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Cardinal functions for k-structures

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Abstract. Cardinal functions for k–structures are defined and studied. Many known results for topological and closure space cardinal invariants are obtained as corollaries.

1. Introduction

In [10,11] R. E. Hodel introduced the notion of k–structure that generalizes the notions of topological and of closure space in the sense of E. Cech ([4]), and considered some conditions that correspond to certain cardinal functions in the topological case.

Here some more cardinal functions, studied for topological spaces, are considered for the more general case of k–structures. Also several known inequalities for topological and closure-space cardinal invariants are transferred to inequalities between cardinal functions of k–structures.

Standard notations are used following [12] and [5].

Let E be a set, k an infinite cardinal number. A k–structure on E is a collection $\boldsymbol{V} = \{V(p,\gamma) : p \in E,\ \gamma \in k\}$ of subsets of E such that $p \in \bigcap\{V(p,\gamma) : \gamma \in k\}$. For each $A \subseteq E$ let $A^* = \{p \in E : V(p,\gamma) \cap A \neq \emptyset$ for each $\gamma \in k\}$. We write $(E, \boldsymbol{V})$ to denote the fact that E is a set with a k–structure $\boldsymbol{V} = \{V(p,\gamma) : p \in E,\ \gamma \in k\}$. Let us consider the following conditions on a k–structure $(E, \boldsymbol{V})$ as defined in [11]:

S) $\bigcap\{V(p,\gamma) : \gamma \in k\} = \{p\}$ for all $p \in E$.

H) If $p \neq q$ there exist $\alpha, \beta \in k$ such that $V(p,\alpha) \cap V(q,\beta) = \emptyset$.

UH) If $p \neq q$ there exists $\gamma \in k$ such that $V(p,\gamma) \cap V(q,\gamma) = \emptyset$.

I) Given $p \in E$ and $\alpha, \beta \in k$ there exists $\gamma \in k$ such that $V(p,\gamma) \subseteq V(p,\alpha) \cap V(p,\beta)$.

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CPT) For each $\boldsymbol{V}_o \subseteq \boldsymbol{V}$ which covers E there is a finite subcollection of $\boldsymbol{V}_o$ which covers E.

C) For each $\boldsymbol{V}_o \subseteq \boldsymbol{V}$ wich covers E there is a subcollection of $\boldsymbol{V}_o$ of cardinality at most k which covers E.

HC) Given $\gamma \in k$ and $A \subseteq E$ there is $B \subseteq A$ with $|B| \leq k$ such that $A \subseteq \bigcup\{V(p,\gamma) : p \in B\}$.

In [10, 11] the following results for a k–structure $(E, \boldsymbol{V})$ are proved:

Theorem I. S) + HC) $\implies |E| \leq 2^k$.

Theorem II. UH) + C) $\implies |E| \leq 2^k$.

Theorem II'. H) + I) + C) $\implies |E| \leq 2^k$.

Theorem G. S) + CPT) $\implies |E| \leq 2^k$.

These results generalize the famous theorems of de Groot [7], Ginsburg–Woods [6], Archangel'skii [1] and Gryzlov [8] for topological cardinal invariants.

In [10] R. E. Hodel also defined a new cardinal invariant called Hausdorff pseudocharacter denoted by $H\psi(X)$. This cardinal function is defined only for a Hausdorff topological space X and is the smallest infinite cardinal τ such that for every $p \in X$ there exists a collection U_p of neighborhoods of p with $|U_p| \leq \tau$ such that if $p \neq q$ there exist $A \in U_p$ and $B \in U_q$ with $A \cap B \neq \emptyset$. For a Hausdorff space X one always has $\psi(X) \leq H\psi(X) \leq \chi(X)$ and in [10] an example is given of a space X with $H\psi(X) = \omega$ and $t(X) > \omega$.

2. Definitions

PD) If $B \subseteq E$ and $\{V(p,\alpha_p) : p \in B\}$ is a pairwise disjoint family of elements of $\boldsymbol{V}$ then $|B| \leq k$.

In [10] R. Hodel considered a condition denoted by PD) but it is different and we shall refer to it here as to UPD). It is as follows:

UPD) If $\gamma \in k$ and A is a subset of E such that $\{V(p,\gamma) : p \in A\}$ is pairwise disjoint then $|A| \leq k$.

