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Unique expansions in integer bases with extended alphabets

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Dedicated to Professors Kálmán Győry, Attila Pethő, János Pintz and András Sárközy on their birthdays

Abstract. Since their introduction by Rényi more than fifty years ago, the investigation of expansions in noninteger bases led to a number of deep and unexpected results. Some of them led to the necessity to study expansions in integer bases on an enlarged alphabet containing the base itself as a possible digit. We show in the present paper how certain recent theorems change in this framework.

1. Introduction

Beginning with Rényi [20] many works have been devoted to expansions of the form

$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i} \tag{1.1}$$

in arbitrary *real* bases $q > 1$ with integer *digits* satisfying $0 \le c_i < q$. It is easy to see that a real number x has an expansion if and only if it belongs to the closed interval

$$J_q := \left[0, \frac{\lceil q \rceil - 1}{q - 1}\right]$$

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where $\lceil q \rceil$ stands for the upper integer part of q.[1]

The case of integer bases is of course well-known: $J_q = [0,1]$ and the expansion of every $x \in [0,1]$ is unique, except if $x = m/q^n$ for some positive integers m and n. These exceptional numbers x have exactly two expansions: a *finite* one ending with 0^∞ and an *infinite* one ending with $(q-1)^\infty$. In particular, the set $\mathcal{U}_q$ of numbers x having a unique expansion is a nonclosed set of Hausdorff dimension one, having full Lebesgue measure in J_q.

For noninteger bases the situation is radically different. The set $\mathcal{U}_q$ has Hausdorff dimension strictly smaller than one and hence zero Lebesgue measure, but it is still an uncountable set. Furthermore, $\mathcal{U}_q$ is closed for almost all values of q, the exceptional bases forming a Cantor set of Hausdorff dimension one but of zero Lebesgue measure. Moreover, the set B of bases for which $\mathcal{U}_q$ is a Cantor set has both interior and exterior points, i.e., both B and $(1,\infty) \setminus B$ contain nondegenerate intervals. In cases where $\mathcal{U}_q$ is not closed, $\overline{\mathcal{U}}_q \setminus \mathcal{U}_q$ is a countably infinite set, each $x \in \overline{\mathcal{U}}_q \setminus \mathcal{U}_q$ has countably many expansions, all of which are explicitly known. We refer to [2], [3], [4], [10], [11], [12], [13], [14], [17] for details.

Sometimes more elegant results are obtained by considering expansions (1.1) with integer *digits* satisfying $0 \le c_i \le q$ (the two definitions differ only for integer bases q). Then J_q is replaced by

$$J_q^* := \left[0, \frac{\lfloor q \rfloor}{q-1}\right]$$

where $\lfloor q \rfloor$ denotes the lower integer part of q as introduced above. The purpose of this work is to investigate the modified *univoque sets* $\mathcal{U}_q^*$ for integer bases by using this extended alphabet. It turns out that their stucture is different from the usual case.

As we will see, the cases $q = 2$ and $q > 2$ are quite different.

Throughout this paper, the index set for all sequences is the set of positive integers: $(c_i) = (c_i)_{i=1}^\infty$, $(\alpha_i) = (\alpha_i)_{i=1}^\infty$, and so on. Hence we will often omit the indication of the index set.

2. Review of univoque bases

In this section we recall some results from [14]–[16]. Given a real number $q > 1$ we consider expansions in base q on the *alphabet* $\{0, \dots, \lceil q \rceil - 1\}$, i.e.

[1]If q is not integer, then its lower and upper integer parts are by definition the consecutive integers satisfying the inequalities $\lfloor q \rfloor < q < \lceil q \rceil$. If q is integer, then we define $\lfloor q \rfloor = \lceil q \rceil := q$.

equalities of the form

$$x = \frac{c_1}{q} + \frac{c_2}{q^2} + \dots \tag{2.1}$$

with integer *digits* satisfying $0 \le c_i < q$. A real number x has at least one expansion if and only if

$$0 \le x \le \frac{\lceil q \rceil - 1}{q-1}.$$

We say that q is a *univoque base* if $x = 1$ has only one expansion in base q. The integer bases are univoque, but there exist other univoque bases too. In order to characterize them, it is convenient to introduce a particular expansion $(\alpha_i) = (\alpha_i)_{i=1}^{\infty}$ of $x = 1$ in each fixed base $q > 1$ by the following algorithm: if α_n has already been defined for some $n \ge 1$ (no hypothesis if $n = 0$), then let α_{n+1} be the biggest nonnegative integer satisfying

$$\frac{\alpha_1}{q} + \dots + \frac{\alpha_{n+1}}{q^{n+1}} < 1.$$

It is called the *quasi-greedy* expansion of 1 in base q.

Remark 2.1. The sequence (α_i) always has the following two properties:

(a) $\alpha_1 = \lceil q \rceil - 1$ is the biggest element of the alphabet;

(b) we have $(\alpha_{n+i}) \le (\alpha_i)$ for all n in the lexicographic sense.

In the following theorem and in the sequel we define the conjugate of a digit c_i by $\overline{c_i} := \alpha_1 - c_i$.

