* **60** (2007), 159–166.

[5] M. P. Drazin, Natural structures on semigroups with involution, *Bull. Amer. Math. Soc.* **84** (1978), 139–141.

[6] S. Gudder, An order for quantum observables, *Math. Slovaca* **56** (2006), 573–589.

[7] R. E. Hartwig and M. P. Drazin, Lattice properties of the *-order for complex matrices, *J. Math. Anal. Appl.* **86** (1982), 359–378.

[8] J. Hamhalter, Quantum Measure Theory, *Kluwer Academic Publishers, Dordrecht*, 2003.

[9] J. Hamhalter, Spectral order of operators and range projections, *J. Math. Anal. Appl.* **331** (2007), 1122–1134.

[10] M. R. Hestenes, Relative hermitian matrices, *Pacific J. Math.* **11** (1961), 225–245.

[11] R. V. Kadison and J. R. Ringrose, Fundamentals of the Theory of Operator Algebras I,II, *Academic Press, New York*, 1983.

[12] J. L. Kelley, General Topology, *Springer, New York*, 1975.

[13] S. K. Mitra, P. Bhimasankaram and S. B. Malik, Matrix Partial Orders, Shorted Operators and Applications, *World Scientific Publishing Co., Singapore*, 2010.

[14] L. Molnár and P. Šemrl, Spectral order automorphisms of the spaces of Hilbert space effects and observables, *Lett. Math. Phys.* **80** (2007), 239–255.

[15] M. P. Olson, The selfadjoint operators of a von Neumann algebra form a conditionally complete lattice, *Proc. Amer. Math. Soc.* **28** (1971), 537–544.

[16] S. Pulmannová and E. Vinceková, Remarks on the order for quantum observables, *Math. Slovaca* **57** (2007), 589–600.

[17] M. H. Stone, Boundedness properties in function-lattices, *Canad. J. Math.* **1** (1949), 176–186.

[18] S. Willard, General Topology, *Addison-Wesley, Reading*, 1970.

MARTIN BOHATA
DEPARTMENT OF MATHEMATICS
FACULTY OF ELECTRICAL ENGINEERING
CZECH TECHNICAL UNIVERSITY IN PRAGUE
TECHNICKÁ 2
166 27 PRAGUE 6
CZECH REPUBLIC

E-mail: bohata@math.feld.cvut.cz

(Received September 21, 2010; revised February 26, 2011)