WC) For each $\boldsymbol{V}_o \subseteq \boldsymbol{V}$ that covers E there exists a subcollection $\boldsymbol{W}_o \subseteq \boldsymbol{V}_o$ of cardinality at most k such that $E = (\bigcup \boldsymbol{W}_o)^*$.

The subset $A \subseteq E$ is called $*$–closed if $A^* = A$ and A is called k–discrete (uniformly k–discrete) if for every $p \in A$ there is a $V(p,\alpha)$ such that $V(p,\alpha) \cap A = \{p\}$ (there is $\gamma \in k$ such that for all x, y in A with $x \neq y$ we have $y \notin V(x,y)$).

D) If $A \subseteq E$ is k–discrete then $|A| \leq k$.

UD) If $A \subseteq E$ is uniformly k–discrete then $|A| \leq k$.

Let us note that D) $\implies$ UD); PD) $\implies$ UPD) and US) $\implies$ S) where US): If $x \neq y$ there exists $\gamma \in k$ such that $y \notin V(x,\gamma)$ and $x \notin V(y,\gamma)$. We also have that UH) $\implies$ H).

In [10] it is proved that if $(E, \boldsymbol{V})$ is a k–structure then: UH) + UPD) $\implies |E| \leq 2^k$ and US) + UD) $\implies |E| \leq 2^k$.

Let us note here that in the same way one can obtain the following results:

Proposition I). H) + PD) $\implies |E| \leq 2^k$ *and*

Proposition II). S) + D) + I) $\implies |E| \leq 2^k$.

Let X be a given set and for every $x \in X$ let us have a filter $\Phi_x \subseteq \exp X$ such that $x \in \bigcap \Phi_x$. If $\Phi = \{\Phi_x : x \in X\}$ then the pair (X, Φ) will be called a neighborhood space, [14]. In [14], [15] and [17] cardinal invariants for such spaces are defined and studied.

If (X, Φ) is an o–Hausdorff neighborhood space we can define $H\psi_o(X)$ as the smallest infinite cardinal τ such that for every $p \in X$ there is a collection $U_x \subseteq \Phi_x$ with $|U_x| \leq \tau$ such that if $x \neq y$ there exist $A \in U_x$ and $B \in U_y$ such that $A \cap B = \emptyset$. Let us also note that for an o–Hausdorff neighborhood space (X, Φ) one always has $\psi_o(X) \leq H\psi_o(X) \leq \chi_o(X)$ and if $H\psi_{\tau(\Phi)}(X)$ is the corresponding cardinal number for the topological space $(X, \tau(\Phi))$ generated on X by the neighborhood structure Φ then $H\psi_{\tau(\Phi)}(X) \geq H\psi_o(X)$. Then from Propositions I) and II) one can obtain the following:

Corollary 1 [9]**.** *For every Hausdorff topological space X we have $|X| \leq \exp c(X) \cdot \chi(X)$.*

Corollary 2 [10]**.** *For every Hausdorff topological space X we have $|X| \leq \exp c(X) \cdot H\psi(X)$.*

Corollary 3 [14]**.** *For every o–Hausdorff neighborhood space (X, Φ) we have $|X| \leq \exp c_o(X) \cdot \chi_o(X)$.*

Corollary 4. *For every o–Hausdorff neighborhood space (X, Φ) we have $|X| \leq \exp c_o(X) \cdot H\psi_o(X)$.*

Proof. Let (X, Φ) be an o–Hausdorff neighborhood space. Let $c_o(X) \cdot H\psi_o(X) = k$. For every $p \in X$ from $H\psi o(X) \leq k$ we could obtain a family $\boldsymbol{V}_p = \{V(p,\gamma) : \gamma \in k\} \subseteq \Phi_p$ such that if $p \neq q$ then there are $\alpha, \beta \in k$ such that $V(p,\alpha) \cap V(q,\beta) = \emptyset$. Let $\boldsymbol{V} = \bigcup \{\boldsymbol{V}_p : p \in X\}$. The pair $(X, \boldsymbol{V})$ is a k–structure satisfying the conditions of Proposition I) and therefore $|X| \leq 2^k$.