Theorem 2.2 ([15, Theorem 3.1]). *A base $q > 1$ is univoque if and only if the following lexicographic inequalities are satisfied:*

$$(\alpha_{n+i}) < (\alpha_i) \quad \textit{whenever} \quad \alpha_n < \alpha_1;$$

$$(\overline{\alpha_{n+i}}) < (\alpha_i) \quad \textit{whenever} \quad \alpha_n > 0.$$

For example, the periodical sequence $(c_i)_{i=1}^{\infty} := 1(10)^{\infty}$ is the unique expansion of $x = 1$ in the base defined by the equality (2.1).

The sequence (α_i) also allows us to characterize the closure $\overline{\mathcal{U}}$ of the set $\mathcal{U}$ of univoque bases:

Theorem 2.3 ([16, Theorem 2.4]). *A base $q > 1$ belongs to $\overline{\mathcal{U}}$ if and only if*

$$(\overline{\alpha_{n+i}}) < (\alpha_i) \quad \textit{whenever} \quad \alpha_n > 0.$$

Comparing the above two theorems it is natural to investigate also the set $\mathcal{V}$ of bases q for which the second lexicographic inequality is satisfied only in the weaker sense:

$$(\overline{\alpha_{n+i}}) \le (\alpha_i) \quad \text{whenever} \quad \alpha_n > 0.$$

The topological properties of these sets are summarized in the following result:

Theorem 2.4 ([16, Theorem 2.5, 2.6])**.**

(a) *We have* $\mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{V} = \overline{\mathcal{V}}$. *All these sets have zero Lebesgue measure and Hausdorff dimension one.*

(b) $\overline{\mathcal{U}} \setminus \mathcal{U}$ *is a countable dense set in* $\overline{\mathcal{U}}$ *and therefore* $\overline{\mathcal{U}}$ *is a Cantor set.*

(c) $\mathcal{V}$ *is a closed set and* $\mathcal{V} \setminus \overline{\mathcal{U}}$ *is a discrete set, dense in* $\mathcal{V}$.

(d) *The set of expansions of* $x = 1$ *is countably infinite in each base* $q \in \mathcal{V} \setminus \mathcal{U}$.

Remarks 2.5.

(a) The smallest element of $\mathcal{V}$ is the golden ratio.

(b) $\mathcal{U}$ has a smallest element: the now so-called *Komornik–Loreti constant* is a transcendental number, its approximate value is 1.787.

Example 2.6. If the base q is an integer, then the quasi-greedy expansion of $x = 1$ is given by $\alpha_i = q - 1$ for all i. It follows from Theorem 2.2 that $q \in \mathcal{U}$.

3. Review of unique expansions

In this section we recall some results from [4]. Given a real number $q > 1$ we consider again expansions in base q on the *alphabet* $\{0, \dots, \lceil q \rceil - 1\}$, i.e. equalities of the form

$$x = \frac{c_1}{q} + \frac{c_2}{q^2} + \dots \tag{3.1}$$

with integer *digits* satisfying $0 \le c_i < q$. We denote by $\mathcal{U}_q$ the set of real numbers x which have only one expansion in base q. For example, $1 \in \mathcal{U}_q$ if and only if $q \in \mathcal{U}$.

Using the sequence (α_i) introduced in the preceding section, the following characterization of $\mathcal{U}_q$ is an easy corollary of a classical theorem of Parry [18]:

Theorem 3.1 ([4, Theorem 1.1 (ii)])**.** *Given a base* $q > 1$ *and an expansion* (3.1), *we have* $x \in \mathcal{U}_q$ *if and only if the following two lexicographic conditions are satisfied:*

$$(c_{n+i}) < (\alpha_i) \quad \textit{whenever} \quad c_n < \alpha_1;$$

$$(\overline{c_{n+i}}) < (\alpha_i) \quad \textit{whenever} \quad c_n > 0.$$

The set $\mathcal{U}_q$ is closed for *almost every* base q with respect to the Lebesgue measure. More precisely and rather surprisingly, $\mathcal{U}_q$ is closed if and only if $q \notin \overline{\mathcal{U}}$. In order to get a complete picture we define the *quasi-greedy* expansion of every real number

$$0 < x \le \frac{\lceil q \rceil - 1}{q - 1}$$

in base q by the following algorithm: if $a_n(x)$ has already been defined for some $n \ge 1$ (no hypothesis if $n = 0$), then let $a_{n+1}(x)$ be the biggest nonnegative integer satisfying

$$\frac{a_1(x)}{q} + \cdots + \frac{a_{n+1}(x)}{q^{n+1}} < x.$$

Furthermore, it is convenient to set $(a_i(x)) := 0^\infty$ if $x = 0$.

Remark 3.2. All quasi-greedy expansions satisfy the condition

$$(a_{n+i}(x)) \le (\alpha_i) \quad \text{whenever} \quad a_n(x) < \alpha_1.$$

Next, analogously to the preceding section, we write $x \in \mathcal{V}_q$ if

$$(\overline{a_{n+i}(x)}) \le (\alpha_i) \quad \text{whenever} \quad a_n(x) > 0.$$

Similarly to Theorem 2.4 (a) we have always $\mathcal{U}_q \subset \overline{\mathcal{U}}_q \subset \mathcal{V}_q = \overline{\mathcal{V}}_q$. However, the finer picture depends on the given value of q. The following results were given in [4, Theorems 1.3, 1.4, 1.5] and in the remarks following their statements.

Theorem 3.3.

