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A note on sum-product estimates

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Dedicated to Professors Kálmán Győry, Attila Pethő, János Pintz and András Sárközy, on the occasion of their birthdays

Abstract. We prove that for any finite set A of positive real numbers one has

$$|AA + AA + AA + AA| \geq \frac{1}{2}\,|A|^2.$$

1. Introduction

Let A and B be finite sets of positive real numbers (denoted by $\mathbb{R}^{>0}$). The *sumset*, *productset*, and *quotientset* are defined by

$$A + B = \{a + b : a \in A,\ b \in B\},$$
$$AB = \{ab : a \in A,\ b \in B\},$$
$$B/A = \{b/a : a \in A,\ b \in B\}.$$

A famous result of Freiman [2] states, if $A + A$ is small then A is *arithmetic progression like* in some sense. Applying the same theorem for $B = \{\log a : a \in A\}$ implies, if AA is small then A is *geometric progression like.* These two structures

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are rather different, A cannot be arithmetic and geometric progression like in the same time. Erdős and Szemerédi [1] expressed this fact in a quantitative conjecture: *If A is a finite set of positive integers, $\epsilon > 0$ is a fixed real number, and $|A| = n > n_0(\epsilon)$, then*

$$\max\{|A + A|, |AA|\} \geq n^{2-\epsilon}.$$

Note that for the set $A = \{1, \dots, n\}$, obviously $A + A = \{2, \dots, 2n\}$ and $AA \subset \{1, \dots, n^2\}$, showing that the above conjecture, if true, is rather tight. Actually an interesting result of multiplicative number theory shows that $n^{-\epsilon}$ cannot be completely omitted, as with the above A one has $|AA| \ll n^2 \log^{-c} n$ for an explicitly given small positive c.

There are several results toward this conjecture. On one hand there is an increasing chain of exponents in place of (but smaller than) $2-\epsilon$, on the other hand the results are extended to other rings. The best exponent is due to Solymosi, in [5] he proves: *If A is a finite set of positive real numbers, and $|A| = n$, then*

$$\max\{|A + A|, |AA|\} \geq \frac{n^{4/3}}{(4 \log n)^{1/3}}. \tag{1}$$

Another manifestation of the same fact is that $AA+AA$ should be always big. Indeed, if $a \in A$ then both $aA + aA$ and $AA + a^2$ are subsets of $AA + AA$, and at least one of them should be big by the sum-product estimate. However, a slightly different philosophy explains why $AA+AA$, or even $AA+A$ should be big, namely AA has a kind of multiplicative structure, therefore it cannot behave nicely in a sum. In this short note we modify the method, developed by Solymosi in [5], to derive results of this spirit. We are going to use only elementary arguments of plane geometric flavor, higher dimensional generalizations may lead to further interesting estimates.

2. Results

Let $A, B \subset \mathbb{R}^{>0}$ be finite sets of positive real numbers. We define the representation function

$$R(q) = R_{B/A}(q) = \#\left\{(a, b) \in A \times B : q = \frac{b}{a}\right\}.$$

The essential step in Solymosi's work is to estimate the 2nd moment of $R(q)$ by means of sumsets, more precisely

Lemma 1 (Solymosi [5], Lemma 2.3)**.** *Let $A, B \subset \mathbb{R}^{>0}$ be finite sets. We have*

$$\sum_q R(q)^2 \le 2\log(|A|\,|B|)\,|A+A|\,|B+B|.$$

Note that Lemma 2.3 of [5] deals with the special case $A = B$ only, which is sufficient for our purposes, however, Remark 2.3 of the same paper extends the lemma to the above result. (1) follows from here with an application of the Cauchy–Schwarz inequality. Using the same argument we rewrite this statement to a form, more suitable for our application. Indeed one has

$$(|A|\,|B|)^2 = \left(\sum_q R(q)\right)^2 \le |B/A| \sum_q R(q)^2,$$

and Lemma 1 implies

$$\frac{|A|^2|B|^2}{2\log(|A|\,|B|)} \le |B/A|\,|A+A|\,|B+B|. \tag{2}$$

As is pointed out earlier, one cannot remove the log factor completely from (1), however, it is possible from (2). Li and Shen proved in [4] that

Lemma 2 (Li and Shen [4], Theorem 1)**.** *Let $A \subset \mathbb{R}^{>0}$ be a finite set. We have*

$$\frac{1}{4}\,|A|^4 \le |A/A|\,|A+A|^2.$$

In the next paragraph we estimate the 1st and the 0th moments of $R(q)$ to get the following results.