Corollary 5 [14]. *For every o–separated neighborhood space (X,Φ) we have $|X| \le \exp s_o(X)\cdot\psi_o(X)$.*

Proof. Let (X,Φ) be an o–separated neighborhood space and let $s_o(X)\cdot\psi_o(X)=k$. For every $p\in X$ by $\psi_o(X)\le k$ we can obtain a family $\boldsymbol{V}_p \subseteq \Phi_p$ with $|\boldsymbol{V}_p|\le k$ and $\{p\}=\bigcap\{V : V\in \boldsymbol{V}_p\}$. We may assume that V_p is closed under finite intersections. Let $V=\bigcup\{V_p : p\in X\}$. Then $(X,\boldsymbol{V})$ is a k–structure satisfying the conditions of Proposition II). Therefore $|X|\le 2^K$.

Let $(E,\boldsymbol{V})$ be a k–structure. We shall define the following properties:

R) If for $A\in E$ we have $A^*=A$ and $p\notin A$ then there exists $V(p,\alpha)$ such that $(V(p,\alpha))^*\cap A=\emptyset$.

QL) For every $L\subseteq E$ with $L^*=L$ and every family $\boldsymbol{V}_o\subseteq\boldsymbol{V}$ that covers L, a subfamily $\boldsymbol{W}\in[\boldsymbol{V}_o]^{\le k}$ can be chosen such that $L\subseteq(\bigcup\boldsymbol{W})^*$.

Let us observe that S) + R) $\implies$ H) — because in that case for every $p\in E$ we have $\{p\}=p^*$ — in fact if $q\ne p$ then from S) there is $V(q,\alpha)\cap\{p\}=\emptyset$, so $q\notin p^*$. Then if $p\ne q$ we have $p\notin q^*=q$ and therefore we can choose $V(p,\alpha)$ such that $V(p,\alpha)^*\cap\{q\}=\emptyset$ i.e. $q\notin V(p,\alpha)^*$ i.e. there exists $\beta\in k$ such that $V(q,\beta)\cap V(p,\alpha)=\emptyset$.

KL) There is an $A\subseteq E$ with $|A|\le 2^k$ such that for each $\boldsymbol{V}_o\subseteq\boldsymbol{V}$ which covers E there is a subcollection $U\subseteq\boldsymbol{V}_o$ of cardinality at most k and $F\in[A]^{\le k}$ such that $\bigcup U\cup F^*=E$.

3. Main theorems

Theorem 1. *Let $(E,\boldsymbol{V})$ be a k–structure that satisfies* S) + KL) + *If $B\in[E]^{\le k}$ then $B^*\in[E]^{\le\exp k}$). Then $|E|\le 2^k$.*

Proof. Let $A\in E$ be such as in the property KL). Construct a sequence $\{E_\alpha:\alpha\in k^+\}$ of subsets of E such that the following conditions hold for every $\alpha\in k^+$:

(1) $|E_\alpha|\le 2^k$.

(2) If $B\subseteq\bigcup\{E_\beta:\beta\in\alpha\}$ and $|B|\le k$ then $B^*\subseteq E_\alpha$.

(3) If W is the union of at most k elements of the collection $\{V(p,\gamma) : p\in\bigcup\{E_\beta:\beta\in\alpha\},\ \gamma\in k\}, B\in[A]^{\le k}$ and $E\setminus(W\cup B^*)\ne\emptyset$ then $E_\alpha\setminus(W\cup B^*)\ne\emptyset$.

Let $\alpha\in k^+$ and let $\{E_\beta:\beta\in\alpha\}$ be already defined with the properties (1)–(3). Let $\boldsymbol{B}_\alpha=\{V(p,\gamma) : p\in\bigcup\{E_\beta:\beta\in\alpha\},\gamma\in k\}$. We have $|\boldsymbol{B}_\alpha|\le 2^k$. If possible let us choose a point $P_{W,B}\in E\setminus(W\cup B^*)$ where $B\in[A]^{\le k}$ and W is the union of at most k elements of $\boldsymbol{B}_\alpha$. Let M_α be the set of all those $P_{W,B}$. Since $|A|\le 2^k$ and $|B|\le k$ then $|M_\alpha|\le 2^k$.

Let $E'_\alpha = \bigcup\{E_\beta : \beta \in \alpha\} \cup M_\alpha$. Then $|E'_\alpha| \le 2^k$. Finally we put $E_\alpha = \bigcup\{B^* : B \in [E'_\alpha]^{\le k}\}$. We have $|E_\alpha| \le 2^k$ and $E_{\alpha'} \subseteq E_\alpha$ for each $\alpha' \in \alpha \in k^+$.