(a) *If $q \in \overline{\mathcal{U}}$, then $\overline{\mathcal{U}}_q = \mathcal{V}_q$ and $\mathcal{V}_q \setminus \mathcal{U}_q$ is a countable dense set in $\mathcal{V}_q$.*

(b) *If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then $\mathcal{U}_q$ is closed: $\overline{\mathcal{U}}_q = \mathcal{U}_q$, and $\mathcal{V}_q \setminus \mathcal{U}_q$ is a discrete set, dense in $\mathcal{V}_q$.*

(c) *If $q \in (1, \infty) \setminus \mathcal{V}$, then the sets $\mathcal{U}_q$ and $\mathcal{V}_q$ are closed and equal: $\mathcal{U}_q = \overline{\mathcal{U}}_q = \mathcal{V}_q = \overline{\mathcal{V}}_q$.*

We have, moreover, the following result concerning the number of expansions of any $x \in \mathcal{V}_q$:

Theorem 3.4.

(a) *If $q \in \mathcal{U}$, then each $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly two expansions.*

(b) *If $q \in \mathcal{V} \setminus \mathcal{U}$, then the set of expansions of each $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ is countably infinite.*

Since the integer bases are univoque, we have the following:

Corollary 3.5. *Let $q \ge 2$ be an integer. Then*

(a) $\overline{\mathcal{U}}_q = \mathcal{V}_q = [0,1]$, *and* $\mathcal{V}_q \setminus \mathcal{U}_q$ *is a countable dense set in* $\mathcal{V}_q$*;*

(b) *each* $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ *has exactly two expansions.*

The corollary follows at once from the well-known fact that all numbers $x \in [0,1]$ belong to $\mathcal{U}_q$ except countably many rational numbers $0 < x < 1$ of the form $x = \frac{m}{q^n}$ with positive integers m and n, for which there are two expansions ending with 0^∞ and $(q-1)^\infty$, respectively.

In the following two sections we investigate what happens with this corollary if we consider the expansions over the enlarged alphabet $\{0, \dots, q\}$.

4. Unique expansions in integer bases with enlarged alphabets

Fix an *integer* $q \ge 2$ and consider the expansions

$$x = \frac{c_1}{q} + \frac{c_2}{q^2} + \dots$$

on the alphabet $\{0, 1, \dots, q\}$. In order to have an expansion, now x has to belong to the interval

$$J_q^* := \left[0, \frac{q}{q-1}\right].$$

Conversely, each $x \in J_q^*$ has at least one expansion, given for example the following modification of the quasi-greedy expansion $(a_i(x))$ of the preceding section. The quasi-greedy expansion of $x = 0$ is by definition 0^∞. If $x > 0$ and if $a_n(x)$ has already been defined for some $n \ge 1$ (no hypothesis if $n = 0$), then let $a_{n+1}(x)$ be the biggest element of $\{0, 1, \dots, q\}$ satisfying the inequality

$$\frac{a_1(x)}{q} + \dots + \frac{a_{n+1}(x)}{q^{n+1}} < x.$$

Remark 4.1. As a special case of a more general result in [1], all quasi-greedy expansions satisfy the condition

$$(a_{n+i}(x)) \le (q-1)^\infty \quad \text{whenever} \quad a_n(x) < q,$$

and conversely, every infinite sequence (c_i) satisfying the condition

$$(c_{n+i}) \le (q-1)^\infty \quad \text{whenever} \quad c_n < q$$

is the quasi-greedy expansion of a suitable real number x.

Let us denote by $\mathcal{U}_q^*$ the set of numbers $x \in J_q^*$ having a unique expansion in base q with digits belonging to the enlarged alphabet $\{0, 1, \dots, q\}$. Instead of Theorem 3.1 we have the following theorem (see also [19]):

Theorem 4.2. *We have $x \in \mathcal{U}_q^*$ if and only if the following two lexicographic conditions are satisfied:*

$$(c_{n+i}) < (q-1)^\infty \quad \textit{whenever} \quad c_n < q;$$
$$(q - c_{n+i}) < (q-1)^\infty \quad \textit{whenever} \quad c_n > 0.$$

Proof. If $(c_{n+i}) \ge (q-1)^\infty$ for some $c_n < q$, then another expansion of x is given by (d_i) where $d_i = c_i$ for all $i < n$, $d_n = c_n + 1$, and (d_{n+i}) is an arbitrary expansion of

$$y := q^n \left(x - \frac{c_1}{q} - \cdots - \frac{c_{n-1}}{q^{n-1}} - \frac{c_n + 1}{q^n} \right).$$

Such an expansion exists because

$$y = q^n \left(\left(\sum_{i=n+1}^{\infty} \frac{c_i}{q^i} \right) - \frac{1}{q^n} \right) = \left(\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} \right) - 1 \in J_q^*;$$

the crucial inequality $y \ge 0$ follows from the condition $(c_{n+i}) \ge (q-1)^\infty$. Indeed, if $(c_{n+i}) = (q-1)^\infty$, then we have

$$y = \frac{q-1}{q-1} - 1 = 0;$$

otherwise there is a first digit $c_{n+m} = q$ and then

$$y \ge \left(\sum_{i=1}^{m} \frac{c_{n+i}}{q^i} \right) - 1 = 0.$$

Similarly, if $(q - c_{n+i}) \ge (q-1)^\infty$ for some $c_n > 0$, then another expansion of x is given by (d_i) where $d_i = c_i$ for all $i < n$, $d_n = c_n - 1$, and (d_{n+i}) is an arbitrary expansion of

$$z := q^n \left(x - \frac{c_1}{q} - \cdots - \frac{c_{n-1}}{q^{n-1}} - \frac{c_n - 1}{q^n} \right).$$