Theorem 1. *Let $A, B, C \subset \mathbb{R}^{>0}$ be finite sets. We have*

$$|A|\,|B| = \sum_q R(q) \le \frac{|AC+A|\,|BC+B|}{|C|}.$$

Theorem 2. *Let $A, B, C, D \subset \mathbb{R}^{>0}$ be finite sets. We have*

$$|B/A| = \sum_{R(q)\ne 0} 1 \le \frac{|AC+AD|\,|BC+BD|}{|C|\,|D|}.$$

Taking $A = B = C$ into Theorem 1 we get

Corollary 1. *Let $A \subset \mathbb{R}^{>0}$ be a finite set. We have*

$$|A|^{3/2} \le |AA+A|.$$

This result in the more general form $|A|\,|B|\,|C| \ll |AB+C|^2$ was earlier proved using incidence geometry, see the book of Tao and Vu [6].

Taking $A=B$ and $C=D=A+A$ into Theorem 2 we get

$$|A/A|\,|A+A|^2 \le |A(A+A)+A(A+A)|^2, \tag{3}$$

and estimating the left hand side of (3) with Lemma 2 we get

Corollary 2. *Let $A \subset \mathbb{R}^{>0}$ be a finite set. We have*

$$\frac{1}{2}\,|A|^2 \le |A(A+A)+A(A+A)|.$$

$A(A+A)+A(A+A)$ is a six variable expression with sums and products having a very favorable lower bound. This suggests that using similar simple arguments may lead to a similar lower bound to $AA+AA+AA$. Unfortunately we are not able to find that argument. A weaker inequality, namely $|AA+AA+AA| \ge |AA+A(A+A)| \gg |A|^{7/4}$ can be derived either from incidence geometry, see [6], or from our Lemma 2 and Theorems 1, 2. Similarly, putting the straightforward relation $A(A+A)+A(A+A) \subset AA+AA+AA+AA$ into Corollary 2 one gets that

Corollary 3. *Let $A \subset \mathbb{R}^{>0}$ be a finite set. We have*

$$\frac{1}{2}|A|^2 \le |AA+AA+AA+AA|.$$

If $A=\{1,\dots,n\}$, as in a previous example, then $AA+AA+AA+AA \subset \{4,\dots,4n^2\}$ showing that Corollary 2 and 3 are rather tight. It is possible, and conceivable that this time a more elegant inequality is also true.

Conjecture. *Let $A \subset \mathbb{R}^{>0}$ be a finite subset. We have*

$$|AA+A| \ge |A|^2.$$

Note added at July 23, 2011. Very recently Alex Iosevich, Oliver Roche-Newton, and Misha Rudnev [3] got close to this Conjecture by proving

$$|AA+AA| \gg \frac{|A|^2}{\log|A|}.$$

This result surpasses Corollary 3 in almost all aspects. They, however, use a different, more involved argument. The author thanks to the referee for drawing his attention to this work in progress.

3. Moments of $R(q)$

The initial steps coincide in the proof of Theorem 1 and Theorem 2. We arrange the quotients q for which $R(q) \neq 0$ by increasing order, that is $B/A = \{q_1 < q_2 < \cdots < q_m\}$, where $m = |B/A|$. Consider $A \times B$, the vectors (a, b) with first coordinate from A and second coordinate from B in the Euclidean plane. They are inside the first quadrant. The line $y = q_j x$ covers exactly $R(q_j)$ of them, and the union of all such lines for $j = 1, \ldots, m$ covers the whole of $A \times B$. We refer to the half line $y = q_j x$, $x > 0$ simply as the *ray* $\mathcal{R}_j$. In each of these rays $\mathcal{R}_j$ we fix one point of $A \times B$, say the one closest to the origin, and we denote by (a_j, b_j). For example, the ray $\mathcal{R}_m$ contains exactly one point of $A \times B$, namely (a_m, b_m), where a_m is the smallest element of A, and b_m is the largest element of B, that is $R(q_m) = 1$.