Let $L = \bigcup\{E_\alpha : \alpha \in k^+\}$. We shall prove that $L^* = L$. Let $p \in L^*$. Then for each $\gamma \in k$ there exists $x_\gamma \in V(p,\gamma) \cap L$. Let $B = \{x_\gamma : \gamma \in k\}$ and let us also note that $p \in B^*$. Since k^+ is regular and $B \subseteq L$ there exists $\alpha \in k^+$ such that $B \in [\bigcup\{E_\beta : \beta \in \alpha\}]^{\le k}$. Then $B^* \subseteq E_\alpha \subseteq L$ and therefore $p \in L$.

We shall prove that $E = A^* \cup L$ i.e. $E \setminus L \subseteq A^*$. Let us fix $q \in E \setminus L$. Then for every $\ell \in L$ there exists $\alpha(\ell,q) \in k$ such that $q \notin V(\ell,\alpha(\ell,q))$. For every $p \in E \setminus L$ there is an $\alpha(p,L) \in k$ such that $V(p,\alpha(p,L)) \cap L = \emptyset$ (from $p \notin L^* = L$). Let $\boldsymbol{W} = \{V(\ell,\alpha(\ell,q)),\ V(p,\alpha(p,L)) : \ell \in L, p \in E \setminus L\}$. We have $\boldsymbol{W} \subseteq \boldsymbol{V}$ and $\bigcup \boldsymbol{W} = E$. From KL) there is $\boldsymbol{U}_o \in [\boldsymbol{W}]^{\le k}$ and $F \in [A]^{\le k}$ such that $E = \bigcup \boldsymbol{U}_o \cup F^*$. Let $\boldsymbol{U} = \{U \in U_o : U \cap L \ne \emptyset\}$. Then $\bigcup \boldsymbol{U} \cup F^* \supseteq L$ and $\boldsymbol{U} \subseteq \{V(\ell,\alpha(\ell,q)) : \ell \in L\}$. If $q \in F^* \subseteq A^*$ we are done. If $q \notin F^*$ then $q \notin \bigcup \boldsymbol{U} \cup F^*$ therefore $E \setminus (\bigcup \boldsymbol{U} \cup F^*) \ne \emptyset$. We also have $\boldsymbol{U} \in [\boldsymbol{B}_\alpha]^{\le k}$ for some $\alpha \in k^+$. This means that we have already chosen a point $P_{\bigcup \boldsymbol{U},F} \in E \setminus (\bigcup \boldsymbol{U} \cup F^*)$ and $P_{\bigcup \boldsymbol{U},F} \in E_{\alpha+1} \subseteq L \subseteq \bigcup \boldsymbol{U} \cup F^*$. This contradiction completes the proof.

Corollary 6. *Let $(E,\boldsymbol{V})$ be a k–structure satisfying* H) + I) + KL). *Then $|E| \le 2^k$.*

Proof. From H) + I) and Theorem P in [11] it follows that if $A \in [E]^{\le k}$ then $A^* \in [E]^{\le \exp k}$ and therefore the conditons of Theorem 1 are fulfilled.

As further corollaries we easily obtain Theorem II from [10] and Theorem II' from [11] because obviously C) $\implies$ KL).

In [13] a new cardinal function for a topological space X is defined that generalizes the Lindelöf number, namely — $k\ell(X) = \omega \cdot \min\{\tau$: there is an $A \in [X]^{\le \exp \tau}$ such that $(*)$: for each open cover γ of X there are $\gamma' \in [\gamma]^{\le \tau}$ and $B \in [A]^{\le \tau}$ such that $X = \bigcup \gamma' \cup \bar{B}\}$.

The next result is a generalization of a theorem proved in [13] for regular topological spaces and $\chi(X)$ instead of $H\psi(X)$.