Such an expansion exists because

$$z = q^n \left(\left(\sum_{i=n+1}^{\infty} \frac{c_i}{q^i} \right) + \frac{1}{q^n} \right) = \left(\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} \right) + 1 \in J_q^*;$$

the crucial inequality $z \le \frac{q}{q-1}$ follows from the condition $(q - c_{n+i}) \ge (q-1)^\infty$. Indeed, if $(q - c_{n+i}) = (q-1)^\infty$, then we have

$$z = \frac{1}{q-1} + 1 = \frac{q}{q-1};$$

otherwise there is a first digit $c_{n+m} = 0$ and then

$$z \le \left(\sum_{i=1}^{m-1} \frac{1}{q^i}\right) + \left(\sum_{i=m+1}^{\infty} \frac{q}{q^i}\right) + 1 = \left(\sum_{i=1}^{\infty} \frac{1}{q^i}\right) + 1 = \frac{1}{q-1} + 1 = \frac{q}{q-1}.$$

Now assume that both lexicographic conditions are satisfied and let (d_i) be an arbitrary sequence on the alphabet $\{0, \dots, q\}$. We claim that if $(d_i) \ne (c_i)$, then

$$\sum_{i=1}^{\infty} \frac{d_i}{q^i} \ne \sum_{i=1}^{\infty} \frac{c_i}{q^i}.$$

To prove this we consider the first index n at which the sequences differ: $d_i = c_i$ for all $i < n$ but $d_n \ne c_n$.

If $d_n > c_n$, then $c_n < q$. It follows from our first lexicographic assumption that if $c_n < q$ for some n, then $c_i < q$ for all $i > n$ too. Since, moreover, the equality $(c_{n+i}) = (q-1)^\infty$ is also excluded, it follows that

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} < \left(\sum_{i=1}^{n} \frac{c_i}{q^i}\right) + \left(\sum_{i=n+1}^{\infty} \frac{q-1}{q^i}\right) = \frac{c_1}{q} + \dots + \frac{c_{n-1}}{q^{n-1}} + \frac{c_n + 1}{q^n} \le \sum_{i=1}^{\infty} \frac{d_i}{q^i}.$$

If $d_n < c_n$, then $c_n > 0$. It follows from our second lexicographic assumption that if $c_n > 0$ for some n, then $c_i > 0$ for all $i > n$ too. Since, moreover, the equality $(q - c_{n+i}) = (q-1)^\infty$, i.e., $(c_{n+i}) = 1^\infty$ is also excluded, it follows that

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} > \left(\sum_{i=1}^{n} \frac{c_i}{q^i}\right) + \left(\sum_{i=n+1}^{\infty} \frac{1}{q^i}\right)$$

$$= \frac{c_1}{q} + \dots + \frac{c_{n-1}}{q^{n-1}} + \frac{c_n - 1}{q^n} + \left(\sum_{i=n+1}^{\infty} \frac{q}{q^i}\right) \ge \sum_{i=1}^{\infty} \frac{d_i}{q^i}. \quad \square$$

Remarks 4.3.

(a) Since the sequence $(q-1)^\infty$ does not satisfy the first condition (we have equalities instead of strict inequalities), the expansion of $x = 1$ is not unique in base q any more if we use the enlarged alphabet $\{0, 1, \dots, q\}$ instead of the earlier digit set $\{0, 1, \dots, q-1\}$. Indeed, it is straightforward to check that $x = 1$ has for example the different expansions

$$\frac{q-1}{q} + \frac{q-1}{q^2} + \frac{q-1}{q^3} + \dots = \frac{q}{q} + \frac{0}{q^2} + \frac{0}{q^3} + \dots.$$

(b) The list of all expansions of $x = 1$ in integer bases for the enlarged alphabet has been determined in [16]:

– in base $q = 2$ the expansions of $x = 1$ are 1^∞ and the sequences

$$1^n 20^\infty \quad \text{and} \quad 1^n 02^\infty, \quad n = 0, 1, \dots;$$

– in bases $q = 3, 4, \dots$ the expansions of $x = 1$ are $(q-1)^\infty$ and the sequences

$$(q-1)^n q 0^\infty, \quad n = 0, 1, \dots.$$

Now, analogously to the preceding section, we write $x \in \mathcal{V}_q^*$ if, using the notation $(a_i(x))$ for the quasi-greedy expansion of x,

$$(q - a_{n+i}(x)) \le (q-1)^\infty \quad \text{whenever} \quad a_n(x) > 0$$

or equivalently

$$(a_{n+i}(x)) \ge 1^\infty \quad \text{whenever} \quad a_n(x) > 0.$$

In the next two sections we clarify the structure of these new sets $\mathcal{U}_q^*$ and $\mathcal{V}_q^*$. As we will see, the cases $q = 2$ and $q \ge 3$ are rather different.