Now we concentrate on the proof of Theorem 1. Pick a vector $(a, b) \in A \times B$ on the ray $\mathcal{R}_j$ and another vector $(a_{j+1}c, b_{j+1}c) \in AC \times BC$ on the next ray $\mathcal{R}_{j+1}$. There are $R(q_j)$ choices of the first and $|C|$ choices of the second pick. Observe that their sum $(a_{j+1}c + a, b_{j+1}c + b)$ is inside the sector defined by the two rays, indeed one can quickly check that

$$q_j = \frac{b}{a} < \frac{b_{j+1}c + b}{a_{j+1}c + a} < \frac{b_{j+1}}{a_{j+1}} = q_{j+1}.$$

The vector $(u, v) = (a_{j+1}c + a, b_{j+1}c + b)$ is an element of $(AC + A) \times (BC + B)$, and (u, v) determines the two initial vectors, as is clear from the parallelogram rule of adding vectors. Alternatively, one can see this in a more formal way. Observe that (u, v) is a sum of a vector (a, b) on the ray $\mathcal{R}_j$ and another vector $(a_{j+1}c, b_{j+1}c)$ on the ray $\mathcal{R}_{j+1}$ iff

$$a = \frac{q_{j+1} - v}{q_{j+1} - q_j}, \quad b = q_j \frac{q_{j+1} - v}{q_{j+1} - q_j}, \quad \text{and } c = \frac{1}{a_{j+1}} \frac{v - q_j u}{q_{j+1} - q_j}.$$

This yields the next inequality.

$$|C| R(q_j) \leq \# \left\{ (u, v) \in (AC + A) \times (BC + B) : q_j < \frac{v}{u} < q_{j+1} \right\}.$$

As for different j-s these sectors are disjoint, we have

$$|C| \sum_{j=1}^{m-1} R(q_j) \leq \# \left\{ (u, v) \in (AC + A) \times (BC + B) : q_1 < \frac{v}{u} < q_m \right\}. \quad (4)$$

To prove Theorem 1 we have to find $|C| = |C| R(q_m)$ more elements of $(AC + A) \times (BC + B)$. We list them, they are $(u, v) = (a_m c + a_m, b_m c + b_m)$ for all

$c \in C$. They are all different and as $v/u = b_m/a_m$, they also differ from all (u, v) in (4). This proves Theorem 1 since

$$|C| \sum_{j=1}^{m} R(q_j) = |C| + |C| \sum_{j=1}^{m-1} R(q_j) \leq |(AC + A) \times (BC + B)|.$$

Next we prove Theorem 2, which is rather similar. Pick a vector $(a_j c, b_j c) \in AC \times BC$ on the ray $\mathcal{R}_j$ and another vector $(a_{j+1}d, b_{j+1}d) \in AD \times BD$ on the next ray $\mathcal{R}_{j+1}$. There are $|C|$ choices of the first and $|D|$ choices of the second pick. Observe that their sum $(a_j c + a_{j+1}d, b_j c + b_{j+1}d)$ is inside the sector defined by the two rays, indeed one can quickly check that

$$q_j = \frac{b_j}{a_j} < \frac{b_j c + b_{j+1}d}{a_j c + a_{j+1}d} < \frac{b_{j+1}}{a_{j+1}} = q_{j+1}.$$

The vector $(u, v) = (a_j c + a_{j+1}d, b_j c + b_{j+1}d)$ is an element of $(AC + AD) \times (BC + BD)$, and (u, v) determines the two initial vectors, as is clear from the parallelogram rule of adding vectors. This yields the next inequality.

$$|C|\,|D| \leq \# \left\{ (u, v) \in (AC + AD) \times (BC + BD) : q_j < \frac{v}{u} < q_{j+1} \right\}.$$

For different j-s these sectors are disjoint, so we have

$$|C|\,|D| \sum_{j=1}^{m-1} 1 \leq \# \left\{ (u, v) \in (AC + AD) \times (BC + BD) : q_1 < \frac{v}{u} < q_m \right\}. \tag{5}$$

To prove Theorem 2 we have to find $|C|\,|D|$ more elements of $(AC+AD) \times (BC+BD)$. We list them. Let c_0 be the smallest element of C and d_0 be the largest element of D respectively. Consider the vectors $(u, v) = (a_m c_0 + a_m d, b_m c + b_m d_0)$ for all $c \in C$ and $d \in D$. They are all different and as $v/u = b_m(c + d_0)/a_m(c_0 + d) \geq b_m/c_m$, they also differ from all (u, v) in (5). This proves Theorem 2 since

$$|C|\,|D| \sum_{j=1}^{m} 1 = |C|\,|D| + |C|\,|D| \sum_{j=1}^{m-1} 1 \leq |(AC + AD) \times (BC + BD)|.$$

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