Corollary 7. *For every Hausdorff topological space X we have $|X| \le \exp k\ell(X) H\psi(X)$.*

Proof. Let $k\ell(X) \cdot H\psi(X) = k$ and let for each $p \in X$ the collection $\{V(p,\gamma) : \gamma \in k\}$ of open neighborhoods of p be such that if $p \ne q$ then there exist $\alpha, \beta \in k$ such that $V(p,\alpha) \cap V(q,\beta) = \emptyset$. We may assume that $\{V(p,\gamma) : \gamma \in k\}$ is closed under finite intersections. Then $\boldsymbol{V} = \{V(p,\gamma) : \gamma \in k,\ p \in X\}$ is a k–structure on X satisfying the conditions of Corollary 6.

If $k\ell_o(X)$ is the corresponding invariant for neighborhood spaces (introduced and studied in [15]) then we get the following

Corollary 8. *For every o–Hausdorff neighborhood space* (X,Φ) *we have* $|X| \leq \exp k\ell_o(X) \cdot H\psi_o(X)$.

Proof. Let (X,Φ) be an o–Hausdorff neighborhood space. Let $k\ell_o(X) \cdot H\psi_o(X) = k$ and for every $p \in X$ let us take the collection $\{V(p,\gamma) : \gamma \in k\}$ of elements of Φ_p, closed under finite intersections in such a way that if $p \neq q$ then there are $\alpha, \beta \in k$ such that $V(p,\alpha) \cap (q,\beta) = \emptyset$. Then $\boldsymbol{V} = \{V(p,\gamma) : \gamma \in k,\ p \in X\}$ is a k–structure on X satisfying the conditions of Corollary 6. Therefore $|X| \leq 2^k$.

Theorem 2. *Let* $(E,\boldsymbol{V})$ *be a k–structure that satisfies* S) + R) + QL) + *If* $B \in [E]^{\leq k}$ *then* $B^* \in [E]^{\leq \exp k}$*). Then* $|E| \leq 2^k$.

Proof. As in Theorem 1 let us construct a sequence $\{E_\alpha : \alpha \in k^+\}$ of subsets of E such that the following conditions hold for $\alpha \in k$:

(1) $|E_\alpha| \leq 2^k$.

(2) If $A \in [\bigcup\{E_\beta : \beta \in \alpha\}]^{\leq k}$ then $A^* \subseteq E_\alpha$.

(3) If W is the union of at most k elements of the family $\{V(p,\gamma) : p \in \bigcup\{E_\beta : \beta \in \alpha\},\ \gamma \in k\}$ and $E \setminus W^* \neq \emptyset$ then $E_\alpha \setminus W^* \neq \emptyset$.

Let $\alpha \in k^+$ and let $\{E_\beta;\ \beta \in \alpha :\}$ be already defined with the properties (1)–(3). Let $\boldsymbol{B}_\alpha = \{V(p,\gamma) : p \in \bigcup\{E_\beta : \beta \in \alpha\},\ \gamma \in k\}$. We have $|\boldsymbol{B}_\alpha| \leq 2^k$. If possible let us choose a point $p_W \in E \setminus W^* \neq \emptyset$ where W is the union of at most k elements of $\boldsymbol{B}_\alpha$. Let M_α be the set of all those p_W. We have $|M_\alpha| \leq 2^k$. Let $E'_\alpha = \bigcup\{E_\beta : \beta \in \alpha\} \cup M_\alpha$. Then $|E'_\alpha| \leq 2^k$. Finally let $E_\alpha = \bigcup\{B^* : B \in [E'_\alpha]^{\leq k}\}$. Since $\left|[E'_\alpha]^{\leq k}\right| \leq (2^k)^k = 2^k$ and $|B| \leq k$, we have $|B^*| \leq 2^k$ and $|E_\alpha| \leq 2^k$.

Let $L = \bigcup\{E_\alpha : \alpha \in k^+\}$ and let us show that $L = L^*$. Let $p \in L^*$. Then for each $\gamma \in k$ we have $V(p,\gamma) \cap L \neq \emptyset$ and let us choose $x_\gamma \in L \cap V(p,\gamma)$. Let $B = \{x_\gamma : \gamma \in k\}$ and let us also note that $p \in B^*$. Since k^+ is regular and $B \subseteq L$, there exists $\alpha \in k^+$ such that $B \in [\bigcup\{E_\beta : \beta \in \alpha\}]^{\leq k}$. Now $B \in [E'_\alpha]^{\leq k}$ and therefore $B^* \subseteq E_\alpha \subseteq L$. Hence $L = L^*$.