It will be convenient to use in the sequel the following notations: we denote by $\mathcal{U}_q'^*$ (resp. by $\mathcal{V}_q'^*$) the set of quasi-greedy expansions of the numbers of $\mathcal{U}_q^*$ (resp. of $\mathcal{V}_q^*$) on the alphabet $\{0, \dots, q\}$. It follows from Theorem 4.2 and from the definition of $\mathcal{V}_q^*$ that a sequence (c_i) on the alphabet $\{0, \dots, q\}$ belongs to $\mathcal{U}_q'^*$ if and only if

$$(c_{n+i}) < (q-1)^\infty \quad \text{whenever} \quad c_n < q \tag{4.1}$$

and

$$(c_{n+i}) > 1^\infty \quad \text{whenever} \quad c_n > 0, \tag{4.2}$$

and it belongs to $\mathcal{V}_q'^*$ if and only if

$$(c_{n+i}) \le (q-1)^\infty \quad \text{whenever} \quad c_n < q \tag{4.3}$$

and

$$(c_{n+i}) \ge 1^\infty \quad \text{whenever} \quad c_n > 0. \tag{4.4}$$

5. Unique expansions in base $q = 2$ with digits $0, 1, 2$

If we add the new digit 2 in base $q = 2$, then we have the following variant of Corollary 3.5:

Proposition 5.1. *We consider the expansions in base $q = 2$ on the alphabet $\{0, 1, 2\}$.*

(a) *We have*

$$\mathcal{U}_2^* = \{0,2\} \quad \textit{and} \quad \mathcal{V}_2^* = \{0,1,2\} \cup \left\{2^{-N}, 2-2^{-N} : N = 1,2,\dots\right\}.$$

Hence both $\mathcal{U}_2^*$ *and* $\mathcal{V}_2^*$ *are countable closed sets, and* $\mathcal{V}_2^* \setminus \mathcal{U}_2^*$ *is a countable discrete set, dense in* $\mathcal{V}_2^*$.

(b) *The set of expansions of each* $x \in \mathcal{V}_2^* \setminus \mathcal{U}_2^*$ *is countably infinite.*

The following proof will also provide the lists of all expansions in part (b).

Proof.

(a) By definition (see (4.3)–(4.4)) $\mathcal{V}_2'^*$ consists of the sequences (c_i) on the alphabet $\{0,1,2\}$ satisfying

$$(c_{n+i}) \le 1^\infty \quad \text{whenever} \quad c_n < 2$$

and

$$(c_{n+i}) \ge 1^\infty \quad \text{whenever} \quad c_n > 0.$$

Hence if $(c_i) \in \mathcal{V}_2'^*$, then each 1 digit is followed by another 1 digit, and (c_i) cannot contain both 0 and 2 digits. This leaves us only the possibilities 0^∞, 1^∞, 2^∞, $0^N 1^\infty$ $(N = 1,2,\dots)$ and $2^N 1^\infty$ $(N = 1,2,\dots)$, and a direct inspection shows that they belong to $\mathcal{V}_2'^*$ indeed. Hence

$$\mathcal{V}_2^* = \{0,2\} \cup \left\{2^{-N}, 2-2^{-N} : N = 0,1,\dots\right\}$$

which is equivalent to the first statement on $\mathcal{V}_2^*$.

It is clear that $0,2 \in \mathcal{U}_2^*$. The other elements of $\mathcal{V}_2^*$ are not univoque. Indeed, for any fixed $N = 0,1,\dots$, two different expansions of 2^{-N} and of $2-2^{-N}$ are given by the equalities

$$\frac{1}{2^N} = \frac{1}{2^{N+1}} + \frac{1}{2^{N+2}} + \dots$$

and

$$\frac{2}{2} + \dots + \frac{2}{2^{N-1}} + \frac{1}{2^N} + \frac{2}{2^{N+1}} + \frac{2}{2^{N+2}} + \dots \\ = \frac{2}{2} + \dots + \frac{2}{2^{N-1}} + \frac{2}{2^N} + \frac{1}{2^{N+1}} + \frac{1}{2^{N+2}} + \dots.$$

Hence $\mathcal{U}_2^* = \{0,2\}$.

The rest of part (a) readily follows from these explicit descriptions.

(b) We recall from Remark 4.3 (b) that the list of all expansions of $x = 1$ is given by 1^∞ and the sequences

$$1^n 2 0^\infty \quad \text{and} \quad 1^n 0 2^\infty, \quad n = 0,1,\dots.$$

Now fix a positive integer N. Every expansion (c_i) of $x = 2^{-N}$ must begin with $c_1 = \cdots = c_{N-1} = 0$ and $c_N \in \{0,1\}$. If $c_N = 1$, then we have necessarily $c_i = 0$ for all $i \neq N$. Otherwise $c_{N+1}, c_{N+2}, \ldots$ is an expansion of $x = 1$. Using the list of expansions of $x = 1$ we conclude that the list of all expansions of $x = 2^{-N}$ is given by $0^{N-1}10^\infty$, 0^N1^∞ and the sequences

$$0^N1^n20^\infty \quad \text{and} \quad 0^N1^n02^\infty, \quad n = 0, 1, \ldots.$$

Furthermore, (c_i) is an expansion of $2 - 2^{-N}$ if and only if $(2 - c_i)$ is an expansion of 2^{-N}. Using this correspondence we conclude from the preceding result that the list of all expansions of $x = 2 - 2^{-N}$ is given by $2^{N-1}12^\infty$, 2^N1^∞ and the sequences

$$2^N1^n02^\infty \quad \text{and} \quad 2^N1^n20^\infty, \quad n = 0, 1, \ldots.$$

In particular, the set of expansions of each $x \in \mathcal{V}_2^* \setminus \mathcal{U}_2^*$ is countably infinite. □

6. Unique expansions in bases $q = 3, 4, \ldots$ with digits $0, 1, \ldots, q$

For $q = 3, 4, \ldots$ we have the following variant of Corollary 3.5 for the enlarged digit set $\{0, 1, \ldots, q\}$. We recall that a Cantor set is a nonempty set having neither interior, nor isolated points.

Theorem 6.1. *Let $q \geq 3$ be an integer and consider the expansions in base q on the alphabet $\{0, 1, \ldots, q\}$.*

(a) *$\overline{\mathcal{U}_q^*} = \mathcal{V}_q^*$ is a Cantor set, and $\mathcal{V}_q^* \setminus \mathcal{U}_q^*$ is a countable dense set in $\mathcal{V}_q^*$.*

(b) *The set of expansions of each $x \in \mathcal{V}_q^* \setminus \mathcal{U}_q^*$ is countably infinite.*

(c) *Both $\mathcal{U}_q^*$ and $\mathcal{V}_q^*$ have Hausdorff dimension $\frac{\log(q-1)}{\log q}$ and hence zero Lebesgue measure.*

Remarks 6.2.

(a) The results differ very much from Corollary 3.5. Indeed, for the original digit set $\{0, 1, \ldots, q-1\}$
   - $\overline{\mathcal{U}_q} = \mathcal{V}_q = [0, 1]$ is not a Cantor set because it has interior points;
   - $\mathcal{U}_q$ and $\mathcal{V}_q$ have full Lebesgue measure and Hausdorff dimension one;
   - each $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly two expansions.

(b) The proof of the theorem will also provide the lists of all expansions in part (b) of the theorem.

Lemma 6.3.

(a) $\mathcal{V}_q'^*$ *consists of* 0^∞, q^∞ *and of the sequences on the alphabet* $\{1,\dots,q-1\}$ *preceded by* 0^n *or* q^n *for some nonnegative integer* n.

(b) $\mathcal{U}_q'^*$ *consists of the elements of* $\mathcal{V}_q'^*$ *which do not end with* 1^∞ *or* $(q-1)^\infty$.

(c) $\mathcal{V}_q'^* \setminus \mathcal{U}_q'^*$ *consists of the elements of* $\mathcal{V}_q'^*$ *which end with* 1^∞ *or* $(q-1)^\infty$.

Proof. (a) If a sequence (c_i) on the alphabet $\{0,1,\dots,q\}$ belongs to $\mathcal{V}_q'^*$, then it follows from (4.3)–(4.4) that

- after a nonzero digit we never have a zero digit;
- after a digit different from q we never have a digit q.

This leaves only the candidates 0^∞, q^∞ and $0^n c_{n+1}c_{n+2}\dots$ and $q^n c_{n+1}c_{n+2}\dots$ with $n=0,1,\dots$ and $0<c_i<q$ for all $i>n$. Conversely, all these sequences satisfy (4.3)–(4.4), so that they belong to $\mathcal{V}_q'^*$ indeed.

(b) The comparison of conditions (4.1)–(4.2) and (4.3)–(4.4) shows that $\mathcal{U}_q'^* \subset \mathcal{V}_q'^*$. The assertion now follows from part (a) and from conditions (4.1)–(4.2).

(c) This follows from parts (a) and (b). □

Proof of Theorem 6.1. (a) In order to prove that $\mathcal{V}_q^*$ is closed we show that $J_q^* \setminus \mathcal{V}_q^*$ is open. Since $\mathcal{V}_q^*$ is symmetric with respect to the midpoint of J_q^* by its definition, it is sufficient to find for each $x \in J_q^* \setminus \mathcal{V}_q^*$ a point $y<x$ such that $(y,x] \cap \mathcal{V}_q^* = \varnothing$. So fix $x \in J_q^* \setminus \mathcal{V}_q^*$ arbitrarily and let (a_i) be its quasi-greedy expansion. Since $x \notin \mathcal{V}_q^*$, there exists n such that $a_n>0$ and $(a_{n+i}) < 1^\infty$. Fix $m>n$ such that $a_m=0$ and set

$$y := \sum_{i=1}^{m} \frac{a_i}{q^i}.$$

Since (a_i) is an infinite sequence, we have $y<x$, and the quasi-greedy expansion of every $z \in (y,x]$ begins with $a_1\dots a_m$. Hence $(a_{n+i}(z)) < 1^\infty$ and therefore $z \notin \mathcal{V}_q^*$.

It follows from Lemma 6.3 that $\mathcal{U}_q^* \subset \mathcal{V}_q^*$ and that $\mathcal{V}_q^* \setminus \mathcal{U}_q^*$ is countably infinite.