Suppose that there exists a $q \in E \setminus L$. By R) there exists $\alpha \in k$ such that $V(q,\alpha) \cap L = \emptyset$. Now for every $\ell \in L$ we have $\ell \notin V(q,\alpha)^*$, so there is $\beta(\ell,q) \in k$ such that $V(\ell,\beta(\ell,q)) \cap V(q,\alpha) = \emptyset$. Let us consider $\boldsymbol{U} = \{V(\ell,\beta(\ell,q)) : \ell \in L\}$. It covers L and $L = L^*$, so by QL) there exists a subfamily $\boldsymbol{W} \in [\boldsymbol{U}]^{\leq k}$ such that $L \subseteq (\bigcup \boldsymbol{W})^*$. By the regularity of k^+ and $|\boldsymbol{W}| \leq k$ there exists $\alpha_o \in k^+$ such that $\boldsymbol{W} \subseteq \{V(p,\gamma) : \gamma \in k,$ $p \in \bigcup\{E_\beta : \beta \in \alpha_o\}\}$. We have $V(q,\alpha) \cap (\bigcup \boldsymbol{W}) = \emptyset$ therefore $q \notin$

$(\bigcup \boldsymbol{W})^*$ hence $E \setminus (\bigcup \boldsymbol{W})^* \neq \emptyset$. Then by (3) $E_{\alpha o} \setminus (\bigcup \boldsymbol{W})^* \neq \emptyset$ and hence $L \setminus (\bigcup \boldsymbol{W})^* \neq \emptyset$ and this contradicts $L \subseteq (\bigcup \boldsymbol{W})^*$.

Corollary 10. *For every k–structure $(E, \boldsymbol{V})$ we have that* S) + R) + I) + QL) *entail* $|E| \leq 2^k$.

Proof. From S) + R) + I) and Theorem P in [11] it follows that the conditions of Theorem 2 are fulfilled and therefore $|X| \leq 2^k$.

Corollary 11. *If (X, Φ) is an o–regular neighborhood space then* $|X| \leq \exp q\ell_o(X) \cdot H\psi_o(X)$.

Corollary 12 [17]**.** *If (X, Φ) is an o–regular neighborhood space then* $|X| \leq \exp q\ell_o(X) \cdot \chi_o(X)$.

Corollary 13. *If X is a regular topological space then* $|X| \leq \exp q\ell(X) \cdot H\psi(X)$.

Corollary 14 [2]**.** *If X is a regular topological space then* $|X| \leq \exp q\ell(X) \cdot \chi(X)$.

Corollary 15. *If X is a normal topological space then* $|X| \leq \exp w\ell(X) \cdot H\psi(X)$.

Corollary 16 [3]**.** *If X is a normal topological space then* $|X| \leq \exp w\ell(X) \cdot \chi(X)$.

Finally let us discuss one more property for a given k–structure $(E, \boldsymbol{V})$:

KQL) There is $A \subseteq E$ with $|A| \leq 2^k$ such that for each $L \subseteq E$ such that $L = L^*$ and each $\boldsymbol{V}_o \subseteq \boldsymbol{V}$ that covers L there is a subcollection $\boldsymbol{U} \subseteq \boldsymbol{V}_o$ of cardinality at most k and $F \in [A]^{\leq k}$ such that $L = (\bigcup \boldsymbol{U})^* \cup F^*$.

This is a natural common generalization of KL) and QL).

Using the same ideas as in the proofs of Theorem 1 and 2 one obtains

Theorem 3. *Let $(E, \boldsymbol{V})$ be a k–structure that satisfies* S) + R) + KQL) + *If $B \in [E]^{\leq k}$ then $B^* \in [E]^{\leq \exp k}$). Then $|E| \leq 2^k$.*

Corollary 17. *For every k–structure $(E, \boldsymbol{V})$ we have that* S) + R) + I) + KQL) *entail* $|E| \leq 2^k$.

Corollary 18. *For every regular topological space X we have* $|X| \leq \exp qk\ell(X) \cdot H\psi(X)$.

Corollary 19 [16]**.** *For every regular topological space X we have* $|X| \leq \exp qk\ell(X) \cdot \chi(X)$.

We can also get:

Corollary 20. *If (X, Φ) is an o–regular neighborhood space then* $|X| \leq \exp qk\ell_o(X) \cdot H\psi_o(X)$.

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