Next we show that both $\mathcal{U}_q^*$ and $\mathcal{V}_q^* \setminus \mathcal{U}_q^*$ are dense in $\mathcal{V}_q^*$. Given $(c_i) \in \mathcal{V}_q'^*$ and

$$x := \sum_{i=1}^{\infty} \frac{c_i}{q^i} \in \mathcal{V}_q^*$$

arbitrarily, it follows from Lemma 6.3 that the formulae

$$x_k := \frac{c_1}{q} + \cdots + \frac{c_k}{q^k} + \frac{1}{q^{k+1}} + \frac{q-1}{q^{k+2}} + \frac{1}{q^{k+3}} + \frac{q-1}{q^{k+4}} + \dots,$$

$$y_k := \frac{c_1}{q} + \cdots + \frac{c_k}{q^k} + \frac{1}{q^{k+1}} + \frac{1}{q^{k+2}} + \frac{1}{q^{k+3}} + \frac{1}{q^{k+4}} + \ldots$$

and

$$z_k := \frac{c_1}{q} + \cdots + \frac{c_k}{q^k} + \frac{q-1}{q^{k+1}} + \frac{q-1}{q^{k+2}} + \frac{q-1}{q^{k+3}} + \frac{q-1}{q^{k+4}} + \ldots$$

define three sequences $(x_k) \in \mathcal{U}_q^*$ and $(y_k), (z_k) \in \mathcal{V}_q^* \setminus \mathcal{U}_q^*$ converging to x. This shows that both $\mathcal{U}_q^*$ and $\mathcal{V}_q^* \setminus \mathcal{U}_q^*$ are dense in $\mathcal{V}_q^*$.

It follows also from this proof that $\mathcal{V}_q^* \setminus \mathcal{U}_q^*$ has no isolated points. Indeed, if $x \in \mathcal{V}_q^* \setminus \mathcal{U}_q^*$, then (c_i) ends either with 1^∞ or $(q-1)^\infty$. In the first case $z_k \neq x$ for all k, while in the second case $y_k \neq x$ for all k. Hence its closure $\mathcal{V}_q^*$ has no isolated points either.

For the Cantor property it remains to prove that $\mathcal{V}_q^*$ has no interior points. Consider an element (c_i) of $\mathcal{V}_q'^*$ for which $1 \leq c_i \leq q-1$ for all $i > n$. If we insert between c_k and c_{k+1} a zero digit, then for every $k > n$ we obtain the quasi-greedy expansion of a number x_k (by Remark 4.1) which does not belong to $\mathcal{V}_q^*$ by Lemma 6.3 (a). Since

$$x_k \to x := \frac{c_1}{q} + \frac{c_2}{q^2} + \cdots \in \mathcal{V}_q^*,$$

this shows that x is not an interior point of $\mathcal{V}_q^*$. We have thus shown that the interior of $\mathcal{V}_q^*$ is a subset of $\left\{0, \frac{q}{q-1}\right\}$. If one of these two points would belong to the interior of $\mathcal{V}_q^*$, then it would be an isolated point of $\mathcal{V}_q^*$, which we have already excluded. Hence $\mathcal{V}_q^*$ has no interior points.

(b) Let $(c_i) \in \mathcal{V}_q'^* \setminus \mathcal{U}_q'^*$ be the quasi-greedy expansion of some $x \in \mathcal{V}_q^* \setminus \mathcal{U}_q^*$. Then by Lemma 6.3 (c) there exist two integers $m \geq k \geq 0$ such that $0 < c_i < q$ for $i = k+1, \ldots, m$, and that the remaining digits satisfy one of the following four conditions:

$$c_i = 0 \quad \text{for all} \quad i \leq k \quad \text{and} \quad c_i = q-1 \quad \text{for all} \quad i > m;$$

$$c_i = q \quad \text{for all} \quad i \leq k \quad \text{and} \quad c_i = q-1 \quad \text{for all} \quad i > m;$$

$$c_i = 0 \quad \text{for all} \quad i \leq k \quad \text{and} \quad c_i = 1 \quad \text{for all} \quad i > m;$$

$$c_i = q \quad \text{for all} \quad i \leq k \quad \text{and} \quad c_i = 1 \quad \text{for all} \quad i > m.$$

By taking the minimal possible value of m we may also assume that in case $m > 1$ we have $c_m \neq c_{m+1}$.

First we consider the cases where $c_{m+1} = q-1$. Let (d_i) be an arbitrary sequence on the alphabet $\{0, 1, \ldots, q\}$. Using the equality

$$\frac{q-1}{q} + \frac{q-1}{q^2} + \cdots = 1$$

we see that

$$\frac{d_1}{q} + \frac{d_2}{q^2} + \cdots = x$$

can only happen if $d_i = c_i$ for all $i < m$, and $d_m \in \{c_m, c_m + 1\}$. Indeed:

- if $d_i = c_i$ for all $i < n$ and $d_n < c_n$ for some $n \le m$, then

$$\begin{aligned}\frac{d_1}{q} + \frac{d_2}{q^2} + \ldots &\le \frac{c_1}{q} + \cdots + \frac{c_{n-1}}{q^{n-1}} + \frac{c_n - 1}{q^n} + \frac{q-1}{q^{n+1}} + \frac{q-1}{q^{n+2}} + \ldots \\ &= \frac{c_1}{q} + \cdots + \frac{c_{n-1}}{q^{n-1}} + \frac{c_n}{q^n} < x;\end{aligned}$$

- if $d_i = c_i$ for all $i < n$ and $d_n > c_n$ for some $n < m$, then

$$\begin{aligned}\frac{d_1}{q} + \frac{d_2}{q^2} + \ldots &\ge \frac{c_1}{q} + \cdots + \frac{c_{n-1}}{q^{n-1}} + \frac{c_n + 1}{q^n} \\ &= \frac{c_1}{q} + \cdots + \frac{c_n}{q^n} + \frac{q-1}{q^{n+1}} + \frac{q-1}{q^{n+2}} + \ldots \\ &> \frac{c_1}{q} + \cdots + \frac{c_m}{q^m} + \frac{q-1}{q^{m+1}} + \frac{q-1}{q^{m+2}} + \ldots \\ &= x;\end{aligned}$$

- if $d_i = c_i$ for all $i < m$ and $d_m \ge c_m + 1$, then

$$\begin{aligned}\frac{d_1}{q} + \frac{d_2}{q^2} + \ldots &\ge \frac{c_1}{q} + \cdots + \frac{c_{m-1}}{q^{m-1}} + \frac{c_m + 1}{q^n} \\ &= \frac{c_1}{q} + \cdots + \frac{c_m}{q^m} + \frac{q-1}{q^{m+1}} + \frac{q-1}{q^{m+2}} + \ldots \\ &= x,\end{aligned}$$

  with equality only if $d_m = c_m + 1$ and $d_i = 0$ for all $i > m$. This is only possible if $c_m < q$.

Apart from this last one, all the other expansions (d_i) of x start with $c_1, \ldots, c_m$, so that

$$\frac{d_{m+1}}{q^{m+1}} + \frac{d_{m+2}}{q^{m+2}} + \cdots = \frac{c_{m+1}}{q^{m+1}} + \frac{c_{m+2}}{q^{m+2}} + \ldots$$

whence

$$\frac{d_{m+1}}{q} + \frac{d_{m+2}}{q^2} + \cdots = \frac{q-1}{q} + \frac{q-1}{q^2} + \cdots = 1.$$

Using Remark 4.3 (b) we conclude that the list of expansions of x is as follows:

- $c_1 \ldots c_m (q-1)^\infty$;

- $c_1 \dots c_m (q-1)^n q 0^\infty, \quad n = 0, 1, \dots;$
- $c_1 \dots c_{m-1}(c_m + 1)0^\infty$ if $c_m < q$.

Since $\mathcal{V}_q^*$ is symmetric with respect to its midpoint and since (c_i) is an expansion of x if and only if $(q - c_i)$ is an expansion of the reflection of x, the case $c_{m+1} = 1$ follows from the preceding one. We conclude that the list of expansions of x is now as follows:

- $c_1 \dots c_m 1^\infty;$
- $c_1 \dots c_m 1^n 0 q^\infty, \quad n = 0, 1, \dots;$
- $c_1 \dots c_{m-1}(c_m - 1)q^\infty$ if $c_m < q$.

In particular,we have proved that the set of expansions of each $x \in \mathcal{V}_q^* \setminus \mathcal{U}_q^*$ is countably infinite.

(c) Since $\mathcal{V}_q'^* \setminus \mathcal{U}_q'^*$ is countable, $\mathcal{U}_q^*$ and $\mathcal{V}_q^*$ have the same Hausdorff dimension. Since $\mathcal{V}_q^*$ is the union of the two points 0, $\frac{q}{q-1}$ and of countably many sets, each of which is similar to the set

$$\mathcal{Z} := \left\{ \sum_{i=1}^{\infty} \frac{c_i}{q^i} : c_i \in \{1, \dots, q-1\}, i = 1, 2, \dots \right\},$$

we have $\dim_{\mathrm{H}} \mathcal{U}_q^* = \dim_{\mathrm{H}} \mathcal{V}_q^* = \dim_{\mathrm{H}} \mathcal{Z}$.

Let us compute the similarity dimension s of $\mathcal{Z}$. Since $\mathcal{Z}$ is the attractor of the iterated function system defined by

$$f_k(x) := \frac{k + x}{q}, \quad x \in \mathcal{Z}, \ k = 1, \dots, q-1,$$

we have $(q-1)q^{-s} = 1$ whence $s = \frac{\log(q-1)}{\log q}$.

The images $f_k(\mathcal{Z})$ have disjoint closures. Indeed, if $k < n$, then

$$\sup f_k(\mathcal{Z}) = \frac{k}{q} + \frac{1}{q} \sum_{i=1}^{\infty} \frac{q-1}{q^i} = \frac{k+1}{q}$$

and

$$\inf f_n(\mathcal{Z}) = \frac{n}{q} + \frac{1}{q} \sum_{i=1}^{\infty} \frac{1}{q^i} = \frac{n}{q} + \frac{1}{q(q-1)},$$

so that

$$\inf f_n(\mathcal{Z}) - \sup f_k(\mathcal{Z}) = \frac{n-k-1}{q} + \frac{1}{q(q-1)} \geq \frac{1}{q(q-1)} > 0.$$

Since Moran's open set condition is thus satisfied, the Hausdorff dimension of $\mathcal{Z}$ is equal to its similarity dimension, so that

$$\dim_{\mathrm{H}} \mathcal{U}_q^* = \dim_{\mathrm{H}} \mathcal{V}_q^* = \dim_{\mathrm{H}} \mathcal{Z} = \frac{\log(q-1)}{\log q}.$$

As sets of Hausdorff dimension < 1, all these sets have zero Lebesgue measure. $\square$

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